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ON PARA-SASAKIAN MANIFOLD ADMITTING ZAMKOVOY CONNECTION

A. GOYAL [1], Swati JAIN *[2] and M. K. PANDEY [3]

Abstract

The purpose of the present paper is to study some properties of para-Sasakian manifold admitting Zamkovoy connection. We obtain some interesting result on para-Sasakian manifold. It is shown that M-projectively flat para-Sasakian manifold is η-Einstein manifold.

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1 Introduction

The notion of the almost para-contact structure on a differentiable manifold was defined by I. Sato [22, 23]. The para-contact metric manifolds have been studied by many authors in recent years. The structure is an analogue of the almost contact structure [7, 21] and is closely related to almost product structure (in contrast to almost contact structure, which is related to almost complex structure). Every differentiable manifold with almost para-contact structure defined by I. Sato has a compatible Riemannian metric.

An almost para-contact structure on a pseudo-Riemannian manifold M of dimension $(2n+1)$ is defined by S. Kaneyuki and M. Konzai [11] and they constructed the almost para-complex structure on $M^{2n+1} \times \mathbb{R}$. Recently, S. Zamkovoy [27] has associated the almost para-contact structure given in [11] to a pseudo-Riemannian metric of signature $(n+1, n)$ and showed that any almost para-contact

[1]Department of Mathematics, University Institute of Technology, Rajiv Gandhi Proudyogiki Vishwavidyalaya, Bhopal, Madhya Pradesh 462033, India, e-mail: anil_goyal03@rediffmail.com
[2]*Corresponding author, Department of Mathematics, University Institute of Technology, Rajiv Gandhi Proudyogiki Vishwavidyalaya, Bhopal, Madhya Pradesh 462033, India, e-mail: swatijain2884@gmail.com
[3]Department of Mathematics, University Institute of Technology, Rajiv Gandhi Proudyogiki Vishwavidyalaya, Bhopal, Madhya Pradesh 462033, India, e-mail: mkp_apsu@rediffmail.com

structure admits such a pseudo-Riemannian metric.

The study of M-projective curvature tensor has been a very attractive field for investigations in the past many decades. M-projective curvature tensor was introduced by G. P. Pokhariyal and R. S. Mishra [19] in 1971. Also, in 1986, R. H. Ojha [17] extended some properties of the M-projective curvature tensor in Sasakian and Kahler manifolds. In 2010, the study of M-projective curvature tensor in Riemannian manifolds and also in Kenmotsu manifolds was resumed by S. K. Chaubey and R. H. Ojha [8]. Further, R. N. Singh and S. K. Pandey [26] have studied various geometric properties of M-projective curvature tensor on $N(k)$-contact metric manifolds. In 2020, A. Mandal and A. Das [12] studied some properties of the M-projective curvature tensor in Sasakian manifolds. Afterwards, several researchers have carried out the study of M-projective curvature tensor in a variety of directions such as [9, 10, 11, 13, 14, 16, 24, 25, 28]. The M-projective curvature tensor defined by G. P. Pokhariyal and R. S. Mishra [19] is given below

$$\begin{aligned} M(X,Y,Z) = R(X,Y,Z) &- \frac{1}{2(n-1)}[S(Y,Z)X - S(X,Z)Y] \\ &- \frac{1}{2(n-1)}[g(Y,Z)QX - g(X,Z)QY], \end{aligned} \tag{1}$$

for all $X, Y, Z \in \chi(M^{2n+1}))$, where $\chi(M^{2n+1}))$ is the set of all vector fields of the manifold M^{2n+1} and R is the Riemannian curvature tensor of type $(0,3)$ and S is the Ricci tensor, i.e.,

$$S(X,Y) = g(QX,Y),$$

where Q is a Ricci operator of type $(1,1)$.

In 2008, the notion of Zamkovoy connection was introduced by S. Zamkovoy [27] for a para-contact manifold. And this was connection defined as a canonical para-contact connection whose torsion is the obstruction of para-contact manifold to be a para-Sasakian manifold. For an n-dimensional almost contact metric manifold M equipped with an almost contact metric structure (ϕ, ξ, η, g) consisting of a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η and a Riemannian metric g, the Zamkovoy connection is defined by [27]

$$\tilde{\nabla}_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi + \eta(X)\phi Y \tag{2}$$

for all $X, Y, Z \in \chi(M^{2n+1})$.

This connection was further studied by A. M. Blaga [6] in para-Kenmotsu manifolds and A. Biswas, K. K. Baishya [4, 5] in Sasakian manifolds. In a para-Sasakian manifold M of dimension $(2n+1)$, the M-projective curvature tensor

$\tilde{M}$ with respect to the Zamkovoy connection $\tilde{\nabla}$ is given by

$$\begin{aligned}\tilde{M}(X,Y,Z) = \tilde{R}(X,Y,Z) - \frac{1}{2(n-1)}[\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y] \\ - \frac{1}{2(n-1)}[g(Y,Z)\tilde{Q}X - g(X,Z)\tilde{Q}Y],\end{aligned} \tag{3}$$

where $\tilde{R}$, $\tilde{S}$ and $\tilde{Q}$ are the curvature tensor, Ricci tensor and Ricci operator with respect to Zamkovoy connection $\tilde{\nabla}$.

The present paper is organized as follows. Section 2 is devoted to preliminaries and we give some relations between curvature tensor (resp. Ricci tensor) with respect to the Zamkovoy connection and curvature tensor (resp. Ricci tensor) with respect to the Levi-Civita connection. In section 3, we have discussed M-projectively flat para-Sasakian manifold and ξ-M-projectively para-Sasakian manifold with respect to the connection and obtained some relations that hold on $(2n+1)$-dimensional para-Sasakian manifold. In section 4, we have discussed ϕ-M-projectively flat para-Sasakian manifold with respect to the Zamkovoy connection. In section 5, the goal is to examine quasi-M-projectively flat para-Sasakian manifold with respect to the Zamkovoy connection.

2 Preliminaries

An $(2n+1)$-dimensional differentiable manifold M is said to have almost para-contact structure (ϕ, ξ, η), where ϕ is a tensor field of type $(1,1)$, ξ is a vector field known as characteristic vector field and η is a 1-form on M^{2n+1} satisfying the following relations

$$\phi^2 = I - \eta \otimes \xi, \tag{4}$$

$$\eta(\xi) = 1, \quad \phi(\xi) = 0, \tag{5}$$

$$\eta(\phi X) = 0, \tag{6}$$

and

$$\mathrm{rank}(\phi) = 2n,$$

where I denotes the identity transformation. A differentiable manifold with almost para-contact structure (ϕ, ξ, η) is called an almost para-contact manifold.

Moreover, the tensor field ϕ induces an almost para-complex structure on the para-contact distribution $D = \ker(\eta)$, i.e, the eigen distributions $D^{\pm}$ corresponding to the eigen values ± 1 of ϕ are both n-dimensional.

If an almost para-contact manifold M^{2n+1} with an almost para-contact structure (ϕ, ξ, η) admits a pseudo-Riemannian metric g such that [27]

$$g(\phi X, \phi Y) = -g(X,Y) + \eta(X)\eta(Y) \tag{7}$$

for $X, Y \in \chi(M^{2n+1})$, then we say that M^{2n+1} is an almost para-contact metric manifold with an almost para-contact metric structure (ϕ, ξ, η, g) and such metric g is called compatible metric. Any compatible metric g is necessarily of signature $(n+1, n)$.

From (7), one can see that [27]

$$g(X, \phi Y) = -g(\phi X, Y). \tag{8}$$

Also, we take

$$\eta(X) = g(X, \xi) \tag{9}$$

for any $X, Y \in \chi(M^{2n+1})$. The fundamental 2-form of M^{2n+1} is defined by

$$\alpha(X, Y) = g(X, \phi Y).$$

The structure (ϕ, ξ, η, g) satisfying conditions (4) to (9) is called an almost para-contact metric structure and the manifold M^{2n+1} with such a structure is called an almost para-contact Riemannian manifold [3, 23].

An almost para-contact metric structure becomes a para-contact metric structure [27] if

$$g(X, \phi Y) = d\eta(X, Y),$$

for all vector field $X, Y \in \chi(M^{2n+1})$, where

$$d\eta(X, Y) = \frac{1}{2}[X\eta(Y) - Y\eta(X) - \eta([X, Y])].$$

Now, we briefly present an account of an analogue of the Sasakian manifold in para-contact geometry which will be called para-Sasakian manifold.

Definition 1. *An almost para-contact metric structure (ϕ, ξ, η, g) on M^{2n+1} is para-Sasakian manifold if and only if the Levi-Civita connection ∇ of g satisfies [27]*

$$(\nabla_X \phi)Y = -g(X, Y)\xi + \eta(Y)X \tag{10}$$

for any $X, Y \in \chi(M^{2n+1})$.

From (10), it can be seen that

$$(\nabla_X \xi) = -\phi X. \tag{11}$$

Example 1. *[2]. Let $M = \mathbb{R}^{2n+1}$ be the $(2n+1)$-dimensional real number space with $(x_1, y_1, x_2, y_2, \ldots.., x_n, y_n, z)$ standard coordinate system. Defining*

$$\phi\frac{\partial}{\partial x_\alpha} = \frac{\partial}{\partial y_\alpha}, \phi\frac{\partial}{\partial y_\alpha} = \frac{\partial}{\partial x_\alpha}, \phi\frac{\partial}{\partial z} = 0,$$

$$\xi = \frac{\partial}{\partial z}, \eta = dz,$$

$$g = \eta \otimes \eta + \sum_{\alpha=1}^{n} dx_\alpha \otimes dx_\alpha - \sum_{\alpha=1}^{n} dy_\alpha \otimes dy_\alpha,$$

where $\alpha = 1, 2, \ldots, n$, *then the set* (M, ϕ, ξ, η, g) *is an almost para-contact metric manifold.*

On a para-Sasakian manifold, the following relations also hold [27]:

$$g(R(X, Y, Z), \xi) = \eta(R(X, Y)Z), \tag{12}$$

which can also be given as

$$g(R(X, Y, Z), \xi) = g(X, Z)\eta(Y) - g(Y, Z)\eta(X),$$

$$R(X, Y, \xi) = \eta(X)Y - \eta(Y)X, \tag{13}$$

$$R(X, \xi, Y) = -R(\xi, X, Y) = -g(X, Y)\xi + \eta(Y)X, \tag{14}$$

$$R(\xi, X, \xi) = X - \eta(X)\xi, \tag{15}$$

and

$$S(X, \xi) = -2n\eta(X) \tag{16}$$

for any vector fields $X, Y, Z \in \chi(M^{2n+1})$. Here, R is Riemannian curvature tensor and S is the Ricci tensor defined by $S(X, Y) = g(QX, Y)$, where Q is the Ricci operator.

In view of (11), the equation (2) becomes

$$\tilde{\nabla}_X Y = \nabla_X Y + \eta(X)\phi Y + \eta(Y)\phi X + g(X, \phi Y)\xi, \tag{17}$$

with torsion tensor

$$\tilde{T}(X, Y) = 2g(X, \phi Y)\xi.$$

On a para-Sasakian manifold, the connection $\tilde{\nabla}$ has the following properties

$$\tilde{\nabla}\eta = 0, \tilde{\nabla}g = 0, \tilde{\nabla}\xi = 0, \tag{18}$$

and

$$(\tilde{\nabla}_X \phi)Y = (\nabla_X \phi)Y + g(X, Y)\xi - \eta(Y)X. \tag{19}$$

Definition 2. *An* $(2n + 1)$*-dimensional para-Sasakian manifold* M *is said to be* η*-Einstein manifold if the Ricci tensor of type* $(0, 2)$ *is of the form*

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y)$$

for any $X, Y \in \chi(M^{2n+1})$, where a and b are scalars.

Definition 3. *An* $(2n + 1)$*-dimensional para-Sasakian manifold* M *is said to be generalized* η*-Einstein manifold if the Ricci tensor of type* $(0, 2)$ *is of the form*

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y) + cA(X, Y)$$

for any $X, Y \in \chi(M^{2n+1})$, where a, b and c are scalars and A is a 2-form.

Definition 4. *An para-Sasakian manifold is said to be M-projectively flat with respect to the Zamkovoy connection if*

$$\tilde{M}(X,Y,Z)=0$$

for any $X,Y,Z\in\chi(M^{2n+1})$.

Definition 5. *An $(2n+1)$-dimensional para-Sasakian manifold M is said to be ξ-M-projectively flat with respect to the Zamkovoy connection if*

$$\tilde{M}(X,Y)\xi=0.$$

It is known that the curvature tensor $\tilde{R}$ of a para-Sasakian manifold M with respect to the Zamkovoy connection $\tilde{\nabla}$ defined by

$$\tilde{R}(X,Y)Z=\tilde{\nabla}_X\tilde{\nabla}_Y Z-\tilde{\nabla}_Y\tilde{\nabla}_X Z-\tilde{\nabla}_{[X,Y]}Z \tag{20}$$

satisfies the following [1]

$$\begin{aligned}\tilde{R}(X,Y)Z=R(X,Y)Z&+g(Y,Z)\eta(X)\xi-g(X,Z)\eta(Y)\xi\\&+\eta(Y)\eta(Z)X-\eta(X)\eta(Z)Y+2g(X,\phi Y)\phi Z\\&+g(X,\phi Z)\phi Y-g(Y,\phi Z)\phi X,\end{aligned} \tag{21}$$

where R , S and r are curvature tensor, Ricci tensor and scalar curvature relative to ∇ respectively and $\tilde{R}$, $\tilde{S}$ and $\tilde{r}$ are curvature tensor, Ricci tensor and scalar curvature relative to $\tilde{\nabla}$.

$$\tilde{S}(X,Y)=S(X,Y)-2g(X,Y)+(2n+2)\eta(X)\eta(Y), \tag{22}$$

and

$$\tilde{r}=r-2n \tag{23}$$

for any $X,Y,Z\in\chi(M^{2n+1})$. From (22), it is easy to note that $\tilde{S}$ is symmetric.

Further, it is known that [1] on a para-Sasakian manifold, the following relation hold

$$g(\tilde{R}(X,Y)Z,\xi)=\eta\tilde{R}((X,Y)Z)=0, \tag{24}$$

$$\tilde{R}(X,Y)\xi=\tilde{R}(\xi,X)Y=\tilde{R}(\xi,X)\xi=0, \tag{25}$$

and

$$\tilde{S}(X,\xi)=0 \tag{26}$$

for any $X,Y,Z\in\chi(M^{2n+1})$.

From (22), it can also be noted that if $\tilde{S}(X,Y)=0$, then

$$S(X,Y)=2g(X,Y)+(-2n-2)\eta(X)\eta(Y),$$

which proves that if a para- Sasakian manifold M^{2n+1} is Ricci-flat with respect to the Zamkovoy connection, then it is an η-Einstein manifold.

3 M-Projectively flat para-Sasakian manifold

In this section, we consider a para-Sasakian manifold admitting a Zamkovoy connection with

$$\tilde{M}(X,Y,Z)=0,$$

i.e, we assume that para-Sasakian manifold is M- projectively flat with respect to the Zamkovoy connection.

Theorem 1. *If M- projective curvature tensor of para-Sasakian manifold with respect to the Zamkovoy connection vanishes, then it is an η-Einstein manifold.*

Proof. Let M be an $(2n+1)$-dimensional M- projectively flat para-Sasakian manifold with respect to the Zamkovoy connection, i.e, $\tilde{M}=0$, then from (3) we have

$$\begin{aligned}\tilde{R}(X,Y,Z) &= \frac{1}{2(n-1)}[\tilde{S}(Y,Z)X-\tilde{S}(X,Z)Y] \\ &+ \frac{1}{2(n-1)}[g(Y,Z)\tilde{Q}X-g(X,Z)\tilde{Q}Y].\end{aligned} \tag{27}$$

Taking inner product of (27), we get

$$\begin{aligned}{}'\tilde{R}(X,Y,Z,V) &= \frac{1}{2(n-1)}[\tilde{S}(Y,Z)g(X,V)-\tilde{S}(X,Z)g(Y,V)] \\ &+ \frac{1}{2(n-1)}[g(Y,Z)\tilde{S}(X,V)-g(X,Z)\tilde{S}(Y,V)],\end{aligned}$$

where

$${}'\tilde{R}(X,Y,Z,V)=g(\tilde{R}(X,Y,Z),V),$$

and

$${}'R(X,Y,U,V)=g(R(X,Y,U),V)$$

for the arbitrary vector fields $X,Y,Z,V\in\chi(M^{2n+1})$.

Contracting the above equation over X and V, we get

$$S(Y,Z)=(2n+2-r)g(Y,Z)-(2n+2)\eta(Y)\eta(Z). \tag{28}$$

Therefore, M is an η-Einstein manifold. □

We may also write the following corollary:

Corollary 1. *If a para-Sasakian manifold is M-projectively flat with respect to the Zamkovoy connection then its scalar curvature is zero.*

Theorem 2. *On para-Sasakian manifold admitting a Zamkovoy connection, we have the following relation:*

$$\begin{aligned}\tilde{M}(X,Y,Z) &= M(X,Y,Z) + g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X + 2g(X,\phi Y)\phi Z \\ &+ g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X \\ &- \eta(X)\eta(Z)Y + \frac{1}{2(n-1)}[4g(Y,Z)X - 4g(X,Z)Y] \\ &- \frac{(n+1)}{(n-1)}[\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y] \\ &+ \frac{(n+1)}{(n-1)}[g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi].\end{aligned} \tag{29}$$

Proof. Using equation (25) in the equation (3), we get

$$\begin{aligned}\tilde{M}(X,Y,Z) &= R(X,Y,Z) + g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X \\ &- \eta(X)\eta(Z)Y + 2g(X,\phi Y)\phi Z + g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X \\ &- \frac{1}{2(n-1)}S(Y,Z)X + \frac{g(Y,Z)X}{(n-1)} - \frac{(n+1)}{(n-1)}\eta(Y)\eta(Z)X \\ &+ \frac{S(X,Z)Y}{2(n-1)} - \frac{g(X,Z)Y}{(n-1)} + \frac{(n+1)}{(n-1)}\eta(X)\eta(Z)Y \\ &- \frac{g(Y,Z)QX}{2(n-1)} + \frac{g(Y,Z)X}{(n-1)} - \frac{(n+1)}{(n-1)}g(Y,Z)\eta(X)\xi \\ &+ \frac{g(X,Z)QY}{2(n-1)} - \frac{g(X,Z)Y}{(n-1)} + \frac{(n+1)}{(n-1)}\eta(Y)g(X,Z)\xi,\end{aligned}$$

which, in view of (1) gives the desired result. □

Corollary 2. *On a para-Sasakian manifold admitting a Zamkovoy connection, we have*

$$\tilde{M}(X,Y)\xi = M(X,Y)\xi + \eta(Y)X - \eta(X)Y + \frac{(n+3)}{(n-1)}[\eta(Y)X - \eta(X)Y].$$

Corollary 3. *In a para-Sasakian manifold admitting a Zamkovoy connection the following relations hold:*

(i) $\tilde{M}(\tilde{X},Y,Z) = M(\tilde{X},Y,Z) - g(X,\tilde{Z})\tilde{X} + g(\tilde{X},\tilde{Z})\tilde{Y} + 2g(\tilde{X},\tilde{Y})\tilde{Z}$
$- g(\tilde{X},Z)\eta(Y)\xi + g(Y,\tilde{Z})\eta(X)\xi + \eta(Y)\eta(Z)\tilde{X}$
$+ \frac{2}{(n-1)}[g(Y,Z)\tilde{X} - g(\tilde{X},Z)Y] + \frac{(n+1)}{(n-1)}[g(\tilde{X},Z)\eta(Y)\xi - \eta(Y)\eta(Z)\tilde{X}],$

(ii) $\tilde{M}(X,\tilde{Y},Z) = M(X,\tilde{Y},Z) + g(\tilde{Y},Z)\eta(X)\xi - \eta(X)\eta(Y)\tilde{Y} + 2g(X,Y)\tilde{Z}$
$- 2\eta(X)\eta(Y)\tilde{Z} + g(X,\tilde{Z})Y - g(X,\tilde{Z})\eta(Y)\xi - g(\tilde{Y},\tilde{Z})\tilde{X}$
$+ \frac{1}{(n-1)}[g(\tilde{Y},\tilde{Z})X - g(X,Z)\tilde{Y}] + \frac{(n+1)}{(n-1)}[\eta(X)\eta(Z)\tilde{Y}],$

(iii) $\tilde{M}(X,Y)\tilde{Z} = M(X,Y)\tilde{Z} + g(Y,\tilde{Z})\eta(X)\xi - g(X,\tilde{Z})\eta(Y)\xi + g(\tilde{X},\tilde{Z})Y + 2g(X,\tilde{Y})Z - 2g(X,\tilde{Y})\eta(Z)\xi - g(X,Z)\tilde{Y} - \eta(X)\eta(Z)\tilde{Y} - g(Y,Z)\tilde{X} + \eta(Y)\eta(Z)\tilde{X} + \frac{2}{(n-1)}[g(Y,\tilde{Z})X - g(X,\tilde{Z})Y] + \frac{(n+1)}{(n-1)}[g(X,\tilde{Z})\eta(Y)\xi - g(Y,\tilde{Z})\eta(X)\xi]$,

(iv) $\tilde{M}(\tilde{X},\tilde{Y})Z = M(\tilde{X},\tilde{Y})Z + 2g(\tilde{X},Y)\tilde{Z} - g(\tilde{X},\tilde{Z})\eta(Y)\xi - g(\tilde{Y},\tilde{Z})X + g(\tilde{Y},\tilde{Z})\eta(X)\xi + \frac{2}{(n-1)}[g(\tilde{Y},Z)\tilde{X} - g(\tilde{X},Z)\tilde{Y}]$,

(v) $\tilde{M}(X,\tilde{Y})\tilde{Z} = M(X,\tilde{Y})\tilde{Z} + g(\tilde{Y},\tilde{Z})\eta(X)\xi + 2g(X,Y)Z - 2\eta(X)\eta(Y)Z - 2g(X,Y)\eta(Z) + g(X,Z)Y - \eta(X)\eta(Z)Y - g(X,Z)\eta(Y)\xi + 3\eta(X)\eta(Y)\eta(Z)\xi + \frac{1}{(n-1)}[g(\tilde{Y},\tilde{Z})X - X] - \frac{1}{(n-1)}[g(X,\tilde{Z})\tilde{Y} - \tilde{Y}]$,

(vi) $\tilde{M}(\tilde{X},Y)\tilde{Z} = M(\tilde{X},Y)\tilde{Z} - g(\tilde{X},\tilde{Z})\eta(Y)\xi + 2g(\tilde{X},\tilde{Y})Z - 2g(\tilde{X},\tilde{Y})\eta(Z)\xi + g(\tilde{X},Z)\tilde{Y} - g(Y,Z)X + \eta(Y)\eta(Z)X - \eta(X)\eta(Y)\eta(Z)\xi + \frac{2}{(n-1)}[g(Y,\tilde{Z})\tilde{X} - g(\tilde{X},\tilde{Z})Y] + \frac{(n+1)}{(n-1)}g(\tilde{X},\tilde{Z})\eta(Y)\xi - g(Y,Z)\eta(X)\xi$,

(vii) $\tilde{M}(\tilde{X},\tilde{Y})\tilde{Z} = M(\tilde{X},\tilde{Y})\tilde{Z} + 2g(\tilde{X},Y)Z - 2g(\tilde{X},Y)\eta(Z)\xi + g(\tilde{X},Z)Y - g(X,Z)\eta(Y)\xi - g(\tilde{Y},Z)X + g(\tilde{Y},Z)\eta(X)\xi + \frac{1}{(n-1)}[g(\tilde{Y},\tilde{Z})\tilde{X} - \tilde{X}] - \frac{1}{(n-1)}[g(\tilde{X},\tilde{Z})\tilde{Y} - \tilde{Y}]$.

4 ϕ-M-projectively flat para-Sasakian manifold

In this section, we study the implication of the condition

$$\phi^2\tilde{M}(\phi X,\phi Y)\phi V = 0. \tag{30}$$

A differentiable manifold M^{2n+1} satisfying the above condition is called ϕ-M-projectively flat [18].

It can be easily seen that $\phi^2\tilde{M}(\phi X,\phi Y)\phi V$ =0 holds if and only if [18],

$$g(\tilde{M}(\phi X,\phi Y)\phi V,\phi U) = 0 \tag{31}$$

for all $X,Y,Z \in \chi(M^{2n+1})$. By using the equations (21) and (22), we obtain from equation (3),

$$\begin{aligned}\tilde{M}(\phi X,\phi Y)\phi V &= R(\phi X,\phi Y)\phi V + g(\phi Y,\phi V)\eta(\phi X)\xi \\ &\quad - g(\phi X,\phi V)\eta(\phi Y)\xi + \eta(\phi Y)\eta(\phi V)\phi X \\ &\quad - \eta(\phi X)\eta(\phi V)\phi Y + 2g(\phi X,\phi^2 Y)\phi^2 V \\ &\quad + g(\phi X,\phi^2 V)\phi^2 Y - g(\phi Y,\phi^2 V)\phi^2 X.\end{aligned} \tag{32}$$

In view of equation (31), ϕ-M-projectively flatness gives

$$\begin{aligned}0 &= g(R(\phi X,\phi Y)\phi V,\phi U)+2g(\phi X,\phi^2 Y)g(\phi^2 V,\phi U)+g(\phi X,\phi^2 V)g(\phi^2 Y,\phi U)\\&-\frac{1}{2(n-1)}[S(\phi Y,\phi V)g(\phi X,\phi U)-S(\phi X,\phi V)g(\phi Y,\phi U)+g(\phi Y,\phi V)S(\phi X,\phi U)\\&-g(\phi X,\phi V)S(\phi Y,\phi U)+4g(\phi X,\phi V)g(\phi Y,\phi U)-4g(\phi Y,\phi V)g(\phi X,\phi U)].\end{aligned} \tag{33}$$

Putting $X=U=\xi$ in the equation (33) and using the equations (4) and (6), we obtain

$$S(\phi Y,\phi V)+3g(\phi Y,\phi V)-2ng(\phi Y,V)=0, \tag{34}$$

for any vector fields Y and V on M^{2n+1}.

Replacing Y by ϕY and V by ϕV in the above equation and in view of equation (4), we have

$$S(Y,V)=3g(Y,V)+3\eta(Y)\eta(V)-2n\phi(Y,V).$$

Therefore, we conclude the following theorem.

Theorem 3. *If ϕ-M-projectively flat para-Sasakian manifold is ϕ-M-projectively flat with respect to the Zamkovoy connection, then the manifold is generalized η-Einstein manifold.*

5 Quasi-M-projectively flat para-Sasakian manifold

In this section, we study a differentiable manifold M^{2n+1} satisfying the condition

$$g(\tilde{M}(X,Y)V,\phi U)=0 \tag{35}$$

for all $X,Y,V,U\in\chi(M^{2n+1})$. Such a manifold is called quasi-M-projectively flat.

Using the equations (21) and (22) in the equation (27), we get

$$\begin{aligned}0 &= g(R(X,Y)V,\phi U)+2g(X,\phi Y)g(\phi V,\phi U)+g(X,\phi V)g(\phi Y,\phi U)\\&-g(Y,\phi V)g(\phi X,\phi U)+g(Y,V)g(\xi,\phi U)\eta(X)-g(X,V)g(\xi,\phi U)\eta(Y)\\&+\eta(Y)\eta(V)g(X,\phi U)-\eta(X)\eta(V)g(Y,\phi U)\\&-\frac{1}{2(n-1)}[S(Y,V)g(X,\phi U)-2g(Y,V)g(X,\phi U)+(2n+2)\eta(Y)\eta(V)g(X,\phi U)\\&-S(X,V)g(Y,\phi U)+2g(X,V)g(Y,\phi U)-(2n+2)\eta(X)\eta(V)g(Y,\phi U)]\\&-\frac{1}{2(n-1)}[g(Y,V)S(X,\phi U)-2g(X,\phi U)g(Y,V)+(2n+2)\eta(X)g(Y,V)g(\xi,\phi U)\\&-g(X,V)S(Y,\phi U)+2g(X,V)g(Y,\phi U)-(2n+2)\eta(Y)g(X,V)g(\xi,\phi U)].\end{aligned} \tag{36}$$

Putting $Y = V = \xi$ in equation (36), we get

$$\begin{aligned}
0 &= S(X,\phi U) + 2g(\phi X,U) + \frac{r}{2(n-1)}g(\phi X,U) - (4n+2)g(\phi X,U) \\
&+ \frac{(n+1)}{(n-1)}g(\phi X,U) + \frac{1}{(n-1)}S(X,\phi U) + \frac{2}{(n-1)}g(\phi X,U) \\
&- \frac{(2n+1)}{(2n-1)}S(X,\phi U) - \frac{(2n+1)}{(n-1)}g(\phi X,U).
\end{aligned} \tag{37}$$

Using the equations (6), (7) and (8) in the above equation, we obtain

$$S(X,\phi U)(3n+2) = (2r+n)g(X,\phi U). \tag{38}$$

Replacing U by ϕU and using the equations (4), (9) and (16) in the above equation, we get

$$S(X,U) = \frac{(2r+n)}{(3n+2)}g(X,U) + \frac{(2r-6n^2-5n)}{(3n+2)}\eta(X)\eta(U),$$

which implies that para-Sasakian manifold is an η-Einstein Manifold. This leads us to state the following theorem:

Theorem 4. *If a para-Sasakian manifold is a quasi-M-projectively flat with respect to the Zamkovoy connection, then it is an η-Einstein manifold.*

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