Bulletin of the *Transilvania* University of Braşov
Series III: Mathematics and Computer Science, Vol. 4(66), No. 1 - 2024, 73-86
https://doi.org/10.31926/but.mif.2024.4.66.1.5

PSEUDO-SPECTRUM OF NON-ARCHIMEDEAN MATRIX PENCILS

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Abstract

In this paper, we define the notions of the C-trace pseudo-spectrum, the M-determinant pseudo-spectrum and the pseudo-spectrum of non-Archimedean matrix pencils. Many results are proved about the C-trace pseudo-spectrum, the M-determinant pseudo-spectrum and the pseudo-spectrum of non-Archimedean matrix pencils. Examples are given to support our work.

2000 *Mathematics Subject Classification:* 47A10, 47A56, 47S10.
Key words: Non-Archimedean matrix pencil, determinant, trace, spectrum.

1 Introduction and results

Throughout this paper, $\mathbb{K}$ is a non-Archimedean (n.a) non trivially complete valued field with valuation $|\cdot|$, $\mathbb{Q}_p$ is the field of p-adic numbers ($p \geq 2$ being a prime) equipped with p-adic valuation $|.|_p$, X is a non-Archimedean Banach space over $\mathbb{K}$, $\mathcal{L}(X)$ denotes the set of all bounded linear operators on X and $X' = \mathcal{L}(X, \mathbb{K})$ is the dual space of X. For more details, we refer to [11]. We denote the completion of algebraic closure of $\mathbb{Q}_p$ under the p-adic valuation $|\cdot|_p$ by $\mathbb{C}_p$ [11]. For more details on non-Archimedean pseudo-spectrum of operator pencils or matrix pencils, we refer to [4], [5] and [6]. In this paper, we study the problem of finding the eigenvalues of the generalized eigenvalue problem

$$Ax = \lambda Bx$$

for $\lambda \in \mathbb{K}$ and $x \in \mathbb{K}^n$, $\mathcal{M}_n(\mathbb{K})$ denotes the algebra of all $n \times n$ (n.a) matrices and I is the $n \times n$ identity matrix. Let $A \in \mathcal{M}_n(\mathbb{K})$, the trace and the determinant of A are denoted by $Tr(A)$ and $\det(A)$ respectively. For more details, we refer to [1], [2], [3], [7] and [9]. We have the following definitions.

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Definition 1. *[6] Let X be a non-Archimedean Banach space over $\mathbb{K}$. For a pair (A,B) of operators in $\mathcal{L}(X)$, the spectrum $\sigma(A,B)$ of the linear operator pencil (A,B) is defined by*

$$\begin{aligned}\sigma(A,B) &= \{\lambda\in\mathbb{K}:\ A-\lambda B \text{ is not invertible in } \mathcal{L}(X)\}\\ &= \{\lambda\in\mathbb{K}: 0\in\sigma(A-\lambda B)\}.\end{aligned}$$

The resolvent set $\rho(A,B)$ of the linear operator pencil (A,B) is the complement of $\sigma(A,B)$ in $\mathbb{K}$ given by

$$\rho(A,B)=\{\lambda\in\mathbb{K}:\ R_\lambda(A,B)=(A-\lambda B)^{-1} \text{ exists in } \mathcal{L}(X)\}.$$

$R_\lambda(A,B)$ is called the resolvent of the linear operator pencil (A,B).

Ingleton [8] showed that:

Theorem 1. *[8] Let X be a non-Archimedean Banach space over a spherically complete field $\mathbb{K}$. For all $x\in X\backslash\{0\}$, there is $\xi\in X'$ such that $\xi(x)=1$ and $\|\xi\|=\|x\|^{-1}$.*

Example 1. *[11] $\mathbb{Q}_p$ is spherically complete.*

2 P-adic spectral sets of matrix pencils

We introduce the following definition.

Definition 2. *[2] Let $A\in\mathcal{M}_n(\mathbb{K})$, the trace of A is*

$$Tr(A)=\sum_{k=1}^{n}a_{k,k},$$

where for all $k\in\{1,\cdots,n\}$, $a_{k,k}\in\mathbb{K}$ are diagonal coefficients of A.

We have the following proposition.

Proposition 1. *[2] Let $A,B\in\mathcal{M}_n(\mathbb{K})$ and $\lambda\in\mathbb{K}$. Then*

(i) $Tr(A+\lambda B)=Tr(A)+\lambda Tr(B)$,

(ii) $Tr(AB)=Tr(BA)$.

Furthermore, the map $Tr:\mathcal{M}_n(\mathbb{K})\to\mathbb{K}$ is a continuous linear functional with $|Tr(A)|\leq\|A\|$.

We have the following definitions.

Definition 3. *[2] Let $A,B\in\mathcal{M}_n(\mathbb{K})$ and $\varepsilon>0$. The trace pseudo-spectrum of the matrix pencil (A,B) of the form $A-\lambda B$ is denoted by $Tr_\varepsilon(A,B)$ and is defined as*

$$Tr_\varepsilon(A,B)=\sigma(A,B)\cup\{\lambda\in\mathbb{K}:|Tr(A-\lambda B)|\leq\varepsilon\}.$$

The trace pseudo-resolvent of the matrix pencil of the form $A-\lambda B$ is denoted by $Tr\rho_\varepsilon(A,B)$ and is defined by

$$Tr\rho_\varepsilon(A,B)=\rho(A,B)\cap\{\lambda\in\mathbb{K}:|Tr(A-\lambda B)|>\varepsilon\}.$$

Definition 4. *Let $A, B, C \in \mathcal{M}_n(\mathbb{K})$ and $\varepsilon > 0$. The C-trace pseudo-spectrum of the matrix pencil (A, B) of the form $A - \lambda B$ is denoted by $Tr_\varepsilon^C(A, B)$ and is defined as*

$$Tr_\varepsilon^C(A, B) = \sigma(A, B) \cup \{\lambda \in \mathbb{K} : |Tr({}^tC(A - \lambda B)C)| \leq \varepsilon\}.$$

The C-trace pseudo-resolvent of the matrix pencil of the form $A - \lambda B$ is denoted by $Tr^C\rho_\varepsilon(A, B)$ and is defined by

$$Tr^C\rho_\varepsilon(A, B) = \rho(A, B) \cap \{\lambda \in \mathbb{K} : |Tr({}^tC(A - \lambda B)C)| > \varepsilon\}.$$

If $C = I$, the Definition 4 coincides with the Definition 3. We have the following theorem.

Theorem 2. *Let $A, B \in \mathcal{M}_n(\mathbb{K})$. Then, for any $C \in \mathcal{M}_n(\mathbb{K})$, not null, we have:*

(i) If $0 < \varepsilon_1 \leq \varepsilon_2$, $Tr_{\varepsilon_1}^C(A, B) \subset Tr_{\varepsilon_2}^C(A, B)$,

(ii) If $\alpha \in \mathbb{K}$ and $\beta \in \mathbb{K}\backslash\{0\}$, then for any $\varepsilon > 0$, $Tr_\varepsilon^C(\beta A + \alpha B, B) = \beta Tr_{\frac{\varepsilon}{|\beta|}}^C(A, B) + \alpha$,

(iii) For all $\lambda, \alpha \in \mathbb{K}, \varepsilon > 0$ and $Tr({}^tCC) \neq 0$,

$$Tr_\varepsilon^C(\alpha I, I) = \Big\{\lambda \in \mathbb{K} : |\lambda - \alpha| \leq \frac{\varepsilon}{|Tr({}^tCC)|}\Big\}.$$

Proof. (i) It is clear from the definition of the C-trace pseudo-spectrum of matrix pencils.

(ii) Let $\alpha \in \mathbb{K}$ and $\beta \in \mathbb{K}\backslash\{0\}$, then it is easy to see that

$$\sigma(\beta A + \alpha B, B) = \alpha + \beta\sigma(A, B),$$

and

$$\begin{aligned} Tr_\varepsilon^C(\beta A + \alpha B, B) &= \Big\{\lambda \in \mathbb{K} : |Tr({}^tC(\beta A + \alpha B - \lambda B)C)| \leq \varepsilon\Big\} \\ &= \Big\{\lambda \in \mathbb{K} : |\beta||Tr({}^tC(A - \frac{(\lambda - \alpha))}{\beta}B)C)| \leq \varepsilon\Big\} \\ &= \Big\{\lambda \in \mathbb{K} : |Tr({}^tC(A - \frac{(\lambda - \alpha)}{\beta}B)C)| \leq \frac{\varepsilon}{|\beta|}\Big\}. \end{aligned}$$

Hence $\lambda \in Tr_\varepsilon^C(\beta A + \alpha B, B)$ if, and only if, $\frac{\lambda-\alpha}{\beta} \in Tr_{\frac{\varepsilon}{|\beta|}}^C(A, B)$ i.e., $\lambda \in \beta Tr_{\frac{\varepsilon}{|\beta|}}^C(A, B) + \alpha$.

(iii) Let $\alpha, \lambda \in \mathbb{K}, \varepsilon > 0$ and $Tr({}^tCC) \neq 0$, then

$$|Tr({}^tC(\alpha I - \lambda I)C)| = |\lambda - \alpha||Tr({}^tCC)| \leq \varepsilon.$$

Thus

$$Tr_\varepsilon^C(\alpha I, I) = \Big\{\lambda \in \mathbb{K} : |\lambda - \alpha| \leq \frac{\varepsilon}{|Tr({}^tCC)|}\Big\}.$$

□

We have the following examples.

Example 2. *Let* $\mathbb{K} = \mathbb{Q}_p$ *and* $\varepsilon > 0$. *If*

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \text{ and } C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then

$$\begin{aligned} Tr_\varepsilon^C(A,B) &= \sigma(A,B) \cup \{\lambda \in \mathbb{Q}_p : |Tr({}^tC(A-\lambda B)C)|_p \leq \varepsilon\} \\ &= \sigma(A,B) \cup \{\lambda \in \mathbb{Q}_p : |Tr({}^tC(A-\lambda B)C)|_p \leq \varepsilon\} \\ &= \{\frac{1}{2}, 1\} \cup \{\lambda \in \mathbb{Q}_p : |2-3\lambda|_p \leq \varepsilon\}. \end{aligned}$$

Example 3. *Let* $\alpha, \beta \in \mathbb{K} = \mathbb{Q}_p$ *be nonzero elements and* $\varepsilon > 0$. *If*

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \ B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } C = \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix}.$$

Then

$$\begin{aligned} Tr_\varepsilon^C(A,B) &= \sigma(A,B) \cup \{\lambda \in \mathbb{Q}_p : |Tr({}^tC(A-\lambda B)C)|_p \leq \varepsilon\} \\ &= \{1\} \cup \{\lambda \in \mathbb{Q}_p : |\alpha^2 + \beta^2(1-\lambda)|_p \leq \varepsilon\}. \end{aligned}$$

Let $r > 0$, $B_f(0,r) = \{\lambda \in \mathbb{K} : |\lambda| \leq r\}$ is the closed ball centered at zero with radius r. We have the following theorem.

Theorem 3. *Let* $A, B, C \in \mathcal{M}_n(\mathbb{K})$ *such that* $|Tr({}^tCBC)| \neq 0$ *and* $\varepsilon > 0$. *Then*

$$Tr_\delta^C(A,B) + B_f(0, \frac{\varepsilon}{|Tr({}^tCBC)|}) \subseteq Tr_\gamma^C(A,B)$$

where $\gamma = \max\{\delta, \varepsilon\}$. *If* $\delta < \varepsilon$, *we have*

$$Tr_\delta^C(A,B) + B_f(0, \frac{\varepsilon}{|Tr({}^tCBC)|}) \subseteq Tr_\varepsilon^C(A,B).$$

Proof. Let $A, B, C \in \mathcal{M}_n(\mathbb{K})$ and $\varepsilon > 0$. Let $\lambda \in Tr_\delta^C(A,B) + B_f(0, \frac{\varepsilon}{|Tr({}^tCBC)|})$, then there exists $\lambda_0 \in Tr_\delta^C(A,B)$ and $\lambda_1 \in B_f(0, \frac{\varepsilon}{|Tr({}^tCBC)|})$ such that $\lambda = \lambda_0 + \lambda_1$, hence

$$|Tr({}^tC(A-\lambda_0 B)C)| \leq \delta$$

and

$$|\lambda_1||Tr({}^tCBC)| \leq \varepsilon.$$

Thus,

$$\begin{aligned} |Tr({}^tC(A-\lambda B)C)| &= |Tr({}^tC(A-\lambda_0 B - \lambda_1 B)C)| \\ &\leq \max\Big\{|Tr({}^tC(A-\lambda_0 B)C|, \ |\lambda_1||Tr({}^tCBC)|\Big\} \\ &\leq \max\Big\{\delta, \ \varepsilon\Big\}. \end{aligned}$$

Set $\gamma = \max\Big\{\delta,\ \varepsilon\Big\}$. Then,

$$Tr^C_\delta(A,B) + B_f(0, \frac{\varepsilon}{|Tr(^tCBC)|}) \subseteq Tr^C_\gamma(A,B).$$

If $\delta < \varepsilon$, then $\gamma = \varepsilon$. Consequently,

$$Tr^C_\delta(A,B) + B_f(0, \frac{\varepsilon}{|Tr(^tCBC)|}) \subseteq Tr^C_\varepsilon(A,B).$$

□

Now we introduce the C-trace set of matrix pencils in non-Archimedean case as follows.

Definition 5. *Let $A, B, C \in \mathcal{M}_n(\mathbb{K})$ and $\varepsilon > 0$. The C-trace set of the matrix pencil (A,B) is denoted by $tr^C_\varepsilon(A,B)$ and is defined as*

$$tr^C_\varepsilon(A,B) = \{\lambda \in \mathbb{K} : |Tr(^tC(A-\lambda B)C| \leq \varepsilon\}.$$

Remark 1.

(i) For all $\varepsilon_1, \varepsilon_2 > 0$ such that $\varepsilon_2 \geq \varepsilon_1$, we have $tr^C_{\varepsilon_1}(A,B) \subseteq tr^C_{\varepsilon_2}(A,B)$.

(ii) If $C = I$, we have $tr^C_\varepsilon(A,B) = tr_\varepsilon(A,B)$.

We have the following results.

Theorem 4. *Let $A, B, D \in \mathcal{M}_n(\mathbb{K})$ and $\varepsilon > 0$. Then, for any $C \in \mathcal{M}_n(\mathbb{K})$, not null, we have:*

$$tr^C_\varepsilon(A,D) + tr^C_\varepsilon(B,D) \subseteq tr^C_\varepsilon(A+B,D).$$

Proof. Let $\lambda \in tr^C_\varepsilon(A,D) + tr^C_\varepsilon(B,D)$, then there exists $\lambda_0 \in tr^C_\varepsilon(A,D)$ and $\lambda_1 \in tr^C_\varepsilon(B,D)$ such that $\lambda = \lambda_0 + \lambda_1$. Then

$$|Tr(^tC(A-\lambda_0 D)C)| \leq \varepsilon \text{ and } |Tr(^tC(B-\lambda_1 D)C)| \leq \varepsilon.$$

Thus,

$$\begin{aligned} |Tr(^tC(A+B-\lambda D)C)| &= |Tr(^tC(A+B-\lambda_0 D-\lambda_1 D)C)| \\ &= |Tr(^tC(A-\lambda_0 D)C) + Tr(^tC(B-\lambda_1 D)C)| \\ &\leq \max\Big\{|Tr(^tC(A-\lambda_0 D)C)|,\ |Tr(^tC(B-\lambda_1 D)C)|\Big\} \\ &\leq \varepsilon. \end{aligned}$$

Consequently, $\lambda \in tr^C_\varepsilon(A+B,D)$. Hence,

$$tr^C_\varepsilon(A,D) + tr^C_\varepsilon(B,D) \subseteq tr^C_\varepsilon(A+B,D).$$

□

Proposition 2. *Let $A, B \in \mathcal{M}_n(\mathbb{K})$ and $\varepsilon > 0$. Let $\lambda, \mu \in tr^C_\varepsilon(A,B)$ and $\alpha \in \mathbb{K}$ such that $|\alpha| \leq 1$. Then $\alpha\lambda + (1-\alpha)\mu \in tr^C_\varepsilon(A,B)$.*

Proof. Let $A, B, C \in \mathcal{M}_n(\mathbb{K})$ and $\varepsilon > 0$. Let $\lambda, \mu \in tr_\varepsilon^C(A, B)$ and $\alpha \in \mathbb{K}$ such that $|\alpha| \leq 1$. Then

$$|Tr({}^tC(A - \lambda B)C)| \leq \varepsilon,$$

and

$$|Tr({}^tC(A - \mu B)C)| \leq \varepsilon.$$

Hence

$$\begin{aligned} |Tr({}^tC(A - (\alpha\lambda + (1-\alpha)\mu)B)C)| &= |Tr({}^tC(A - \mu B)C) + \alpha Tr({}^tC(A - \lambda B)C) \\ &\quad -\alpha Tr({}^tC(A - \mu B)C)| \\ &\leq \max\Big\{|Tr({}^tC(A - \mu B)C)|, \\ &\quad |\alpha||Tr({}^tC(A - \lambda B)C)|, \\ &\quad |\alpha||Tr({}^tC(A - \mu B)C)|\Big\} \\ &\leq \varepsilon. \end{aligned}$$

Thus,

$$\alpha\lambda + (1-\alpha)\mu \in tr_\varepsilon^C(A, B).$$

□

Proposition 3. *Let $A, B, C \in \mathcal{M}_n(\mathbb{K})$ and $\varepsilon > 0$ such that $\|{}^tCAC\| < \varepsilon$. Let $\lambda, \mu \in tr_\varepsilon^C(A, B)$. Then*

$$\lambda - \mu \in tr_\varepsilon^C(A, B).$$

Proof. Let $A, B \in \mathcal{M}_n(\mathbb{K})$ and $\varepsilon > 0$ such that $\|{}^tCAC\| < \varepsilon$. Let $\lambda, \mu \in tr_\varepsilon^C(A, B)$. By Proposition 1, we have $|Tr({}^tCAC)| \leq \|{}^tCAC\|$. Then

$$|Tr(A - \lambda B)| \leq \varepsilon,$$

and

$$|Tr(A - \mu B)| \leq \varepsilon.$$

Hence

$$\begin{aligned} |Tr({}^tC(A - (\lambda - \mu)B)C)| &= |Tr({}^tC(A - \lambda B)C) \\ &\quad -Tr({}^tC(A - \mu B)C) + Tr({}^tCAC)| \\ &\leq \max\Big\{|Tr({}^tC(A - \lambda B)C)|, \\ &\quad |Tr({}^tC(A - \mu B)C)|, |Tr({}^tCAC)|\Big\}, \\ &\leq \varepsilon. \end{aligned}$$

Thus,

$$\lambda - \mu \in tr_\varepsilon^C(A, B).$$

□

3 M-determinant pseudo-spectrum of non-Archimedean matrix pencils

We introduce the following definition.

Definition 6. *Let $A, B, M \in \mathcal{M}_n(\mathbb{K})$ such that $\det(M) \neq 0$ and $\varepsilon > 0$. The M-determinant pseudo-spectrum of the matrix pencil (A, B) is denoted by $d_\varepsilon^M(A, B)$ and is defined as*

$$d_\varepsilon^M(A, B) = \{\lambda \in \mathbb{K} : |\det(M(A - \lambda B))| \leq \varepsilon\}.$$

If $B = M = I \in \mathcal{M}_n(\mathbb{K})$, we have the following:

Definition 7. *Let $A \in \mathcal{M}_n(\mathbb{K})$ and $\varepsilon > 0$. The determinant pseudo-spectrum $d_\varepsilon(A)$ of A is*

$$d_\varepsilon(A) = \{\lambda \in \mathbb{K} : |\det(A - \lambda I)| \leq \varepsilon\}.$$

From the definition of the M-determinant spectrum, we have the following remark.

Remark 2.

(i) If $I = M$, then the Definition 6 coincides with the Definition 6. of [2].

(ii) If $M = B = I$, the M-determinant pseudo-spectrum coincides with the determinant pseudo-spectrum, i.e., $d_\varepsilon^M(A, I) = d_\varepsilon(A)$.

(iii) In Definition 6, if B is invertible, then for any $\varepsilon > 0$, $\sigma_\varepsilon(A, B) \subseteq d_\varepsilon^M(A, B)$ and $d_0^M(A, B) = \sigma(A, B)$.

(iv) For all unitary matrix $C \in \mathcal{M}_n(\mathbb{K})$, $d^M(A, B) = d^M({}^tCAC, {}^tCBC)$.

We have the following proposition.

Proposition 4. *Let $A, B, M \in \mathcal{M}_n(\mathbb{K})$. Then*

(i) For all $0 < \varepsilon_1 \leq \varepsilon_2$, we have $d_{\varepsilon_1}^M(A, B) \subseteq d_{\varepsilon_2}^M(A, B)$,

(ii) If $\det(M) \neq 0$, then for any $\varepsilon > 0$, $d_\varepsilon^M(\alpha I, I) = \{\lambda \in \mathbb{K} : |\lambda - \alpha| \leq (\frac{\varepsilon}{|\det(M)|})^{\frac{1}{n}}\}$,

(iii) For all $\alpha, \beta \in \mathbb{K}, \varepsilon > 0$ such that $\beta \neq 0$, $d_\varepsilon^M(\alpha I + \beta A, I) = \alpha + \beta d^M_{\frac{\varepsilon}{|\beta|^n}}(A, I)$.

Proof. Let $A, B \in \mathcal{M}_n(\mathbb{K})$ and $\varepsilon > 0$.

(i) For $0 < \varepsilon_1 \leq \varepsilon_2$. Let $\lambda \in d_{\varepsilon_1}^M(A, B)$, then $|\det(M(A - \lambda B))| \leq \varepsilon_1 \leq \varepsilon_2$. Hence $\lambda \in d_{\varepsilon_2}^M(A, B)$.

(ii) Let $\alpha \in \mathbb{K}$, then $|\det(M(\alpha I - \lambda I))| = |\lambda - \alpha|^n |\det(M)|$, hence

$$\begin{aligned} d_\varepsilon^M(\alpha I, I) &= \{\lambda \in \mathbb{K} : |\det(M(\alpha I - \lambda I))| \leq \varepsilon\} \\ &= \{\lambda \in \mathbb{K} : |\lambda - \alpha|^n |\det(M)| \leq \varepsilon\} \\ &= \{\lambda \in \mathbb{K} : |\lambda - \alpha| \leq (\frac{\varepsilon}{|\det(M)|})^{\frac{1}{n}}\}. \end{aligned}$$

(iii) Let $\alpha, \beta \in \mathbb{K}$ such that $\beta \neq 0$, we have

$$\begin{aligned} d_\varepsilon^M(\alpha I + \beta A, I) &= \{\lambda \in \mathbb{K} : |\det(M(\alpha I + \beta A - \lambda I))| \leq \varepsilon\} \\ &= \{\lambda \in \mathbb{K} : |\det(M(\beta A - (\lambda - \alpha) I))| \leq \varepsilon\} \\ &= \{\lambda \in \mathbb{K} : |\beta|^n |\det(M(A - \frac{(\lambda - \alpha)}{\beta} I))| \leq \varepsilon\} \\ &= \{\lambda \in \mathbb{K} : \det(M(A - \frac{(\lambda - \alpha)}{\beta} I))| \leq \frac{\varepsilon}{|\beta|^n}\}. \end{aligned}$$

Hence,

$$\begin{aligned} \lambda \in d_\varepsilon^M(\alpha I + \beta A, I) &\iff \frac{\lambda - \alpha}{\beta} \in d^M_{\frac{\varepsilon}{|\beta|^n}}(A, I) \\ &\iff \lambda \in \alpha + \beta d^M_{\frac{\varepsilon}{|\beta|^n}}(A, I). \end{aligned}$$

□

We give some examples of the M-determinant pseudo-spectrum.

Example 4. *Let $\mathbb{K} = \mathbb{Q}_p$ and $\varepsilon > 0$. If*

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix} \text{ and } M = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

It is easy to check that

$$\begin{aligned} d_\varepsilon^M(A, B) &= \{\lambda \in \mathbb{Q}_p : |\det(M(A - \lambda B))|_p \leq \varepsilon\} \\ &= \{\lambda \in \mathbb{Q}_p : |(\lambda - 1)(3\lambda - 1)|_p \leq \varepsilon\}. \end{aligned}$$

Example 5. *Let $\mathbb{K} = \mathbb{Q}_p$ with $p \geq 2$ and $\varepsilon > 0$. If*

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \ B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } M = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then

$$\begin{aligned} d_\varepsilon^M(A, B) &= \{\lambda \in \mathbb{Q}_p : |\det(M(A - \lambda B))|_p \leq \varepsilon\} \\ &= \{\lambda \in \mathbb{Q}_p : |2\lambda(\lambda - 2)|_p \leq \varepsilon\}. \end{aligned}$$

Example 6. *Let $\mathbb{K} = \mathbb{C}_p$ with $p \neq 2$ and $\varepsilon > 0$. Let $a, b \in \mathbb{C}_p$ such that $a^2 + b^2 = 1$, we consider*

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \ B = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } M = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}.$$

It is easy to see that

$$\begin{aligned} d_\varepsilon^M(A,B) &= \{\lambda \in \mathbb{C}_p : |\det(M(A-\lambda B))|_p \le \varepsilon\} \\ &= \{\lambda \in \mathbb{C}_p : |3|_p|(a-\lambda)(a-2\lambda)+b^2|_p \le \varepsilon\} \\ &= \{\lambda \in \mathbb{C}_p : |3|_p|2\lambda^2 - 3\lambda a + 1|_p \le \varepsilon\}. \end{aligned}$$

We have the following proposition.

Proposition 5. *Let* $M \in \mathcal{L}(\mathbb{Q}_p^n)$ *such that* $\det(M) \neq 0$ *and* $\varepsilon > 0$, *let* $D \in \mathcal{L}(\mathbb{Q}_p^n)$ *be diagonal operator such that for all* $i \in \{1, \cdots, n\}$, $De_i = \lambda_i e_i$ *and* $\lambda_i \in \mathbb{Q}_p$. *Then*

$$d_\varepsilon^M(D,I) = \{\lambda \in \mathbb{Q}_p : |\lambda - \lambda_1| \cdots |\lambda - \lambda_n| \le \frac{\varepsilon}{|\det(M)|}\}.$$

Proof. Let $\varepsilon > 0$. Then $D - \lambda I$ has the form

$$\text{for all } i \in \{1, \cdots, n\},\ (D-\lambda)e_i = (\lambda - \lambda_i)e_i$$

where $(e_i)_{1 \le i \le n}$ is the canonical base of $\mathbb{Q}_p^n$.

Then, $|\det(M(D-\lambda I))| = |\lambda - \lambda_1| \cdots |\lambda - \lambda_n||\det(M)|$. Hence

$$\begin{aligned} d_\varepsilon^M(D,I) &= \{\lambda \in \mathbb{Q}_p : |\det(M(D-\lambda I))| \le \varepsilon\} \\ &= \{\lambda \in \mathbb{Q}_p : |\lambda - \lambda_1| \cdots |\lambda - \lambda_n| \le \frac{\varepsilon}{|\det(M)|}\}. \end{aligned}$$

□

4 Non-Archimedean pseudo-spectrum of matrix pencils

We begin with the following definition.

Definition 8. *[5] Let* $A, B \in \mathcal{M}_n(\mathbb{K})$ *and* $\varepsilon > 0$. *The pseudo-spectrum* $\sigma_\varepsilon(A,B)$ *of a matrix pencil* (A,B) *is defined by*

$$\sigma_\varepsilon(A,B) = \sigma(A,B) \cup \{\lambda \in \mathbb{K} : \ \|(A-\lambda B)^{-1}\| > \varepsilon^{-1}\}.$$

The pseudo-resolvent $\rho_\varepsilon(A,B)$ *of a matrix pencil* (A,B) *is defined by*

$$\rho_\varepsilon(A,B) = \rho(A,B) \cap \{\lambda \in \mathbb{K} : \ \|(A-\lambda B)^{-1}\| \le \varepsilon^{-1}\},$$

by convention $\|(A-\lambda B)^{-1}\| = \infty$ *if, and only if,* $\lambda \in \sigma(A,B)$.

We have the following results.

Proposition 6. *Let* $A, B \in \mathcal{M}_n(\mathbb{K})$ *and* $\varepsilon > 0$, *we have*

(i) $\sigma(A,B) = \bigcap_{\varepsilon > 0} \sigma_\varepsilon(A,B)$.

(ii) For all ε_1 *and* ε_2 *such that* $0 < \varepsilon_1 < \varepsilon_2$, $\sigma(A,B) \subset \sigma_{\varepsilon_1}(A,B) \subset \sigma_{\varepsilon_2}(A,B)$.

Proof. (i) By Definition 8, we have for all $\varepsilon > 0$, $\sigma(A,B) \subset \sigma_\varepsilon(A,B)$. Conversely, if $\lambda \in \bigcap_{\varepsilon>0} \sigma_\varepsilon(A,B)$, then for all $\varepsilon > 0$, $\lambda \in \sigma_\varepsilon(A,B)$. If $\lambda \in \sigma(A,B)$, there is nothing to prove. If $\|(A-\lambda B)^{-1}\| > \varepsilon^{-1}$, taking limits as $\varepsilon \to 0^+$, we get $\|(A-\lambda B)^{-1}\| = \infty$. Thus $\lambda \in \sigma(A,B)$.

(ii) For ε_1 and ε_2 such that $0 < \varepsilon_1 < \varepsilon_2$. Let $\lambda \in \sigma_{\varepsilon_1}(A,B)$, then $\|(A-\lambda B)^{-1}\| > \varepsilon_1^{-1} > \varepsilon_2^{-1}$. Hence $\lambda \in \sigma_{\varepsilon_2}(A,B)$.

□

Theorem 5. *Let X be a non-Archimedean finite dimensional Banach space over $\mathbb{Q}_p$ such that $\|X\| \subseteq |\mathbb{Q}_p|$, let $A, B \in \mathcal{L}(X)$ and $\varepsilon > 0$. Then,*

$$\sigma_\varepsilon(A,B) = \bigcup_{C\in\mathcal{L}(X):\|C\|<\varepsilon} \sigma(A+C,B).$$

Proof. Let $A, B \in \mathcal{L}(X)$ and $\varepsilon > 0$, let $\lambda \in \bigcup_{C\in\mathcal{L}(X):\|C\|<\varepsilon} \sigma(A+C,B)$. We argue by contradiction. Suppose that $\lambda \in \rho(A,B)$ and $\|(A-\lambda B)^{-1}\| \le \varepsilon^{-1}$.
Consider the bounded linear operator D defined on X by

$$D = \sum_{n=0}^{\infty} (A-\lambda B)^{-1}\Big(-C(A-\lambda B)^{-1}\Big)^n.$$

It is easy to see that D can be written as follows

$$D = (A-\lambda B)^{-1}(I + C(A-\lambda B)^{-1})^{-1}.$$

Hence for all $x \in X$, $D(I + C(A-\lambda B)^{-1})x = (A-\lambda B)^{-1}x$.
Let $y = (A-\lambda B)^{-1}x$, then, for all $y \in X$, $D(A-\lambda B + C)y = y$. Moreover, we have

$$\text{for all } x \in X,\ (A-\lambda B + C)Dx = x.$$

Hence, we conclude that $(A-\lambda B+C)$ is invertible and $D = (A-\lambda B+C)^{-1}$, which is a contradiction. Thus $\lambda \in \sigma_\varepsilon(A,B)$.
Conversely, let $A, B, C \in \mathcal{L}(X)$ and $\varepsilon > 0$, suppose that $\lambda \in \sigma_\varepsilon(A,B)$. We discuss two cases.

First case: If $\lambda \in \sigma(A,B)$, we may put $C = 0$.

Second case: Assume that $\lambda \in \sigma_\varepsilon(A,B)$ and $\lambda \notin \sigma(A,B)$. Then, there exists $y \in X\backslash\{0\}$ such that

$$\frac{\|(A-\lambda B)^{-1}y\|}{\|y\|} > \frac{1}{\varepsilon}. \tag{1}$$

Since $\|X\| \subseteq |\mathbb{Q}_p|$, then there exists $c \in \mathbb{Q}_p\backslash\{0\}$ such $|c| = \|y\|$. Then, setting $z = c^{-1}y$, then $\|z\| = 1$. Hence, we obtain

$$\begin{aligned}\|(A-\lambda B)^{-1}z\| &= \|(A-\lambda B)^{-1}c^{-1}y\| \\ &= \frac{\|(A-\lambda B)^{-1}y\|}{|c|} \\ &= \frac{\|(A-\lambda B)^{-1}y\|}{\|y\|}.\end{aligned}$$

From (1),

$$\|(A-\lambda B)^{-1}z\| > \frac{1}{\varepsilon}. \tag{2}$$

By the same reasoning above, we infer that there exists $c_0 \in \mathbb{Q}_p \backslash \{0\}$ such that $|c_0| = \|(A-\lambda B)^{-1}z\|$. Then, setting $z_0 = c_0^{-1}(A-\lambda B)^{-1}z$ which yields $z_0 \in \mathbb{Q}_p$ and $\|z_0\| = 1$. Consequently, we have

$$\begin{aligned}\|(A-\lambda B)z_0\| &= \|(A-\lambda B)(A-\lambda B)^{-1}c_0^{-1}z\| \\ &= \frac{\|z\|}{|c_0|}.\end{aligned}$$

Using the fact that $\|z\| = 1$, we deduce from (2) that

$$\begin{aligned}\|(A-\lambda B)z_0\| &= \|(A-\lambda B)^{-1}z\|^{-1}, \\ &< \varepsilon.\end{aligned}$$

By Theorem 1, there exists $\phi \in X'$ such that $\phi(z_0) = 1$ and $\|\phi\| = \|z_0\|^{-1} = 1$. We consider the following linear operator given by

$$\text{for all } y \in X,\ Cy = -\phi(y)(A-\lambda B)z_0.$$

Clearly, C is a bounded linear operator on X, since for all $y \in X$,

$$\begin{aligned}\|Cy\| &= \|\phi(y)\|\|(A-\lambda B)z_0\|, \\ &< \varepsilon\|y\|.\end{aligned}$$

Then, $\|C\| < \varepsilon$. Moreover, we have $(A-\lambda B+C)z_0 = 0$. So, $(A-\lambda B+C)$ is not invertible. Consequently,

$$\lambda \in \bigcup_{C \in \mathcal{L}(X) : \|C\| < \varepsilon} \sigma(A+C, B).$$

□

We finish with some examples of the pseudo-spectrum of matrix pencils.

Example 7. *Let* $\mathbb{K}=\mathbb{Q}_p$ *and* $\varepsilon>0$. *If*

$$A=\begin{pmatrix}\lambda_1 & 1\\ 0 & \lambda_2\end{pmatrix} \text{ and } B=\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix},$$

where $\lambda_1,\lambda_2\in\mathbb{Q}_p\backslash\{0\}$. *Then*

$$\begin{aligned}\sigma_\varepsilon(A,B) &= \sigma(A,B)\cup\{\lambda\in\mathbb{Q}_p:\|(A-\lambda B)^{-1}\|>\frac{1}{\varepsilon}\}\\ &= \{\lambda_1,\lambda_2\}\cup\{\lambda\in\mathbb{Q}_p:\|(A-\lambda B)^{-1}\|>\frac{1}{\varepsilon}\},\end{aligned}$$

where $\|(A-\lambda B)^{-1}\|=\max\{\frac{1}{|\lambda_1-\lambda|_p},\frac{|\lambda+1|_p}{|(\lambda_1-\lambda)(\lambda_2-\lambda)|_p},\frac{1}{|\lambda_2-\lambda|_p}\}$.

Example 8. *Let* $\mathbb{K}=\mathbb{Q}_p$ *and* $\varepsilon>0$. *If*

$$A=\begin{pmatrix}\lambda_1 & 0\\ 0 & \lambda_2\end{pmatrix} \text{ and } B=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$

where $\lambda_1,\lambda_2\in\mathbb{Q}_p$. *Then it is easy to see that*

$$\begin{aligned}\sigma_\varepsilon(A,B) &= \sigma(A,B)\cup\{\lambda\in\mathbb{Q}_p:\|(A-\lambda B)^{-1}\|>\frac{1}{\varepsilon}\}\\ &= \{\lambda_1,\lambda_2\}\cup\{\lambda\in\mathbb{Q}_p:\max\{\frac{1}{|\lambda_1-\lambda|_p},\frac{1}{|\lambda_2-\lambda|_p}\}>\frac{1}{\varepsilon}\}.\end{aligned}$$

Example 9. *Let* $\mathbb{K}=\mathbb{Q}_p$ *and* $\varepsilon>0$. *If*

$$A=\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix} \text{ and } B=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$$

where $\lambda_1,\lambda_2\in\mathbb{Q}_p$. *Then it is easy to check that*

$$\begin{aligned}\sigma_\varepsilon(A,B) &= \sigma(A,B)\cup\{\lambda\in\mathbb{Q}_p:\|(A-\lambda B)^{-1}\|>\frac{1}{\varepsilon}\}\\ &= \{0,1\}\cup\{\lambda\in\mathbb{Q}_p:\max\{\frac{1}{|\lambda|_p},\frac{1}{|\lambda(1-\lambda)|_p}\}>\frac{1}{\varepsilon}\}.\end{aligned}$$

Example 10. *Let* $\mathbb{K}=\mathbb{Q}_p$ *and* $\varepsilon>0$. *If*

$$A=\begin{pmatrix}3 & 0\\ 1 & 0\end{pmatrix} \text{ and } B=\begin{pmatrix}1 & 0\\ 2 & 0\end{pmatrix}.$$

Then one can see that

$$\begin{aligned}\sigma_\varepsilon(A,B) &= \sigma(A,B)\\ &= \mathbb{Q}_p.\end{aligned}$$

Acknowledgement. The author would like to thank the editor and the anonymous referee for their valuable efforts, comments and suggestions which improved the quality of this paper.

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