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AN OPTIMIZED CHEN FIRST INEQUALITY FOR SEMI-SLANT SUBMANIFOLDS IN LORENTZ KENMOTSU SPACE FORMS

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Abstract

In this paper, we present necessary and sufficient conditions for a Lorentz contact manifold to be a Lorentz Kenmotsu manifold. Moreover, we obtain the optimal Chen first inequality for semi-slant submanifolds in Lorentz Kenmotsu space forms. Furthermore, the equality case of Chen inequality has been discussed.

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1 Introduction

B. Y. Chen [6] introduced the notion of slant immersions in complex geometry. B. Y. Chen [6] and N. Papaghiuc [17] pioneered the study of slant and semi-slant submanifolds of almost Hermitian manifolds, demonstrating that slant submanifolds are the natural generalization of complex (holomorphic) and totally real submanifolds. F. Etayo [12] investigated the notion of pointwise slant submanifolds in almost Hermitian manifolds as quasi-slant submanifolds and as a generalization of slant and semi-slant submanifolds. K. S. Park [18] recently presented the notion of pointwise slant and pointwise semi-slant submanifolds in almost contact metric manifolds, as well as some classification results and examples.

J. L. Cabrerizo [3] investigated semi-slant submanifolds of almost contact manifolds and Sasakian manifolds. Slant and semi-slant submanifolds of a Kenmotsu

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manifold has been investigated by V. Khan, M. Khan and K. Khan [14]. Furthermore, Lorentz Kenmotsu manifolds were introduced by R. Rosca [20]. S. Uddin and K. Singh [23] have significantly contributed to studying Lorentz β-Kenmotsu manifolds. Slant submanifolds of Lorentz Kenmotsu manifolds were studied by R. Sari [22]. Recently, M. A. Lone and I. F. Harry[15] studied Ricci solitons on Lorentz-Sasakian space forms.

Curvature invariants are intrinsic properties of Riemannian manifolds that generally affect the behaviour of the Riemannian manifold. Curvature invariants are well-known for playing a crucial role in Riemannian geometry. Curvature invariants are the most natural and vital Riemannian invariants that are widely used in the field of differential geometry, as well as in physics.

B. Y. Chen defined new sorts of Riemannian invariants on Riemannian manifolds called δ-invariants (or Chen invariants) in the early 1990s. The δ-invariants are very different from the classical scalar and Ricci curvatures, simply because both the scalar and Ricci curvatures are *total sum* of sectional curvatures on a Riemannian manifold. All non-trivial δ-invariants, on the other hand, are generated from the scalar curvature by deleting a finite number of sectional curvatures. He studied the concept of δ-invariants to discover new required conditions for the existence of minimal immersions in a Euclidean space of any dimension, as well as to apply the famous Nash embedding theorem, making use of subspaces orthogonal to the Reeb vector field ξ. D. Cioroboiu [11] established B. Y. Chen's inequalities for semi-slant submanifolds in Sasakian space forms. R. S. Gupta [13] investigated an inequality similar to B. Y. Chen's inequality for a submanifold of a Kenmotsu manifold. P. K. Panday [16] extended the idea of B. Y. Chen's inequalities for bi-slant submanifold in Kenmotsu space forms. O. Postavaru and I. Mihai[19] proved Chen's first inequality for special slant submanifolds in Lorentz-Sasakian space forms in 2021.

In this paper, our aim is to extend these Chen's inequalities to semi-slant submanifolds of Lorentz Kenmotsu space forms. In Section 2, first we give some basic formulae and definitions of Lorentz Kenmotsu manifolds. In Section 3, we recall the definition of semi-slant submanifolds of a Lorentz Kenmotsu manifold, along with the example. Besides, we obtain some characterization theorems for semi-slant submanifolds of Lorentz Kenmotsu manifolds. In Section 4, we prove an optimal Chen inequality for semi-slant Lorentz Kenmotsu space forms and further, we consider the equality case of the Chen first inequality.

2 Preliminaries

Consider $\bar{\mathcal{M}}$ to be a $(2n+1)$-dimensional differentiable manifold along with an almost contact structure (ϕ, η, ξ), such that ϕ denotes a tensor field of type $(1, 1)$,

η denotes a 1-form, and ξ denotes a vector field on $\bar{\mathcal{M}}$ satisfying,

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1. \tag{1}$$

Then, we say that $\bar{\mathcal{M}}$ is an almost contact manifold. It follows that $\phi(\xi) = 0$, $\eta \circ \phi = 0$, rank$\phi = 2n$. In addition, if there exists a Lorentz metric g satisfying

$$g(\phi\mathcal{X}, \phi\mathcal{Y}) = g(\mathcal{X}, \mathcal{Y}) + \eta(\mathcal{X})\eta(\mathcal{Y}), \quad g(\xi, \xi) = -1, \tag{2}$$

then (ϕ, η, ξ, g) is called a *Lorentz almost contact structure* and $\bar{\mathcal{M}}$ is said to be a *Lorentz almost contact manifold.*

For a Lorentz almost contact manifold, we also have $\eta(\mathcal{X}) = g(\mathcal{X}, \xi)$, where, ξ is the time-like vector field on $\bar{\mathcal{M}}$. We consider a local basis $\{e_1, ..., e_{2n}, \xi\}$ of $\mathcal{T}\bar{\mathcal{M}}$, i.e,

$$g(e_i, e_j) = \delta_{ij} \quad \text{and} \quad g(\xi, \xi) = -1,$$

that is $e_1, ..., e_{2n}$ are spacelike vector fields, and ξ is timelike. The fundamental 2-form Φ is defined by $\Phi(\mathcal{X}, \mathcal{Y}) = g(\mathcal{X}, \phi\mathcal{Y})$, for any $\mathcal{X}, \mathcal{Y} \in \Gamma(\mathcal{T}\bar{\mathcal{M}})$. Moreover, a Lorentz almost contact manifold is normal if

$$N = [\phi, \phi] + 2d\eta \otimes \xi = 0,$$

where $[\phi, \phi]$ denotes the Nijenhuis tensor field associated with ϕ and is given by:

$$[\phi, \phi](\mathcal{X}, \mathcal{Y}) = \phi^2[\mathcal{X}, \mathcal{Y}] + [\phi\mathcal{X}, \phi\mathcal{Y}] - \phi[\phi\mathcal{X}, \mathcal{Y}] - \phi[\mathcal{X}, \phi\mathcal{Y}].$$

Definition 1. *[20] Let $\bar{\mathcal{M}}$ be a $(2n+1)$-dimensional Lorentz almost contact manifold with (ϕ, ξ, η, g). $\bar{\mathcal{M}}$ is said to be a Lorentz almost Kenmotsu manifold if the 1-form η is closed and $d\Phi = -2\eta \wedge \Phi$. A normal Lorentz almost Kenmotsu manifold $\bar{\mathcal{M}}$ is called a Lorentz Kenmotsu manifold.*

Theorem 1. *[20] Let $(\bar{\mathcal{M}}, \phi, \xi, \eta, g)$ be a Lorentz contact manifold. $\bar{\mathcal{M}}$ is a Lorentz Kenmotsu manifold if and only if*

$$(\bar{\nabla}_{\mathcal{X}}\phi)\mathcal{Y} = -g(\phi\mathcal{X}, \mathcal{Y})\xi + \eta(\mathcal{Y})\phi\mathcal{X}, \tag{3}$$

for all $\mathcal{X}, \mathcal{Y} \in \Gamma(T\bar{\mathcal{M}})$.

Let $\mathcal{K}(\mathcal{X}_p, \mathcal{Y}_p)$ be the sectional curvature for the 2 plane spanned by $\mathcal{X}_p$ and $\mathcal{Y}_p$, $p \in \bar{\mathcal{M}}$. $\bar{\mathcal{M}}$ is said to have constant $\phi - \textit{sectional}$ curvature if $\mathcal{K}(\mathcal{X}_p, \phi\mathcal{X}_p)$ is constant for any point p and for any unit vector $\mathcal{X}_p \neq 0$ such that $\eta(\mathcal{X}_p) = 0$.

A Lorentz Kenmotsu manifold is said to be a Lorentz Kenmotsu space form if it has constant ϕ - holomorphic section curvature c and it is denoted by $\bar{\mathcal{M}}(c)$. The curvature tensor field $\mathcal{R}$ of $\bar{\mathcal{M}}(c)$ is given by [21]

$$\begin{aligned}\mathcal{R}(\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{W}) \quad = \quad & \frac{c+3}{4}\{g(\mathcal{X}, \mathcal{W})g(\mathcal{Y}, \mathcal{Z}) - g(\mathcal{X}, \mathcal{Z})g(\mathcal{Y}, \mathcal{W})\} \\ & + \frac{c-1}{4}\{g(\phi\mathcal{X}, \mathcal{W})g(\phi\mathcal{Y}, \mathcal{Z}) - g(\phi\mathcal{X}, \mathcal{Z})g(\phi\mathcal{Y}, \mathcal{W}) \\ & - 2g(\phi\mathcal{X}, \mathcal{Y})g(\phi\mathcal{Z}, \mathcal{W}) + g(\mathcal{X}, \mathcal{Z})\eta(\mathcal{Y})\eta(\mathcal{W}) \\ & - g(\mathcal{Y}, \mathcal{Z})\eta(\mathcal{X})\eta(\mathcal{W}) + g(\mathcal{Y}, \mathcal{W})\eta(\mathcal{X})\eta(\mathcal{Z})\},\end{aligned} \tag{4}$$

where $\mathcal{X}, \mathcal{Y}, \mathcal{Z}, \mathcal{W} \in \Gamma(\mathcal{T}\bar{\mathcal{M}})$.
The mean curvature vector $\mathcal{H}$ for an orthonormal frame$\{e_1, e_2, ..., e_n\}$ of the tangent space $\mathcal{T}\bar{\mathcal{M}}$ on $\bar{\mathcal{M}}$ is defined by

$$\mathcal{H} = \frac{1}{n} Trace(\sigma) = \frac{1}{n}\sum_{k=1}^{n} \sigma(e_k, e_k) \quad \text{and} \quad \|\mathcal{H}\|^2 = \frac{1}{n^2}\Big(\sum_{k=1}^{n} \sigma(e_k, e_k)\Big)^2,$$

where $n = dim\bar{\mathcal{M}}$ and σ is the second fundamental form. Also, we have

$$\sigma_{ks}^r = g(\sigma(e_k, e_s)e_r) \quad \text{and} \quad \|\sigma\|^2 = \sum_{k,s}^{n} g(\sigma(e_k, e_s), \sigma(e_k, e_s)).$$

Remark 1. *For $\mathcal{X}, \mathcal{Y} \in \mathcal{T}_p\bar{\mathcal{M}}$ space-like vectors, $\bar{\mathcal{R}}(\mathcal{X}, \mathcal{Y})\xi = 0$, therefore*

$$\mathcal{A}_{\mathcal{N}\mathcal{Y}}\mathcal{X} = \mathcal{A}_{\mathcal{N}\mathcal{X}}\mathcal{Y},$$

or equivalently, the coefficients of the second fundamental form satisfy

$$\sigma_{ks}^r = \sigma_{rs}^k = \sigma_{kr}^s, \tag{5}$$

for all $e_r, e_k, e_s \in \mathcal{T}_p\bar{\mathcal{M}}$.

Throughout, we denote by $\bar{\mathcal{M}}$ a Lorentz Kenmotsu manifold, $\mathcal{M}$ a submanifold of $\bar{\mathcal{M}}$ with structure vector field ξ tangent to $\mathcal{M}$. σ and $\mathcal{A}$ denote the second fundamental form and the shape operator of the immersion of $\mathcal{M}$ into $\bar{\mathcal{M}}$ respectively. If ∇ is the induced connection on $\mathcal{M}$, the Gauss and Weingarten formulae of $\mathcal{M}$ into $\bar{\mathcal{M}}$ are given respectively.

$$\bar{\nabla}_{\mathcal{X}}\mathcal{Y} = \nabla_{\mathcal{X}}\mathcal{Y} + \sigma(\mathcal{X}, \mathcal{Y}), \tag{6}$$

$$\bar{\nabla}_{\mathcal{X}}\mathcal{V} = -\mathcal{A}_{\mathcal{V}}\mathcal{X} + \nabla_{\mathcal{X}}^{\perp}\mathcal{V}, \tag{7}$$

for all vector fields $\mathcal{X}, \mathcal{Y}$ on $\mathcal{M}$ and normal vector fields $\mathcal{V}$ on $\mathcal{M}$, where $\nabla^{\perp}$ denotes the connection on the normal bundle $\mathcal{T}^{\perp}\mathcal{M}$ of $\mathcal{M}$. σ and $\mathcal{A}$ are related by

$$g(\sigma(\mathcal{X}, \mathcal{Y}), \mathcal{V}) = g(\mathcal{A}_{\mathcal{V}}\mathcal{X}, \mathcal{Y}), \tag{8}$$

where the induced Riemannian metric on $\mathcal{M}$ is denoted by the same symbol g. Now, for any $x \in \mathcal{M}$, $\mathcal{X} \in \mathcal{T}_x\mathcal{M}$ and $\mathcal{V} \in \mathcal{T}^{\perp}\mathcal{M}$ of $\mathcal{M}$, we put

$$\phi\mathcal{X} = \mathcal{T}\mathcal{X} + \mathcal{N}\mathcal{X}, \tag{9}$$

$$\phi\mathcal{V} = t\mathcal{V} + n\mathcal{V}, \tag{10}$$

where $\mathcal{T}\mathcal{X}$ and $\mathcal{N}\mathcal{X}$ (respectively, $t\mathcal{V}$ and $n\mathcal{V}$) are the tangential and the normal components of $\phi\mathcal{X}$ (respectively, of $\phi\mathcal{V}$).

The releation (9) gives rise to an endomorphism $\mathcal{T} : \mathcal{T}_x\mathcal{M} \to \mathcal{T}_x\mathcal{M}$. The covariant derivatives $\nabla\mathcal{T}$, and $\nabla\mathcal{N}$ are defined by[14]

$$(\nabla_{\mathcal{X}}\mathcal{T})\mathcal{Y} = \nabla_{\mathcal{X}}\mathcal{T}\mathcal{Y} - \mathcal{T}\nabla_{\mathcal{X}}\mathcal{Y}$$

$$(\nabla_{\mathcal{X}}\mathcal{N})\mathcal{Y} = \nabla_{\mathcal{X}}^{\perp}\mathcal{N}\mathcal{Y} - \mathcal{N}\nabla_{\mathcal{X}}\mathcal{Y}.$$

3 Semi-slant submanifolds of a Lorentz Kenmotsu manifold

N. Papaghiuc [17] developed the theory of semi-slant submanifolds of an almost Hermitian manifolds, such that the class of proper CR-submanifolds and the class of slant submanifolds appear as particular cases in the class of semi-slant submanifolds. J. L. Cabrerizo [3] investigated semi-slant submanifolds of Kenmotsu manifold. The semi-slant submanifolds of a Lorentz Kenmotsu manifold are now studied in this section. A submanifold $\mathcal{M}$ of a Lorentz Kenmotsu manifold $\bar{\mathcal{M}}$ is said to be *semi-slant submanifold*, if there exist two orthogonal spacelike distribution $\mathcal{D}_1$ and $\mathcal{D}_2$ on $\mathcal{M}$ such that:

$$\mathcal{TM} = \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus < \xi >,$$

where the spacelike distribution $\mathcal{D}_1$ is an invariant distribution, i.e., $\phi(\mathcal{D}_1) = \mathcal{D}_1$. The spacelike distribution $\mathcal{D}_2$ is slant with slant angle $\theta \neq 0$. In this case, the angle θ is called the *slant angle* of a submanifold $\mathcal{M}$. We note that the invariant distribution of $\mathcal{M}$ is a slant spacelike distribution with zero angle. Moreover, it is clear that, if $\theta = \frac{\pi}{2}$, then semi-slant submanifold is an anti-invariant submanifold. On the other hand, if we denote the dimensions of the distribution $\mathcal{D}_i$ by d_i in a semi-slant submanifold for $i = 1, 2$, then we have the following cases: [22]

1. If $d_2 = 0$, then $\mathcal{M}$ is an invariant submanifold.
2. If $d_1 = 0$, and $\theta = \frac{\pi}{2}$, the $\mathcal{M}$ is an anti-invariant submanifold.
3. If $d_1 = 0$ and $\theta \neq \frac{\pi}{2}$, then $\mathcal{M}$ is a slant Lorentz submanifold.
4. If $d_1 d_2 \neq 0$ and $\theta \neq \frac{\pi}{2}$, then $\mathcal{M}$ is a proper semi-slant Lorentz submanifold.

Example 1. *Suppose $\mathcal{M}$ is a $(2n+1)$-dimensional manifold defined by,*

$$\mathcal{M} = \{(x_1, ..., x_n, y_1, ..., y_n, z) \in \mathbb{R}^{(2n+1)}; z \neq 0\}.$$

Then $(\mathcal{M}, \phi, \xi, \eta, g)$ is a Lorentz Kenmotsu manifold with

$$\eta = dz, \quad \xi = \partial z,$$

$$g = e^{-z}\bigg(\sum_{i=1}^{n}(dx^i \otimes dx^i + dy^i \otimes dy^i)\bigg) + \eta \otimes \eta,$$

$$\phi\bigg(\sum_{i=1}^{n}(\mathcal{X}_i \partial x_i + \mathcal{Y}_i \partial y_i) + \mathcal{Z}\partial z\bigg) = \sum_{i=1}^{n}(-\mathcal{Y}_i \partial x_i + \mathcal{X}_i \partial y_i).$$

Then, we consider a submanifold of $\mathbb{R}^7$ defined by

$$\mathcal{M} = \mathcal{X}(a, b, c, t, u) = (a, 0, c, b, \cos t, \sin t, u).$$

A local frame of $\mathcal{TM}$ is given by

$$e_1 = \frac{\partial}{\partial x_1}, \ e_2 = \frac{\partial}{\partial y_1},$$

$$e_3 = \frac{\partial}{\partial x_3}, \ e_4 = -\sin t\frac{\partial}{\partial y_2} + \cos t\frac{\partial}{\partial y_3},$$

$$e_5 = \frac{\partial}{\partial z} = \xi.$$

We can take $\mathcal{D}_1 = \{e_1, e_2\}$ and $\mathcal{D}_2 = \{e_3, e_4\}$; the $\mathcal{D}_1$ and $\mathcal{D}_2$ are invariant distribution and slant spacelike distribution with slant angle $\theta = t$, respectively. Thus $\mathcal{TM} = \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \{\xi\}$ is a semi-slant Lorentz submanifold of $\mathbb{R}^7$.

On the other hand, let $\mathcal{M}$ be a semi-slant Lorentz submanifold and $\mathcal{X} \in \mathcal{TM}$. Then as $\mathcal{TM} = \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus < \xi >$, we write

$$\mathcal{X} = \mathcal{P}_1\mathcal{X} + \mathcal{P}_2\mathcal{X} + \eta(\mathcal{X})\xi, \tag{11}$$

where $\mathcal{P}_1\mathcal{X} \in \mathcal{D}_1$ and $\mathcal{P}_2\mathcal{X} \in \mathcal{D}_2$. Now by using equations (9) and (11), we obtain

$$\phi\mathcal{X} = \phi\mathcal{P}_1\mathcal{X} + \mathcal{T}\mathcal{P}_2\mathcal{X} + \mathcal{N}\mathcal{P}_2\mathcal{X}. \tag{12}$$

It is easy to see that

$$\phi\mathcal{P}_1\mathcal{X} = \mathcal{T}\mathcal{P}_1\mathcal{X}, \ \mathcal{N}\mathcal{P}_1\mathcal{X} = 0, \ \mathcal{T}\mathcal{P}_2\mathcal{X} \in \mathcal{D}_2. \tag{13}$$

Thus,

$$\mathcal{T}\mathcal{X} = \phi\mathcal{P}_1\mathcal{X} + \mathcal{T}\mathcal{P}_2\mathcal{X} \tag{14}$$

and

$$\mathcal{N}\mathcal{X} = \mathcal{N}\mathcal{P}_2\mathcal{X}. \tag{15}$$

Now, these observations leads to the following proposition for semi-slant submanifolds of a Lorentz Kenmotsu manifold. We assume that the structure vector field is tangent to the submanifold $\mathcal{M}$.

Proposition 1. *Let $\mathcal{M}$ be a Lorentz semi-slant submanifold of a Lorentz Kenmotsu manifold $\bar{\mathcal{M}}$. Then for any $\mathcal{X}, \mathcal{Y} \in \Gamma(\mathcal{TM})$ we have*

$$(\nabla_{\mathcal{X}}\mathcal{T})\mathcal{Y} = g(\mathcal{X}, \mathcal{T}\mathcal{Y})\xi + \eta(\mathcal{Y})\mathcal{T}\mathcal{X} + t\sigma(\mathcal{X}, \mathcal{Y}) + \mathcal{A}_{\mathcal{N}\mathcal{Y}}\mathcal{X},$$

$$(\nabla_{\mathcal{X}}\mathcal{N})\mathcal{Y} = n\sigma(\mathcal{X}, \mathcal{Y}) + \eta(\mathcal{Y})\mathcal{N}\mathcal{X} - \sigma(\mathcal{X}, \mathcal{T}\mathcal{Y}).$$

Proof. For any $\mathcal{X}, \mathcal{Y} \in \Gamma(\mathcal{TM})$, we have

$$(\bar{\nabla}_{\mathcal{X}}\phi)\mathcal{Y} = \bar{\nabla}_{\mathcal{X}}\phi\mathcal{Y} - \phi\bar{\nabla}_{\mathcal{X}}\mathcal{Y}.$$

Now, using equation (3), we have

$$-g(\phi\mathcal{X}, \mathcal{Y})\xi + \eta(\mathcal{Y})\phi\mathcal{X} = \bar{\nabla}_{\mathcal{X}}\phi\mathcal{Y} - \phi\bar{\nabla}_{\mathcal{X}}\mathcal{Y},$$

and using equation (12), we have

$$\begin{aligned} g(\mathcal{X}, \phi\mathcal{P}_1\mathcal{Y} + \mathcal{TP}_2\mathcal{Y} + \mathcal{NP}_2\mathcal{Y})\xi \quad &+ \quad \eta(\mathcal{Y})(\phi\mathcal{P}_1\mathcal{X} + \mathcal{TP}_2\mathcal{X} + \mathcal{NP}_2\mathcal{X}) \\ &= \bar{\nabla}_{\mathcal{X}}(\phi\mathcal{P}_1\mathcal{Y} + \mathcal{TP}_2\mathcal{Y} + \mathcal{NP}_2\mathcal{Y}) - \phi\bar{\nabla}_{\mathcal{X}}\mathcal{Y}. \end{aligned}$$

Using equation (6) and equation (7), we have

$$\begin{aligned} g(\mathcal{X}, \phi\mathcal{P}_1\mathcal{Y} + \mathcal{TP}_2\mathcal{Y} \quad &+ \quad \mathcal{NP}_2\mathcal{Y})\xi + \eta(\mathcal{Y})(\phi\mathcal{P}_1\mathcal{X} + \mathcal{TP}_2\mathcal{X} + \mathcal{NP}_2\mathcal{X}) \\ &= \nabla_{\mathcal{X}}\phi\mathcal{P}_1\mathcal{Y} + \sigma(\mathcal{X}, \phi\mathcal{P}_1\mathcal{Y}) + \nabla_{\mathcal{X}}\mathcal{TP}_2\mathcal{Y} \\ &+\sigma(\mathcal{X}, \mathcal{TP}_2\mathcal{Y})\nabla^{\perp}_{\mathcal{X}}\mathcal{NP}_2\mathcal{Y} - \mathcal{A}_{\mathcal{NP}_2\mathcal{Y}}\mathcal{X} - \phi(\nabla_{\mathcal{X}}\mathcal{Y}) \\ &-t\sigma(\mathcal{X}, \mathcal{Y}) - \eta\sigma(\mathcal{X}, \mathcal{Y}). \end{aligned}$$

By using equation (14) and equation (15), we arrive at

$$\begin{aligned} g(\mathcal{X}, \mathcal{TY})\xi + g(\mathcal{X}, \mathcal{NY})\xi + \eta(\mathcal{Y})\mathcal{TX} + \eta(\mathcal{Y})\mathcal{NX} \quad &= \quad (\nabla_{\mathcal{X}}\mathcal{T})\mathcal{Y} + (\nabla_{\mathcal{X}}\mathcal{N})\mathcal{Y} \\ &+\sigma(\mathcal{X}, \mathcal{TY}) - \mathcal{A}_{\mathcal{NY}}\mathcal{X} \\ &-t\sigma(\mathcal{X}, \mathcal{Y}) - \eta\sigma(\mathcal{X}, \mathcal{Y}). \end{aligned}$$

Now, comparing tangential and normal parts we get the desired result. □

Proposition 2. *Let $\mathcal{M}$ be an invariant Lorentz submanifold of a $(2n+1)$-dimensional Lorentz Kenmotsu manifold $\bar{\mathcal{M}}$. Then*

$$(\nabla_{\mathcal{X}}\mathcal{T})\mathcal{Y} = -g(\mathcal{TX}, \mathcal{Y})\xi + \eta(\mathcal{Y})\mathcal{TX},$$

for any $\mathcal{X}, \mathcal{Y} \in \Gamma(\mathcal{TM})$.

Proof. Since $\mathcal{M}$ is an invariant submanifold, we have $\mathcal{TX} = \phi\mathcal{X}$ for any $\mathcal{X} \in \Gamma(\mathcal{TM})$, then

$$\begin{aligned} (\nabla_{\mathcal{X}}\mathcal{T})\mathcal{Y} &= \nabla_{\mathcal{X}}\mathcal{TY} - \mathcal{T}\nabla_{\mathcal{X}}\mathcal{Y} \\ &= \nabla_{\mathcal{X}}\phi\mathcal{Y} - \phi\nabla_{\mathcal{X}}\mathcal{Y} \\ &= \bar{\nabla}_{\mathcal{X}}\phi\mathcal{Y} - \sigma(\mathcal{X}, \phi\mathcal{Y}) - \phi\bar{\nabla}_{\mathcal{X}}\mathcal{Y} + \phi\sigma(\mathcal{X}, \mathcal{Y}). \end{aligned}$$

Now, using equation (12) we have

$$\begin{aligned} (\nabla_{\mathcal{X}}\mathcal{T})\mathcal{Y} \quad = \quad & \bar{\nabla}_{\mathcal{X}}(\phi\mathcal{P}_1\mathcal{Y} + \mathcal{TP}_2\mathcal{Y} + \mathcal{NP}_2\mathcal{Y}) - \sigma(\mathcal{X}, \phi\mathcal{Y}) - \phi\mathcal{P}_1\bar{\nabla}_{\mathcal{X}}\mathcal{Y} \\ & -\mathcal{TP}_2\bar{\nabla}_{\mathcal{X}}\mathcal{Y} - \mathcal{NP}_2\bar{\nabla}_{\mathcal{X}}\mathcal{Y} + \phi\sigma(\mathcal{X}, \mathcal{Y}). \end{aligned}$$

Which, when further simplified and using equation (12), yields

$$(\nabla_{\mathcal{X}}\mathcal{T})\mathcal{Y} = (\bar{\nabla}_{\mathcal{X}}\phi)\mathcal{Y}.$$

Since, $\mathcal{M}$ is an invariant submanifold, we can rewrite the above equation as

$$(\nabla_{\mathcal{X}}\mathcal{T})\mathcal{Y} = -g(\mathcal{TX}, \mathcal{Y})\xi + \eta(\mathcal{Y})\mathcal{TX}.$$

This completes the proof. □

4 An optimal Chen first inequality for semi-slant submanifold of Lorentz Kenmotsu space form

Let $\mathcal{M}$ be a submanifold of a Lorentz Kenmotsu space form $\bar{\mathcal{M}}$. A submanifold $\mathcal{M}$ has constant ϕ-sectional curvature c if and only if the curvature tensor $\mathcal{R}$ satisfies [22]

$$\begin{aligned}\mathcal{R}(\mathcal{X},\mathcal{Y},\mathcal{Z},\mathcal{W}) &= \frac{c+3}{4}\{g(\mathcal{X},\mathcal{W})g(\mathcal{Y},\mathcal{Z})-g(\mathcal{X},\mathcal{Z})g(\mathcal{Y},\mathcal{W})\}\\ &+\frac{c-1}{4}\{g(\phi\mathcal{X},\mathcal{W})g(\phi\mathcal{Y},\mathcal{Z})-g(\phi\mathcal{X},\mathcal{Z})g(\phi\mathcal{Y},\mathcal{W})\\ &-2g(\phi\mathcal{X},\mathcal{Y})g(\phi\mathcal{Z},\mathcal{W})+g(\mathcal{X},\mathcal{Z})\eta(\mathcal{Y})\eta(\mathcal{W})\\ &-g(\mathcal{Y},\mathcal{Z})\eta(\mathcal{X})\eta(\mathcal{W})+g(\mathcal{Y},\mathcal{W})\eta(\mathcal{X})\eta(\mathcal{Z})\}\\ &+g(\sigma(\mathcal{X},\mathcal{W}),\sigma(\mathcal{Y},\mathcal{Z}))-g(\sigma(\mathcal{Y},\mathcal{W}),\sigma(\mathcal{X},\mathcal{Z})), \qquad (16)\end{aligned}$$

for all $\mathcal{X},\mathcal{Y},\mathcal{Z},\mathcal{W}\in\Gamma(\mathcal{TM})$. Now, we can choose a local field of orthonormal frames $\{e_1,...,e_{2p},e_{2p+1},...,e_{2q},e_{2q+1}\}$ on the semi-slant submanifold $\mathcal{M}$. Then we have

$$\mathcal{D}_1=\{e_1,...,e_{2p}\},\ \mathcal{D}_2=\{e_{2p+1},...,e_{2q}\} \quad \text{and} \quad \xi=\{e_{2q+1}\}$$

where $\dim\mathcal{D}_1=2p$, $\dim\mathcal{D}_2=2q$, and $2p+2q=n$.
Let $\mathcal{M}$ be a semi slant Lorentz submanifold of a Lorents Kenmotsu space form $\bar{\mathcal{M}}$. Then, the scalar curvature is given by [22]

$$\begin{aligned}\tau &= \left\{\frac{c+3}{4}(2p+2q-3)+\frac{c-1}{4}(3+3\cos^2\theta-1)\right\}(2p+2q+1)\\ &-\frac{3(c-1)}{4}+\frac{1}{(2p+2q+1)^2}\|\mathcal{H}\|^2+\|\sigma\|^2. \qquad (17)\end{aligned}$$

Definition 2. *[9] The first Chen invariant is defined by*

$$\delta(p)=\tau(p)-inf\mathcal{K}(p),$$

where $\tau(p)$ is the scalar curvature and $inf\mathcal{K}(p)$ is defined as $inf\mathcal{K}(p)$ $=inf\{\mathcal{K}(\pi)|\pi\in\mathcal{T}_p\mathcal{M},dim\pi=2\}$.

Definition 3. *[4] For a 2-plane section $\pi\subset\mathcal{T}_p\mathcal{M}$ orthonormal to ξ_p, one defines by*

$$\Psi^2(\pi)=g^2(\mathcal{P}e_1,e_2), \qquad (18)$$

where $\{e_1,e_2\}$ is an orthonormal basis of π.

Actually, $\Psi^2(\pi)$ is a real number in the interval $[0,1]$, and is independent of the basis choice.

Theorem 2. *[13] Let $\Psi:\mathcal{M}^{n+1}\to\bar{\mathcal{M}}^{2n+1}$ be an isometric immersion from a Riemannian $(n+1)$-dimensional manifold $\mathcal{M}$ into a Kenmotsu space form $\bar{\mathcal{M}}(c)$*

of dimensions $(2n+1)$ of constant curvature c, such that $\xi \in \mathcal{TM}$. Then, for any point $p \in \mathcal{M}$ and any plane section $\Pi \in \mathcal{D}_p$, we have

$$\begin{aligned}\tau - \mathcal{K}(\Pi) \leq\ & \frac{(n+1)^2(n-1)}{2n}\|\mathcal{H}\|^2 + \frac{1}{2}(n+1)(n-2)\frac{(c-3)}{4} - n \\ & + \frac{3}{2}\|\mathcal{P}\|^2\frac{c+1}{4} - 3g^2(e_1, \phi e_2)\frac{c+1}{4}.\end{aligned}$$

The equality case in the above inequalities hold at a point $p \in \mathcal{M}$ if and only if there exist an orthonormal basis $\{e_1, e_2, ..., e_n, e_{n+1}\}$ of $\mathcal{T}_p\mathcal{M}$ and an orthonormal basis $\{e_{n+2}, e_{n+3}, ..., e_{2n+1}\}$ of $\mathcal{T}_p^{\perp}\mathcal{M}$ such that
(a) $e_{n+1} = \xi_x$,
(b) Π *is spanned by* e_1, e_2 *and*
(c) the shape operator of $\mathcal{M}$ in $\bar{\mathcal{M}}(c)$, at a point x take the following forms:

$$\mathcal{A}_{n+2} = \begin{pmatrix} a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & 0_{n-1} \end{pmatrix},$$

$$\mathcal{A}_{e_r} = \begin{pmatrix} \sigma_{11}^r & \sigma_{12}^r & 0 \\ \sigma_{12}^r & -\sigma_{11}^r & 0 \\ 0 & 0 & 0_{n-1} \end{pmatrix}, r \in \{n+3, ..., 2n+1\}.$$

Corollary 1. *[13] Let $\mathcal{M}$ be an $(n+1)$-dimensional anti-invariant submanifold of a $(2n+1)$-dimensional Kenmotsu space form $\bar{\mathcal{M}}(c)$ such that $\xi \in \mathcal{TM}$. Then, we have*

$$\delta_{\mathcal{M}}^{\mathcal{D}} \leq \frac{(n+1)^2(n-1)}{2n} + \frac{1}{2}(n+2)(n-1)\frac{(c-3)}{4} - n,$$

where $\delta_{\mathcal{M}}^{\mathcal{D}}$ denotes the difference between the scalar curvature and $inf_{\mathcal{D}}\mathcal{K}(p)$ such that $inf_{\mathcal{D}}\mathcal{K}(p)$ is defined as $inf\mathcal{K}(p) = inf\{\mathcal{K}(\pi) | \pi \in \mathcal{T}_p\mathcal{M}, dim\pi = 2\}$.

Theorem 3. *[13] Let $\psi : \mathcal{M}^{n+1} \to \bar{\mathcal{M}}^{2n+1}(c)$ be a θ-slant immersion of a Riemannian $(n+1)$-dimensional manifold into a Kenmotsu space form $\bar{\mathcal{M}}(c)$. Then for any point $x \in \mathcal{M}$ and any plane section $\Pi \subset \mathcal{D}_x$, we have*

$$\begin{aligned}\tau - \mathcal{K}(c) \leq\ & \frac{(n+1)^2(n-1)}{2n}\|\mathcal{H}\|^2 + \frac{1}{2}(n+2)(n-1)\frac{c-3}{4} - n \\ & + \frac{3}{4}(c+1)(\frac{n}{2}\cos^2\theta - g^2(e_1, \phi e_2)).\end{aligned}$$

Now, we can extend these resuts for semi-slant submanifolds of Lorentz Kenmotsu space form:

Theorem 4. *Let $\mathcal{M}$ be an $(n+1)$-dimensional semi-slant Lorentz submanifold into a $(2n+1)$-dimensional Lorentz Kenmotsu space form $\tilde{\mathcal{M}}(C)$ and $p \in \mathcal{M}$,*

$\pi \in \mathcal{T}_p\mathcal{M}$ a 2-plane section orthogonal to ξ_p. Then

$$\begin{aligned}\tau(p) - inf\mathcal{K}(p) \;&\leq\; \frac{(n+1)^2(2n-3)}{2(2n+3)}\|\mathcal{H}\|^2 + \frac{c+3}{4}\Big[(n-3)(n+1)-1\Big] \\ &+\frac{c-1}{4}\Big[(2+3\cos^2\theta)(n+1)\Big] - \frac{3(c-1)}{4}\Psi^2(\pi) + \frac{2(c-1)}{4}.\end{aligned} \tag{19}$$

Moreover, the equality case of the inequality holds for some plane section π at a point $p \in \mathcal{M}$ if and only if there exists an orthonormal basis $\{e_0 = \xi, e_1, e_2, ..., e_n\}$ at p such that $\pi = span\{e_1, e_2\}$, and with respect to this basis the second fundamental form takes the following form

$$\begin{gathered}\sigma(e_1,e_1) = a\mathcal{N}e_1 + 3b\mathcal{N}e_3,\ \sigma(e_1,e_3) = 3b\mathcal{N}e_1,\ \sigma(e_3,e_j) = 4b\mathcal{N}e_j, \\ \sigma(e_2,e_2) = -a\mathcal{N}e_1 + 3b\mathcal{N}e_3,\ \sigma(e_2,e_3) = 3b\mathcal{N}e_2,\ \sigma(e_j,e_k) = 4b\mathcal{N}e_3\delta_{jk}, \\ \sigma(e_1,e_2) = -a\mathcal{N}e_2, \sigma(e_3,e_3) = 12b\mathcal{N}e_3,\ \sigma(e_1,e_j) = \sigma(e_2,e_j) = 0,\end{gathered}$$

for some a, b and for $j, k = 4, ..., n$.

Proof. The sectional curvature $\mathcal{K}(\pi)$ of semi-slant Lorentz submanifold of a Lorentz Kenmotsu space form $\bar{\mathcal{M}}$ can be obtained by putting $\mathcal{X} = \mathcal{W} = e_i$ and $\mathcal{Y} = \mathcal{Z} = e_j$ in equation (16); we have

$$\begin{aligned}\sum_{i,j=1}^{2p+2q+1} \mathcal{R}(e_i,e_j,e_j,e_i) \;&=\; \frac{c+3}{4}\Big[g(e_i,e_i)g(e_j,e_j) - g(e_i,e_j)g(e_j,e_i)\Big] \\ &+\frac{c-1}{4}\Big[g(\phi e_i,e_i)g(\phi e_j,e_j) - g(\phi e_i,e_j)g(\phi e_j,e_i) \\ &-2g(\phi e_i,e_j)g(\phi e_j,e_i) + g(e_i,e_j)g(e_j,\xi)g(e_i,\xi) \\ &-g(e_i,e_j)g(e_i,\xi)g(e_i,\xi) + g(e_i,e_j)g(e_i,\xi)g(e_j,\xi)\Big] \\ &+g(\sigma(e_i,e_i),\sigma(e_j,e_j)) - g(\sigma(e_j,e_i),\sigma(e_i,e_j)),\end{aligned}$$

or,

$$\begin{aligned}\mathcal{K}(\pi) \;&=\; \sum_{i,j=1}^{2p+2q+1}\frac{c+3}{4} + \frac{(c-1)}{4}\sum_{i,j=1}^{2p+2q+1}\{3g^2(\phi e_i,e_j) - 1\} \\ &+\sum_{i,j=1}^{2p+2q+1} g(\sigma(e_i,e_i),\sigma(e_j,e_j)) - \sum_{i,j=1}^{2p+2q+1} g(\sigma(e_j,e_i),\sigma(e_i,e_j)).\end{aligned} \tag{20}$$

Using equation (18), we have

$$\mathcal{K}(\pi) = \frac{c+3}{4} + \frac{3(c-1)}{4}\Psi^2(\pi) - \frac{c-1}{4} + \frac{1}{(2p+2q+1)}\|\mathcal{H}\|^2 - \|\sigma\|^2. \tag{21}$$

From equation (17), we have

$$\begin{aligned}\tau(p) &= \left[\frac{c+3}{4}(2p+2q-3)+\frac{c-1}{4}(2+3\cos^2\theta)\right](2p+2q+1)\\ &-\frac{3(c-1)}{4}+\sum_{i=1}^{2p+2q}\big(g(\sigma(e_i,e_i),\sigma(e_j,e_j)\big)-g\big(\sigma(e_i,e_j),\sigma(e_i,e_j)\big).\end{aligned}$$

We can rewrite the above equation as

$$\begin{aligned}\tau(p) &= \left[\frac{c+3}{4}(2p+2q-3)+\frac{c-1}{4}(2+3\cos^2\theta)\right](2p+2q+1)\\ &-\frac{3(c-1)}{4}+\sum_{r=1}^{2p+2q}\sum_{1\leq i<j\leq 2p+2q}\big(\sigma_{ii}^r\sigma_{jj}^r-(\sigma_{ij}^r)\big).\end{aligned} \tag{22}$$

Similarly, equation (21) can be re-written as

$$\mathcal{K}(\pi)=\frac{c+3}{4}+\frac{3(c-1)}{4}\Psi^2(\pi)-\frac{c-1}{4}+\sum_{r=1}^{2p+2q}\big(\sigma_{11}^r\sigma_{22}^r-(\sigma_{12}^r)\big). \tag{23}$$

Now from equation (22) and equation (23), we get

$$\begin{aligned}\tau(p)-\mathcal{K}(\pi) &= \left[\frac{c+3}{4}(2p+2q-3)+\frac{c-1}{4}(2+3\cos^2\theta)\right](2p+2q+1)\\ &-\frac{3(c-1)}{4}+\sum_{r=1}^{2p+2q}\sum_{1\leq i<j\leq 2p+2q}\big(\sigma_{ii}^r\sigma_{jj}^r-(\sigma_{ij}^r)\big)-\frac{c+3}{4}\\ &-\frac{3(c-1)}{4}\Psi^2(\pi)+\frac{c-1}{4}-\sum_{r=1}^{2p+2q}\big(\sigma_{11}^r\sigma_{22}^r-(\sigma_{12}^r)\big).\end{aligned}$$

$$\begin{aligned}\tau(p)-\mathcal{K}(\pi) &= \frac{c+3}{4}\Big[(2p+2q-3)(2p+2q+1)-1\Big]-\frac{3(c-1)}{4}\Psi^(\pi)\\ &+\frac{c-1}{4}\Big[(2+3\cos^2\theta)(2p+2q+1)\Big]-\frac{2(c-1)}{4}\\ &+\sum_{r=1}^{2p+2q}\Bigg[\sum_{j=3}^{2p+2q}(\sigma_{11}^r+\sigma_{22}^r)\sigma_{jj}^r+\sum_{3\leq i<j\leq 2p+2q}\sigma_{ii}^r\sigma_{jj}^r\\ &-\sum_{j=3}^{2p+2q}[(\sigma_{1j}^r)^2+(\sigma_{2j}^r)^2]-\sum_{2\leq i\neq j\leq n}(\sigma_{ij}^r)^2\Bigg].\end{aligned}$$

It follows that

$$\begin{aligned}\tau(p)-\mathcal{K}(\pi) &= \frac{c+3}{4}\Big[(n-3)(n+1)-1\Big]+\frac{c-1}{4}\Big[(2+3\cos^2\theta)(n+1)\Big]\\ &-\frac{3(c-1)}{4}\Psi^2(\pi)-\frac{2(c-1)}{4}+\sum_{r=1}^{n}\Big[\sum_{j=3}^{n}(\sigma_{11}^r+\sigma_{22}^r)\sigma_{jj}^r\\ &+\sum_{3\le i<j\le n}\sigma_{ii}^r\sigma_{jj}^r-\sum_{j=3}^{n}[(\sigma_{1j}^r)^2+(\sigma_{2j}^r)^2]-\sum_{2\le i\neq j\le n}(\sigma_{ij}^r)^2\Big].\end{aligned}$$

Next, we use the symmetry property from (5). In particular, we have $\sigma_{1j}^1=\sigma_{11}^j$, $\sigma_{1j}^j=\sigma_{jj}^1$ for $3\le j\le n$, and $\sigma_{ij}^j=\sigma_{jj}^i$ for $2\le i\neq j\le n$, and hence

$$\begin{aligned}\tau(p)-\mathcal{K}(\pi) &= \frac{c+3}{4}\Big[(n-3)(n+1)-1\Big]+\frac{c-1}{4}\Big[(2+3\cos^2\theta)(n+1)\Big]\\ &-\frac{3(c-1)}{4}\Psi^2(\pi)-\frac{2(c-1)}{4}+\sum_{r=1}^{n}\Big[\sum_{j=3}^{n}(\sigma_{11}^r+\sigma_{22}^r)\sigma_{jj}^r\\ &+\sum_{3\le i<j\le n}\sigma_{ii}^r\sigma_{jj}^r-\sum_{j=3}^{n}[(\sigma_{1j}^r)^2+(\sigma_{2j}^r)^2]-\sum_{2\le i\neq j\le n}(\sigma_{ij}^r)^2\Big].\end{aligned} \tag{24}$$

To get the desired result, we follow the procedure described in [2]. We have the inequality

$$\begin{aligned}\sum_{j=3}^{n}(\sigma_{11}^r+\sigma_{22}^r)\sigma_{jj}^r \quad &+ \sum_{3\le i<j\le n}\sigma_{ii}^r\sigma_{jj}^r-\sum_{j=3}^{n}(\sigma_{jj}^r)^2\le\frac{n-2}{2(n+1)}\sum_{j=1}^{n}(\sigma_{jj}^r)^2\\ &\le\frac{2n-3}{2(2n+3)}(\sigma_{11}^r+\sigma_{22}^r+...+\sigma_{nn}^r)^2,\end{aligned} \tag{25}$$

for $r=1,2$. The first inequality in (25) is equivalent to

$$\sum_{j=3}^{n}\Big(\sigma_{11}^r+\sigma_{22}^r-3\sigma_{jj}^r\Big)^2+3\sum_{3\le i<j\le n}\Big(\sigma_{ii}^r-\sigma_{jj}^r\Big)^2\ge 0.$$

This equality holds if and only if $3\sigma_{jj}^r=\sigma_{11}^r+\sigma_{22}^r$, for all $j=3,...,n$. The equality also holds for $\sigma_{11}^r+\sigma_{22}^r=0$ and $\sigma_{jj}^r=0$ for $j=3,...,n$ and $r=1,2$.
Also, we have

$$\begin{aligned}\sum_{j=3}^{n}(\sigma_{ii}^r+\sigma_{22}^r)\sigma_{jj}^r \quad &+ \sum_{3\le i<j\le n}\sigma_{ii}^r\sigma_{jj}^r-\sum_{r\neq j=1}^{n}(\sigma_{jj}^r)^2\\ &\le\frac{2n-3}{2(2n+3)}(\sigma_{11}^r+\sigma_{22}^r+...+\sigma_{nn}^r)^2\end{aligned} \tag{26}$$

for $r = 3, ..., n$, which is equivalent to

$$\sum_{3\leq j\leq n; j\neq r} \left(2\sigma_{11}^r + 2\sigma_{22}^r - 3\sigma_{jj}^r\right)^2 + (2n+3)\left(\sigma_{11}^r - \sigma_{22}^r\right)^2 + 2\sum_{j=3}^{n}\left(\sigma_{rr}^r - \sigma_{jj}^r\right)^2$$
$$+6\sum_{3\leq i<j\leq n; i,j\neq r}\left(\sigma_{ii}^r - \sigma_{jj}^r\right)^2 + 3\left(\sigma_{rr}^r - 2\sigma_{11}^r - 2\sigma_{22}^r\right)^2 \geq 0.$$

The equality holds if and only if $\sigma_{11}^r = \sigma_{22}^r = 3\lambda^r$, $\sigma_{jj}^r = 4\lambda^r$, for $j = 3, ..., n, j \neq r$, $r = 3, ..., n$ and $\sigma_{rr}^r = 12\lambda^r$, with $\lambda^r \in \mathbb{R}$.
By summing the inequalities (25) and (26) and using the result in equation (24), we get the result. Combining the above equality cases, we get the desired forms of the second fundamental form. □

As special cases, we obtain the following corollaries:

Corollary 2. *Let $\mathcal{M}$ be an $(n+1)$-dimensional anti-invariant Lorentz submanifold of a $(2n+1)$-dimensional Lorentz Kenmotsu space form $\tilde{\mathcal{M}}(C)$ and $p \in \mathcal{M}$, $\pi \in \mathcal{T}_p\mathcal{M}$ a 2-plane section orthogonal to ξ_p; then*

$$\tau(p) - inf\mathcal{K}(p) \leq \frac{(n+1)^2(2n-3)}{2(2n+3)}\|\mathcal{H}\|^2 + \frac{c+3}{4}\Big[(n-3)(n+1) - 1\Big]$$
$$+2\frac{c-1}{4}(n+2) - \frac{3(c-1)}{4}\Psi^2(\pi).$$

Proof. For anti-invariant submnifold, the dimensions d_i of the distribution $\mathcal{D}_1$ are identically 0, and $\theta = \frac{\pi}{2}$. Substituting these values in (19) the proof follows easily. □

Corollary 3. *Let $\mathcal{M}$ be an $(n+1)$-dimensional invariant Lorentz submanifold of a $(2n+1)$-dimensional Lorentz Kenmotsu space form $\tilde{\mathcal{M}}(C)$ and $p \in \mathcal{M}$, $\pi \in \mathcal{T}_p\mathcal{M}$ a 2-plane section orthogonal to ξ_p; then*

$$\tau(p) - inf\mathcal{K}(p) \leq \frac{(n+1)^2(2n-3)}{2(2n+3)}\|\mathcal{H}\|^2 + \frac{c+3}{4}\Big[(n-3)(n+1) - 1\Big]$$
$$+\frac{c-1}{4}(5n+7) - \frac{3(c-1)}{4}\Psi^2(\pi).$$

Proof. For invariant submnifold, the dimensions d_i of the distribution $\mathcal{D}_2$ are identically 0, and $\theta = 0$. Substituting these values in (19) the proof follows easily. □

Data Availabality Statement:
In this article, the data sharing is not applicable as no data sets were used or generated in the current study.

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