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PASCAL'S CONNECTION AND FRACTIONS CONTAINING SUCCESSIVE PADOVAN NUMBERS IN THEIR DECIMAL REPRESENTATION, READING LEFT TO RIGHT

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Abstract

In this paper, it is given a generalization of the relation of the repeating decimals which displays the successive terms of the Fibonacci sequence and Pascal's rows. Additionally, two types of fractions that contain successive terms of the Padovan sequence in their decimal representation are given, by reading from left to right and giving numerical illustrations.

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1 Introduction

The phenomenon of the appearance of successive terms of the Fibonacci sequence and other well-known sequences in the decimal representation of some remarkable fractions has aroused great interest in many mathematicians and has been the subject of study in various articles. Such fractions been reported for the Fibonacci sequence as in [1, 4-6] and for sequences whose terms are diagonal sums from Pascal's triangle, which is known as ,,Pascal's connection" (see [1] and [2]). Additionaly, J. Smoak and T. J. Osler (see [3]) displayed the first terms of the Fibonacci sequence within the decimal expansion of several fractions.
The most famous example of such a fraction is $\frac{1}{89}$, first noted by Stancliff (1953), which has been discussed in many mathematical forums and can be expanded as follows:

$$\frac{1}{89} = \frac{1}{10}+\frac{1}{10^2}+\frac{2}{10^3}+\frac{3}{10^4}+\frac{5}{10^5}+\frac{8}{10^6}+\frac{13}{10^7}+\frac{21}{10^8}+\frac{34}{10^9}+\frac{55}{10^{10}}+... = 0.0112359550\ldots,$$

in which 1,2,3,4 and 5 are the first five terms of the Fibonacci sequence defined by the recurrence relation $F_{n+1} = F_n + F_{n-1},\ F_0 = 0$ and $F_1 = 1,\ n > 1$.

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In [1] and [7] a generalized mathematical form is given to determine fractions whose decimal expansion contains successive values of Fibonacci numbers, Lucas numbers and Pell numbers.
The aim of this paper is to study the phenomenon of the appearance of the first terms from left to right in well-known sequences such as Fibonacci sequence, generalization of Pascal's connection mentioned in [1], and studying the form of fractions whose decimal representations contain successive values of Padovan sequence from left to right, providing a general form for such fractions.

2 On Pascal's connection and repeating decimals that contain Fibonacci numbers from left to right

It is known that $\left(10^k+1\right)^n$generates rows of Pascal's triangle, and this relationship which also is obtained in the stated result in the following theorem is known as "the Pascal connection" which is discussed in [1] without proof. In order this connection to be generalised we give an alternate proof, the following theorem.
Theorem 2.1. Let $\{F_n\}_{n\geq 0}$ be the sequence of Fibonacci numbers, then
$$\sum_{n=1}^{\infty}\begin{bmatrix} 10^{-(n+2)k}F_{n+1} \\ 10^{-(n+3)k}F_{n+2}\end{bmatrix} = 10^{-2k}\sum_{n=0}^{\infty}\frac{\left(10^k+1\right)^n}{\left(10^{2k}\right)^n}\begin{bmatrix} 1 \\ 10^{-2k}+10^{-k}\end{bmatrix},\ k\in N.$$
Proof: Let us consider the following 2x2 matrix:

$$A=\begin{bmatrix} 0 & 1 \\ 10^{-2k} & 10^{-k}\end{bmatrix}.$$

We notice that

$$A\times\begin{bmatrix} 10^{-(n+1)k}F_n \\ 10^{-(n+2)k}F_{n+1}\end{bmatrix}=\begin{bmatrix} 0 & 1 \\ 10^{-2k} & 10^{-k}\end{bmatrix}\times\begin{bmatrix} 10^{-(n+1)k}F_n \\ 10^{-(n+2)k}F_{n+1}\end{bmatrix}=$$
$$=\begin{bmatrix} 10^{-(n+2)k}F_{n+1} \\ 10^{-(n+3)k}\left(F_n+F_{n+1}\right)\end{bmatrix}=\begin{bmatrix} 10^{-(n+2)k}F_{n+1} \\ 10^{-(n+3)k}F_{n+2}\end{bmatrix}$$

Summing the terms of right side we get

$$\sum_{n=0}^{\infty}\begin{bmatrix} 10^{-(n+2)k}F_{n+1} \\ 10^{-(n+3)k}F_{n+2}\end{bmatrix}=\left(\sum_{n=0}^{\infty}A^n\right)\begin{bmatrix} 10^{-2k} \\ 10^{-3k}\end{bmatrix}=\left(A^0+A+A^2+...\right)\begin{bmatrix} 10^{-2k} \\ 10^{-3k}\end{bmatrix},$$

where A^0 represents the identity matrix of order 2x2, and when we diagonalize the matrix A, i.e.,

$$A=\begin{bmatrix} -\frac{10^k\left(1+\sqrt{5}\right)}{2} & \frac{10^k\left(-1+\sqrt{5}\right)}{2} \\ 1 & 1\end{bmatrix}\times\begin{bmatrix} \frac{10^{-k}\left(1-\sqrt{5}\right)}{2} & 0 \\ 0 & \frac{10^{-k}\left(1+\sqrt{5}\right)}{2}\end{bmatrix}\times$$
$$\times\begin{bmatrix} -\frac{10^k\left(1+\sqrt{5}\right)}{2} & \frac{10^k\left(-1+\sqrt{5}\right)}{2} \\ 1 & 1\end{bmatrix}^{-1}.$$

Obviously, the series converges, i.e.,

$$A^0 + A + A^2 + ... = \left(A^0 - A\right)^{-1} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 10^{-2k} & 10^{-k} \end{bmatrix}\right)^{-1} =$$

$$\begin{bmatrix} 1 & -1 \\ -10^{-2k} & 1-10^{-k} \end{bmatrix}^{-1} = \frac{1}{10^{2k}-10^k-1}\begin{bmatrix} 10^{2k}-10^k & 10^{2k} \\ 1 & 10^{2k} \end{bmatrix}.$$

Substituting this result, since $\left|\frac{10^k+1}{10^{2k}}\right| < 1, k \in N$ we get

$$\sum_{n=0}^{\infty}\begin{bmatrix} 10^{-(n+2)k}F_{n+1} \\ 10^{-(n+3)k}F_{n+2} \end{bmatrix} = \frac{1}{10^{2k}-10^k-1}\begin{bmatrix} 10^{2k}-10^k & 10^{2k} \\ 1 & 10^{2k} \end{bmatrix} \times \begin{bmatrix} 10^{-2k} \\ 10^{-3k} \end{bmatrix} =$$
$$\frac{1}{10^{2k}-10^k-1}\begin{bmatrix} 1 \\ 10^{-2k}+10^{-k} \end{bmatrix}$$

Since $\left|\frac{10^k+1}{10^{2k}}\right| < 1, k \in N$, we may write

$$\sum_{n=0}^{\infty}\begin{bmatrix} 10^{-(n+2)k}F_{n+1} \\ 10^{-(n+3)k}F_{n+2} \end{bmatrix} = \frac{1}{10^{2k}-10^k-1}\begin{bmatrix} 1 \\ 10^{-2k}+10^{-k} \end{bmatrix} =$$
$$= \frac{1}{10^{2k}}\frac{1}{1-\frac{10^k+1}{10^{2k}}}\begin{bmatrix} 1 \\ 10^{-2k}+10^{-k} \end{bmatrix} = \sum_{n=0}^{\infty}\frac{\left(10^k+1\right)^n}{\left(10^{2k}\right)^n}\begin{bmatrix} 10^{-2k} \\ 10^{-4k}+10^{-3k} \end{bmatrix}$$

Which completes the proof of the theorem.

As can be seen from the proof, the form of the fractions containing the Fibonacci numbers in their decimal representation from left to right has the form $\frac{1}{10^{2k}-10^k-1}$, $k \in N$.

For k=1, we get $\dfrac{1}{10^2-10-1} = \frac{1}{89} = 0.0112359550561797...$,
where successive terms of the Fibonacci sequence occurred and are summed.

$$\begin{aligned}
\frac{1}{89} = \;& 0.0112358 \\
& 0.00000013 \\
& 0.000000021 \\
& 0.0000000034 \\
& 0.00000000055 \\
+ & \ldots\ldots\ldots\ldots\ldots\ldots \\
\hline
= \;& 0.01123595505\ ...
\end{aligned}$$

Moreover, the above fraction can also be represented as the sum of successive powers of 11, as

$$
\begin{array}{r}
\frac{1}{89} = 0.01 \\
0.0011 \\
0.000121 \\
0.00001331 \\
0.0000014641 \\
+\ldots\ldots\ldots\ldots\ldots\ldots \\
\hline
= \ 0.0112359550\ \ldots
\end{array}.
$$

Another example for k=6 is given. By using WolframAlpha we get

$$
\begin{array}{l}
\frac{1}{10^{12}-10^{6}-1} = \frac{1}{999998999999} = \\
000002100000340000055000008900014400023300037\,7 \\
0006100000987001597002584004181006765 00\ldots \times 10^{-12}
\end{array}.
$$

If the zeros are ignored, it is obvious that the decimal digits represent the numbers of the Fibonacci sequence from left to right.

3 Some fractions that contain the Padovan numbers in their decimal representation from left to right

The Padovan sequence is named after Richard Padovan who attributed its discovery to Dutch architect Hans van der Laan in his 1994 essay Dom and later were studied by many researchers [9,10,11]. This is sequence A000931 on the On-Line Encyclopedia of Integer Sequences (OEIS)[8].
Padovan number sequence by the recurrence relation below
$P_{n+3} = P_{n+1} + P_n$, for all $n \geq 0$
with $P_0 = 0,\ P_1 = 1,\ P_2 = 1$.
In order to discuss the form of fractions which contain Padovan numbers in their decimal representation from left to right the following theorems are given:

Theorem 3.1. The decimal representation of the fraction $\frac{\left(10^k+1\right)^2}{10^{4k}-10^{2k}-10^k},\ k \in N$, contains successive terms of Padovan sequence, starting from the third term, reading from the left to the right.

Proof: Let us take any integers p and q that satisfy the recurrence relation of the form

$P_{n+3} = pP_{n+1} + qP_n,\ P_n \geq 0$, for $n \geq 1$.

Using the fact that $P_{n+3} - pP_{n+1} - qP_n = 0$, we consider the product

$$
\begin{aligned}
&\left(1 - px^2 - qx^3\right)\sum_{i=2}^{\infty} P_{i+1}x^i = \\
&= P_3x^2 + P_4x^3 + P_5x^4 - pP_3x^4 + \sum_{i=3}^{\infty}\left(P_{i+3} - pP_{i+1} - qP_i\right)x^{i+2} = \\
&= P_3x^2 + P_4x^3 + qP_2x^4
\end{aligned}.
$$

Thus

$$\sum_{i=2}^{\infty} P_{i+1}x^i = \frac{P_3x^2 + P_4x^3 + qP_2x^4}{1 - px^2 - qx^3}.$$

Because we want to look at decimal expansions, change the form by replacing x by $\frac{1}{x}$ to make

$$\sum_{i=2}^{\infty} \frac{P_{i+1}}{x^i} = \frac{P_3x^2 + P_4x + qP_2}{x^4 - px^2 - qx}.$$

For $p = q = 1$ and $x = 10^k,\ k \in N$ we get the stated result.
Example:
For k=2, we have

$$\frac{\left(10^2+1\right)^2}{10^8-10^4-10^2} = \frac{10201}{99989900} =$$
$$0.00010202030405070912162128374965 8$$
$$715530268557124269551224673692 0.....$$

For k=3, we have $\frac{\left(10^3+1\right)^2}{10^{12}-10^6-10^3} = \frac{1002001}{999998999000} =$

$$= 1.002002003004005007009012016021028037049065086114151200 2653... \cdot 10^{-12}.$$

Ignoring the zeros it is obvious that the decimal digits represent the numbers of the Padovan sequence from left to right.

Theorem 3.2. The decimal representation of the fraction $\frac{10^{3k}+10^{2k}}{10^{3k}-10^k-1},\ k \in N$, contains successive terms of Padovan sequence, reading from the left to the right.

*P*roof: Let's take the function

$$f\left(x\right) = \sum_{k=0}^{\infty} P_k x^k = P_0 + P_1 x + P_2 x^2 + \sum_{k=3}^{\infty}\left(P_{k-2} + P_{k-3}\right)x^k =$$
$$= 1 + x + x^2 + \sum_{k=1}^{\infty} P_k x^{k+2} + \sum_{k=0}^{\infty} P_k x^{k+3} =$$
$$= 1 + x + x^2 + x^2\left(f\left(x\right) - 1\right) + x^3 f\left(x\right)$$

From where we get that $f\left(x\right) = \frac{1+x}{1-x^2-x^3}$, and replacing x by $\frac{1}{x}$ we get

$$f\left(\frac{1}{x}\right) = \frac{x^3 + x^2}{x^3 - x - 1}.$$

Thus for $x = 10^k,\ k \in N$ we get the stated result.
Example
For k=2 we get

$$\frac{10^6+10^4}{10^6-10^2-1} = \frac{1010000}{999899} =$$
$$= 1.010102020304050709121621283749 65$$
$$87155302685571242695512246736920...$$

For k=3 we get

$$\frac{10^9+10^6}{10^9-10^3-1} = \frac{1001000000}{999998999} =$$
$$= 1.001001002002003004005007009012016021028037049065086114151200265 3....$$

Ignoring the zeros it is obvious that the decimal digits represent the numbers of the Padovan sequence from left to right.

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