Bulletin of the *Transilvania* University of Braşov
Series III: Mathematics and Computer Science, Vol. 4(66), No. 1 - 2024, 53-72
https://doi.org/10.31926/but.mif.2024.4.66.1.4

ON THE TANGENT BUNDLES OVER F-KÄHLERIAN MANIFOLDS

Nour Elhouda DJAA*,1 and Aydin GEZER 2

Abstract

The main purpose of the present paper is to study all forms of Riemannien curvatures and the harmonic Killing vector fields of a tangent bundle over an $F-$Kählerian manifold endowed with a Berger type deformed Sasaki metric g_{BS} .

2000 *Mathematics Subject Classification:* 53A45, 53C07, 53C55..
Key words: Berger type deformed Sasaki metric, $F-$Kählerian manifold, Riemannian curvature tensor, tangent bundle, Killing field.

1 Introduction

The first thing that comes to mind when the tangent bundle of any Riemannian manifold is mentioned is the Sasaki metric. When the geometric properties of the tangent bundle with the Sasaki metric are investigated, we usually encounter the flatness of the base manifold. This is the reason why a number of researchers proposed to deform the Sasaki metric in order to get some kind of fexibility of its properties . In recent years Yampolsky [20], A. Gezer and all [3, 11] (resp. Abbassi Kaddaoui [1, 2], M. Djaa [23, 12]) are introduced and studied a new deformation on tangent bundle TM, called Berger type deformed Sasaki metric (resp. g-natural metrics, Mus-Sasaki metric).

In this present paper we study all forms of Riemannien curvatures of the tangent bundle over an $F-$Kählerian manifold endowed with Berger type deformed Sasaki metric g_{BS}. First, we define the Berger type deformed Sasaki metric g_{BS} on a tangent bundle over an $F-$Kählerian manifold (Definition 1) and we give the formulas describing the Levi-Civita connection of this metric (Theorem 1). Secondly we obtain the tensor curvature (Theorem 2), the sectional curvature (Theorem 3) and the scalar curvature (Theorem 4, corollary 1 and Theorem 5), also we give some examples of scalar curvatures (Exemple 4.1 and Exemple 4.2). In the last section, we give the characterization of a harmonic Killing vector fields (Theorem 10 and Theorem 11).

1* *Corresponding author*, Departement of Mathematics, *Relizane* University, Algeria, e-mail: Djaanour@gmail.com
2 Faculty of Science, *Erzurum* Ataturk University, Turkey, e-mail: aydingzr@gmail.com

2 Preliminaries

Let M be an $m-$dimensional manifold. We point out here and once that all geometric objects considered in this paper are supposed to be of class C^{∞}.

An $F-$structure is a $(1,1)-$tensor field F which satisfies

$$F^3 + F = 0. \tag{1}$$

An almost complex structure $(F^2 + I = 0)$ is an example of $F-$structures. Also, note that an $F-$structure is a polynomial structure with the structural polynomial $Q(F) = F^3 + F$.
A polynomial structure is integrable if the Nijenhuis tensor vanishes [19]. Then, the integrability of an $F-$structure is equivalent to the vanishing of the Nijenhuis tensor N_F:

$$N_F(X,Y) = [FX, FY] - F\,[FX, Y] - F\,[X, FY] + F^2\,[X, Y]\,.$$

Recall that an $F-$structure on a Riemannian manifold (M, g) is called a metric $F-$structure, if it satisfies

$$g(FX, Y) = -g(X, FY). \tag{2}$$

for any vector fields X, Y on M (see [4, 5, 18]).
The manifold (M, F, g) equipped with a metric $F-$structure g is called an almost $F-$Hermitian manifold [17]. According to Bures and Vanzura in [7], the manifold (M, F, g) is a metric polynomial manifold. We will use the terminology in [17].

The $2-$covariant skew-symmetric tensor field ω defined by $\omega(X,Y) = g(FX,Y)$ is the fundamental $2-$form of the almost $F-$Hermitian manifold (M, F, g). If the fundamental $2-$form ω is closed, i.e., $d\omega = 0$, then the triple (M, F, g) will be called an almost $F-$Kählerian manifold. Moreover, if $d\omega = 0$ and $N_F = 0$, the triple (M, F, g) will be called an $F-$Kählerian manifold. In [17], Opozda proved that $d\omega = 0$ and $N_F = 0$ is equivalent to $\nabla F = 0$, where ∇ is the Levi-Civita connection of g.

Let M be an $m-$dimensional Riemannian manifold with a Riemannian metric g and TM be its tangent bundle denoted by $\pi : TM \to M$. A system of local coordinates (U, x^i) in M induces on TM a system of local coordinates $\left(\pi^{-1}(U), x^i, x^{\bar{i}} = u^i\right)$, $\bar{i} = n + i = n + 1, ..., 2n$, where (u^i) is the cartesian coordinates in each tangent space $T_P M$ at $P \in M$ with respect to the natural base $\left\{\frac{\partial}{\partial x^i}\,|_P\right\}$, P being an arbitrary point in U whose coordinates are (x^i).

Given a vector field $X = X^i \frac{\partial}{\partial x^i}$ on M, the vertical lift ${}^V X$ and the horizontal lift ${}^H X$ of X are given, with respect to the induced coordinates, by

$$X^V = X^i \partial_{\bar{i}}, \tag{3}$$

$$X^H = X^i \partial_i - u^s \Gamma^i_{sk} X^k \partial_{\bar{i}}, \tag{4}$$

where $\partial_i = \frac{\partial}{\partial x^i}$, $\partial_{\bar{i}} = \frac{\partial}{\partial u^i}$ and Γ^i_{sk} are the coefficients of the Levi-Civita connection ∇ of g [21]. In particular, we have the vertical spray ${}^V u$ and the horizontal spray ${}^H u$ on TM defined by

$$u^V = u^i(\partial_i)^V = u^i\partial_{\bar{i}}, u^H = u^i(\partial_i)^H = u^i\overline{\partial}_i.$$

u^V is also called the canonical or Liouville vector field on TM.

Let f be smooth function of M to $\mathbb{R}$ and X,Y,Z be any vector fields on M. We have [21]

$$\begin{aligned} X^H\left(f^V\right) &= X\left(f\right), \\ X^V\left(f^V\right) &= 0. \\ X^H\left((g(Y,Z))^V\right) &= X\left(g(Y,Z)\right), \\ X^V\left((g(Y,Z))^V\right) &= 0. \end{aligned}$$

where $f^V = f \circ \pi$.

The bracket operation of vertical and horizontal vector fields is given by the formulas [13, 21]

$$\left\{ \begin{array}{l} \left[X^H, Y^H\right] = [X,Y]^H - (R(X,Y)u)^V, \\ \left[X^H, Y^V\right] = (\nabla_X Y)^V, \\ \left[X^V, Y^V\right] = 0 \end{array} \right. \tag{5}$$

for all vector fields X and Y on M, where R is the Riemannian curvature tensor of g defined by

$$R\left(X,Y\right) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

Proposition 1. *Let (M,g) be a Riemannian manifold and F be a $(1,1)-$tensor field on M such that $\nabla F = 0$. If $u = u^i\frac{\partial}{\partial x^i} \in T_xM$ for all $x \in M$, then we have the followings*

1. $X^H(g(Fu,Fu))_{(x,u)} = 0$,
2. $X^H(g(Y,Fu))_{(x,u)} = g(\nabla_X Y, Fu)_x$,
3. $X^V(g(Fu,Fu)_{(x,u)} = X^V(|Fu|^2)_{(x,u)} = 2g(FX,Fu)_x$,
4. $X^V(g(Y,Fu)_{(x,u)} = g(FX,Y)_x$,
5. $X^H(f(r^2)) = 0$,
6. $X^V(f(r^2)) = 2f'(r^2)g(FX,Fu)_x$,

where $r^2 = g(Fu,Fu) = |Fu|^2$.

3 The Berger type deformed Sasaki metric on the tangent bundle

Definition 1. *Let (M, F, g) be an almost $F-$Kählerian manifold and TM be its tangent bundle. A Berger type deformed Sasaki metric on TM is defined as followings*

$$\begin{aligned} g_{BS}(X^H, Y^H) &= g(X,Y), \\ g_{BS}(X^V, Y^H) &= g_{BS}(X^H, Y^V) = 0, \\ g_{BS}(X^V, Y^V) &= g(X,Y) + \delta^2 g(X, Fu) g(Y, Fu) \end{aligned} \tag{6}$$

for all vector fields X, Y on M, where δ is some constant. The metric is said to be a Berger type deformed Sasaki metric.

In the particular case when F is invertible, the metric g_{BS} coincide with the Berger type deformed Sasaki metric over a Kählerian manifold. Thus the geometric results of g_{BS} generalize the other deformed Sasaki metrics.

Lemma 1. *Let (M, F, g) be an $F-$Kählerian manifold. Then we have*

$$g(FX, X) = 0, \tag{7}$$

$$g(F^2X, F^2Y) = g(FX, FY), \tag{8}$$

$$R(FX, Y) = -R(X, FY). \tag{9}$$

for all vector fields X and Y on M, where R is the Riemannian curvature of the Levi-Civita connection ∇ of g defined by

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

Proof. The first and second relation comes directly from (1) and (2). We will only show the second relation. For all vector fields X, Y, Z, W, we have

$$\begin{aligned} g(R(FX,Y)Z,W) &= g(R(Z,W)FX,Y) \\ &= g([\nabla_Z, \nabla_W] FX - \nabla_{[Z,W]} FX, Y) \\ &= g(F\big([\nabla_Z, \nabla_W] X\big) - F\big(\nabla_{[Z,W]} X\big), Y) \\ &= g\big(F(R(Z,W)X), Y\big) \\ &= -g\big(R(Z,W)X, F(Y)\big) \\ &= -g\big(R(X, F(Y))Z, W\big). \end{aligned}$$

$\square$

Lemma 2. *Let (M,F,g) be an F-Kählerian manifold and g_{BS} be the Berger type deformed Sasaki on TM. Then*

$$\begin{aligned} g(Z,Fu) &= \frac{1}{\lambda} g_{BS}(Z^V,(Fu)^V), \\ g(Z,FX) &= g_{BS}(Z^V,(FX)^V) - \frac{\delta^2}{\lambda} g(FX,Fu) g_{BS}(Z^V,(Fu)^V) \\ &= g_{BS}\Big((FX)^V - \frac{\delta^2}{\lambda} g(FX,Fu)(Fu)^V, Z^V\Big), \end{aligned}$$

where $\lambda = 1+\delta^2|Fu|^2$, X,Z are vector fields and $u \in TM$.

Proof. Using Definition 1, it follows that

$$\begin{aligned} g_{BS}(Z^V,(Fu)^V) &= g(Z,Fu) + \delta^2 g(Z,Fu) g(Fu,Fu) \\ &= g(Z,Fu) + \delta^2 |Fu|^2 g(Z,Fu) \\ &= (1+\delta^2|Fu|^2) g(Z,Fu) \\ &= \lambda\, g(Z,Fu), \end{aligned}$$

$$\begin{aligned} g_{BS}(Z^V,(FX)^V) &= g(Z,FX) + \delta^2 g(Z,Fu) g(FX,Fu) \\ &= g(Z,FX) + \frac{\delta^2}{\lambda} g(FX,Fu) g_{BS}(Z^V,(Fu)^V). \end{aligned}$$

Hence

$$g(Z,FX) = g_{BS}(Z^V,(FX)^V) - \frac{\delta^2}{\lambda} g(FX,Fu) g_{BS}(Z^V,(Fu)^V).$$

□

Using Definition 1, Proposition 1, Lemma 2 and Koszul formula, we obtain the following theorem.

Theorem 1. *Let (M,F,g) be an $F-$Kählerian manifold and g_{BS} be the Berger type deformed Sasaki metric on TM. If $\overline{\nabla}$ and ∇ denote the Levi-Civita connection of (TM,g_{BS}) and (M,g) respectively, then*

$$\begin{aligned} &(1) \quad \overline{\nabla}_{X^H} Y^H = \big(\nabla_X Y\big)^H - \frac{1}{2}(R(X,Y)u)^V, \\ &(2) \quad \overline{\nabla}_{X^H} Y^V = (\nabla_X Y)^V + \frac{1}{2}\big[R(u,Y)X + \delta^2 g(Y,F(u)) R(u,F(u))X\big]^H, \\ &(3) \quad \overline{\nabla}_{X^V} Y^H = \frac{1}{2}\big[R(u,X)Y + \delta^2 g(X,F(u)) R(u,F(u))Y\big]^H, \\ &(4) \quad \overline{\nabla}_{X^V} Y^V = \delta^2\big[g(Y,F(u))(FX)^V + g(X,F(u))(FY)^V\big], \\ &\qquad - \frac{\delta^4}{\lambda}\big[g\big(FX,Fu\big)g\big(Y,Fu\big) + g\big(FY,Fu\big)g\big(X,Fu\big)\big](Fu)^V, \end{aligned}$$

where $\lambda = 1+\delta^2|Fu|^2$.

Lemma 3. *Let (M,F,g) be an $F-$Kählerian manifold and g_{BS} be the Berger type deformed Sasaki metric on TM. If $\overline{\nabla}$ and ∇ denote the Levi-Civita connection of (TM,g_{BS}) and (M,g) respectively, then*

$$\begin{aligned}
\widetilde{\nabla}_{^HX}{}^V(Fu) &= \frac{\lambda}{2}{}^H(R(u,Fu)X),\\
\widetilde{\nabla}_{^HX}{}^V(FY) &= {}^V(\nabla_X FY)+\frac{1}{2}{}^H\big(R(u,FY)X+\delta^2 g(FY,Fu)R(u,Fu)X\big),\\
\widetilde{\nabla}_{^VX}{}^V(Fu) &= \lambda^V(FX)+\delta^2 g(X,Fu)^V(F^2u)-\frac{\delta^2(\lambda-1)}{\lambda}g(FX,Fu)^V(Fu),\\
\widetilde{\nabla}_{^VX}{}^V(FY) &= \delta^2\big(g(FY,Fu)^V(FX)+g(X,Fu)^V F^2Y\big)\\
&\quad -\frac{\delta^4}{\lambda}\big(g(FX,Fu)g(FY,Fu)-g(X,Fu)g(Y,Fu)\big)^V(Fu)
\end{aligned}$$

for all vector fields X,Y on M.

Definition 2. *Let (M,F,g) be an $F-$Kählerian manifold and g_{BS} be the Berger type deformed Sasaki metric on TM, $K:TM\to TM$ be a smooth bundle endomorphism of TM and $\bar{K}:TM\times TM\to TM$ be a differential map preserving the fibers and bilinear on each them. Then the vertical and horizontal vector fields ${}^VK$, ${}^HK$, ${}^V\bar{K}$ and ${}^H\bar{K}$ respectively are defined on TM by respectively are defined on TM by*

$$\begin{array}{cccc}
{}^VK: & TM & \to & TTM\\
& (x,u) & \mapsto & {}^V(K_xu)
\end{array}
\qquad
\begin{array}{cccc}
{}^HK: & TM & \to & TTM\\
& (x,u) & \mapsto & {}^H(K_xu),
\end{array}$$

$$\begin{array}{cccc}
{}^V\bar{K}: & TM & \to & TTM\\
& (x,u) & \mapsto & {}^V(\bar{K}_x(u,Fu))
\end{array}
\qquad
\begin{array}{cccc}
{}^H\bar{K}: & TM & \to & TTM\\
& (x,u) & \mapsto & {}^H(\bar{K}_x(u,Fu))
\end{array}$$

Locally, we have

$$\begin{aligned}
{}^V(K(u)) &= u^j{}^V(K\partial j), &(10)\\
{}^H(K(u)) &= u^j{}^H(K\partial j). &(11)\\
{}^V(\bar{K}(u)) &= u^iu^sF_s^j{}^V(\bar{K}(\partial i,\partial j))=u^iu^s{}^V(\bar{K}(\partial i,F(\partial s))), &(12)\\
{}^H(\bar{K}(u)) &= u^iu^sF_s^j{}^H(\bar{K}(\partial i,\partial j)))=u^iu^s{}^H(\bar{K}(\partial i,F(\partial s))). &(13)
\end{aligned}$$

Proposition 2. *Let (M,F,g) be an $F-$Kählerian manifold and g_{BS} be the Berger* *type deformed Sasaki metric, then we have the following formulas:*

$$
\begin{aligned}
1.\ \widetilde{\nabla}_{^HX}{}^HK_{(x,u)} &= {}^H(\nabla_X K)(u) - \frac{1}{2}{}^V(R_x(X,Ku)u),\\
2.\ \widetilde{\nabla}_{^HX}{}^VK_{(x,u)} &= {}^V(\nabla_X K)(u) + \frac{1}{2}{}^H\big(R_x(u,Ku)X + \delta^2 g(Fu,Ku)R_x(u,Fu)X\big),\\
3.\ \widetilde{\nabla}_{^VX}{}^HK_{(x,u)} &= {}^H(KX) + \frac{1}{2}{}^H\big(R_x(u,X)Ku + \delta^2 g(X,Fu)R_x(u,Fu)Ku\big),\\
4.\ \widetilde{\nabla}_{^VX}{}^VK_{(x,u)} &= {}^V(KX) + \delta^2\big(g(X,Fu){}^V(FKu) + g(Fu,Ku){}^V(FX)\big)\\
&\quad - \frac{\delta^4}{\lambda}\big(g(FX,Fu)g(Fu,Ku) + g(X,Fu)g(Fu,FKu)\big){}^V(Fu).
\end{aligned}
$$

$$
\begin{aligned}
5.\ (\widetilde{\nabla}_{^HX}{}^H\bar{K})_{(x,u)} &= {}^H\big((\nabla_X\bar{K})(u,Fu)\big) - \frac{1}{2}{}^V(R_x(X,\bar{K}(u,Fu)u),\\
6.\ (\widetilde{\nabla}_{^HX}{}^V\bar{K})_{(x,u)} &= {}^V\big((\nabla_X\bar{K})(u,Fu)\big) + \frac{1}{2}{}^H\big(R_x(u,\bar{K}(u,Fu))X\\
&\quad + \delta^2 g(\bar{K}(u,Fu),Fu)R_x(u,Fu)X\big),\\
7.\ (\widetilde{\nabla}_{^VX}{}^H\bar{K})_{(x,u)} &= \frac{1}{2}{}^H\big(R_x(u,X)\bar{K}(u,Fu) + \delta^2 g(X,Fu)R_x(u,Fu)\bar{K}(u,Fu)\big)\\
&\quad + {}^H(\bar{K}(X,Fu)) + {}^H(\bar{K}(u,FX)),\\
8.\ (\widetilde{\nabla}_{^VX}{}^V\bar{K})_{(x,u)} &= +\delta^2\big[g(X,Fu){}^V(F(\bar{K}(u,Fu)) + g(\bar{K}(u,Fu),Fu){}^V(FX)\big]\\
&\quad - \frac{\delta^4}{\lambda}\big[g(X,Fu)g(F\bar{K}(u,Fu),Fu)\\
&\quad + g(FX,Fu)g(\bar{K}(u,Fu),Fu)\big]{}^V(Fu)\\
&\quad + {}^V(\bar{K}(X,Fu)) + {}^V(\bar{K}(u,FX))
\end{aligned}
$$

for any vector field X on M, where ∇ is the Levi-Civita connection, R is its curvature tensor of (M,g,F).

4 The Riemannian curvatures of Berger type deformed Sasaki metric

We shall calculate the Riemannian curvature tensor $\widetilde{R}$ of TM with the Berger type deformed Sasaki metric g_{BS}. The Riemannian curvature tensor is characterized by the formula

$$\widetilde{R}(\widetilde{X},\widetilde{Y})\widetilde{Z} = \widetilde{\nabla}_{\widetilde{X}}\widetilde{\nabla}_{\widetilde{Y}}\widetilde{Z} - \widetilde{\nabla}_{\widetilde{Y}}\widetilde{\nabla}_{\widetilde{X}}\widetilde{Z} - \widetilde{\nabla}_{[\widetilde{X},\widetilde{Y}]}\widetilde{Z}$$

for all vector fields $\widetilde{X},\widetilde{Y},\widetilde{Z}$ on TM.

Theorem 2. *Let (M,F,g) be an $F-$Kählerian manifold and (TM,g_{BS}) its tangent bundle equipped with the Berger type deformed Sasaki metric. Then, we have* *the following formulas:*

$$\begin{aligned}
\widetilde{R}({}^{H}X,{}^{H}Y){}^{H}Z &= \frac{\delta^2}{2}g(R(X,Y)u,Fu)^{H}(R(u,Fu)Z)+\frac{1}{2}{}^{H}(R(u,R(X,Y)u)Z)\\
&+\frac{\delta^2}{4}g(R(X,Z)u,Fu)^{H}(R(u,Fu)Y)+\frac{1}{4}{}^{H}(R(u,R(X,Z)u)Y)\\
&-\frac{\delta^2}{4}g(R(Y,Z)u,Fu)^{H}(R(u,Fu)X)-\frac{1}{4}{}^{H}(R(u,R(Y,Z)u)X)\\
&+{}^{H}(R(X,Y)Z)+\frac{1}{2}{}^{V}((\nabla_Z R)(X,Y)u),
\end{aligned}\tag{14}$$

$$\begin{aligned}
\widetilde{R}({}^{H}X,{}^{V}Y){}^{V}Z &= +\frac{\delta^2}{4}g(Y,Fu)^{H}\big(2R(u,FZ)X-R(u,Fu)R(u,Z)X\big)\\
&+\frac{\delta^2}{4}g(Z,Fu)^{H}\big(2R(FY,u)X-R(u,Y)R(u,Fu)X\big)\\
&-\frac{\delta^4}{4}g(Y,Fu)g(Z,Fu)^{H}(R(u,Fu)R(u,Fu)X)\\
&-\frac{1}{2}{}^{H}(R(Y,Z)X)-\frac{1}{4}{}^{H}(R(u,Y)R(u,Z)X)\\
&-\frac{\delta^2}{2}g(Z,FY)^{H}(R(u,Fu)X),
\end{aligned}\tag{15}$$

$$\begin{aligned}
\widetilde{R}({}^{V}X,{}^{V}Y){}^{V}Z &= \delta^4 g(Z,Fu)\big(g(X,Fu)^{V}F^2Y-g(Y,Fu)^{V}F^2X\big)\\
&+\delta^2\big(g(Y,FZ)^{V}(FX)-g(X,FZ)^{V}(FY)\big)-2\delta^2 g(X,FY)^{V}(FZ)\\
&+\frac{\delta^6}{\lambda}g(Z,Fu)\big(g(FX,Fu)g(Y,Fu)-g(X,Fu)g(FY,Fu)\big)^{V}F^2u\\
&+\Big(\frac{\delta^6}{\lambda^2}g(FZ,Fu)\big(g(FX,Fu)g(Y,Fu)-g(X,Fu)g(FY,Fu)\big)\\
&+\frac{\delta^4}{\lambda}\big(g(X,Fu)g(FY,FZ)-g(Y,Fu)g(FX,FZ)\big)\\
&+\frac{\delta^4}{\lambda}\big(g(X,FZ)g(FY,Fu)-g(Y,FZ)g(FX,Fu)\big)\\
&+\frac{2\delta^4}{\lambda}g(FZ,Fu)g(X,FY)\Big)^{V}(Fu)
\end{aligned}\tag{16}$$

for all vector fields X, Y, Z on M_{2k}.

Proof. In the proof, we used the Theorem 1 and Lemma 3, Proposition 2. □

Now, we consider the sectional curvature $\widetilde{K}$ on (TM, g_{BS}) for P is given by

$$\widetilde{K}(\widetilde{X},\widetilde{Y})=\frac{g_{BS}(\widetilde{R}(\widetilde{X},\widetilde{Y})\widetilde{Y},\widetilde{Y})}{g_{BS}(\widetilde{X},\widetilde{X})g_{BS}(\widetilde{Y},\widetilde{Y})-g_{BS}(\widetilde{X},\widetilde{Y})^2},\tag{17}$$

where $P=P(\widetilde{X},\widetilde{Y})$ denotes the plane spanned by $\{\widetilde{X},\widetilde{Y}\}$, for all linearly independent vector fields $\widetilde{X},\widetilde{Y}$ on TM.

Let $\widetilde{K}({}^HX,{}^HY)$, $\widetilde{K}({}^HX,{}^VY)$ and $\widetilde{K}({}^VX,{}^VY)$ denote the sectional curvature of the plane spanned by $\{{}^HX,{}^HY\}$, $\{{}^HX,{}^VY\}$ and $\{{}^VX,{}^VY\}$ on (TM,g_{BS}) respectively, where X,Y are orthonormal vector fields on M.

Proposition 3. *Let (M,F,g) be an $F-$Kählerian manifold and (TM,g_{BS}) its tangent bundle equipped with the Berger type deformed Sasaki metric. Then we have the followings:*

$$\begin{aligned} i)\ g_{BS}(\widetilde{R}({}^HX,{}^HY){}^HY,{}^HX) &= g(R(X,Y)Y,X)-\frac{3}{4}\|R(X,Y)u\|^2\\ &\quad -\frac{3\delta^2}{4}g(R(X,Y)u,Fu)^2,\end{aligned}$$

$$\begin{aligned} ii)\ g_{BS}(\widetilde{R}({}^HX,{}^VY){}^VY,{}^HX) &= \frac{1}{4}\|R(u,Y)X\|^2+\frac{\delta^4}{4}g(Y,Fu)^2\|R(u,Fu)X\|^2\\ &\quad +\frac{\delta^2}{2}g(Y,Fu)g(R(u,Fu)X,R(u,Y)X),\end{aligned}$$

$$\begin{aligned} iii)\ g_{BS}(\widetilde{R}({}^VX,{}^VY){}^VY,{}^VX) &= -\frac{\delta^6}{\lambda}\big(g(FX,Fu)g(Y,Fu)-g(X,Fu)g(FY,Fu)\big)^2\\ &\quad +\delta^4\big(g(X,Fu)^2\|FY\|^2+g(Y,Fu)^2\|FX\|^2\big)\\ &\quad -2\delta^4 g(X,Fu)g(Y,Fu)g(FX,FY)\\ &\quad -3\delta^2 g(X,FY)^2.\end{aligned}$$

From the Proposition 3 and the formula (17), we obtain the following result.

Theorem 3. *Let (M,F,g) be an $F-$Kählerian manifold and (TM,g_{BS}) its tangent bundle equipped with the Berger type deformed Sasaki metric. Then the sectional curvature $\widetilde{K}$ satisfy the following equations:*

$$\begin{aligned} (1)\ \widetilde{K}_p({}^HX,{}^HY) &= K_x(X,Y)-\frac{3}{4}\|R_x(X,Y)u\|^2-\frac{3\delta^2}{4}g_x(R(X,Y)u,Fu)^2,\\ (2)\ \widetilde{K}_p({}^HX,{}^VY) &= \frac{1}{1+\delta^2 g_x(Y,Fu)^2}\Big(\frac{\delta^4}{4}g_x(Y,Fu)^2\|R_x(u,Fu)X\|^2\\ &\quad +\frac{\delta^2}{2}g_x(Y,Fu)g_x(R(u,Fu)X,R(u,Y)X)\\ &\quad +\frac{1}{4}\|R_x(u,Y)X\|^2\Big),\end{aligned}$$

$$\begin{aligned} (3)\ \widetilde{K}_p({}^VX,{}^VY) &= \frac{1}{1+\delta^2\big(g_x(X,Fu)^2+g_x(Y,Fu)^2\big)}\big(-3\delta^2 g_x(X,FY)^2\\ &\quad +\delta^4\big(g_x(X,Fu)^2\|Y\|^2+g_x(Y,Fu)^2\|X\|^2\big)\\ &\quad -\frac{\delta^6}{\lambda}\big(g_x(FX,Fu)g_x(Y,Fu)-g_x(X,Fu)g_x(FY,Fu)\big)^2\\ &\quad -2\delta^4 g(X,Fu)g(Y,Fu)g(FX,FY)\big).\end{aligned}$$

where $p=(x,u)\in TM$ and K denotes the sectional curvature of (M,F,g).

Remark 1. *Let $p = (x, u) \in TM$ such as $u \in T_xM \backslash \{0\}$ and $\{e_i\}_{i=\overline{1,m}}$ be an orthonormal basis of the vector space T_xM, such that $e_1 = \dfrac{Fu}{\|Fu\|}$ if $Fu \neq 0$ (resp. $e_1 = \dfrac{u}{\|u\|}$ if $Fu = 0$), then*

$$\left\{E_i = {}^{H}e_i,\ E_{m+1} = \frac{1}{\sqrt{\lambda}}{}^{V}(e_1),\ E_{m+j} = {}^{V}(e_j)\right\}_{i=\overline{1,m},j=\overline{2,m}} \tag{18}$$

resp.

$$\left\{E_i = {}^{H}e_i,\ E_{m+j} = {}^{V}(e_j)\right\}_{i,j=\overline{1,m}} \tag{19}$$

is an orthonormal basis of $T_p(TM)$.

Lemma 4. *Let (M, F, g) be an $F-$Kählerian manifold and (TM, g_{BS}) its tangent bundle equipped with the Berger type deformed Sasaki metric, $p = (x, u) \in TM$ and $(E_a)_{a=\overline{1,2m}}$ be anorthonormal basis of $T_p(TM)$ defined by (18), then the sectional curvatures $\widetilde{K}$ satisfy the following equations:*

$$\begin{aligned}
\widetilde{K}_p(E_i, E_j) &= K_x(e_i, e_j) - \frac{3}{4}\|R_x(e_i, e_j)u\|^2 - \frac{3\delta^2}{4}g_x(R(e_i, e_j)u, Fu)^2,\\
\widetilde{K}_p(E_i, E_{m+1}) &= \frac{\lambda\delta^2}{4(\lambda-1)}|R_x(u, Fu)e_i\|^2,\\
\widetilde{K}_p(E_i, E_{m+l}) &= \frac{1}{4}\|R_x(u, e_l)e_i\|^2,\\
\widetilde{K}(E_{m+t}, E_{m+1}) &= \frac{\delta^2(\lambda-1)}{\lambda}\|Fe_t\|_2 - \frac{\delta^4(\lambda^2+\lambda+1)}{\lambda^2(\lambda-1)}\big(g_x(Fe_t, Fu)\big)^2,\\
\widetilde{K}_p(E_{m+t}, E_{m+l}) &= -3\delta^2 g_x(e_t, Fe_l)^2
\end{aligned}$$

for $i, j = \overline{1, m}$ and $t, l = \overline{2, m}$, where K is a sectional curvature of (M, F, g).

Proof. The results comes directly from Theorem 3 and Remark 1. □

We now consider the scalar curvature $\widetilde{\sigma}$ of (TM, g_{BS}), with standard calculations we have the following result.

Theorem 4. *Let (M, F, g) be an $F-$Kählerian manifold and (TM, g_{BS}) its tangent bundle equipped with the Berger type deformed Sasaki metric. If σ (resp., $\widetilde{\sigma}$) denote the scalar curvature of (M, F, g) (resp., (TM, g_{BS})), then we have*

$$\begin{aligned}
\widetilde{\sigma}_p &= \sigma_x - \frac{1}{4}\sum_{i,j=1}^{m}\|R(e_i, e_j)u\|^2 - \frac{\delta^2}{4}\sum_{i=1}^{m}\|R(u, Ju)e_i\|^2\\
&\quad -\frac{\delta^2(\lambda+2)}{\lambda}\sum_{i=2}^{m}\|Fe_i\|^2 + \frac{\delta^2}{\lambda^2}\big(\lambda^2 - 2\lambda - 2\big),
\end{aligned}$$

where $p = (x, u) \in TM$ and $(e_i)_{i=\overline{1,m}}$ is an orthonormal basis of T_pM defined by (18).

Proof. From the definition of scalar curvature, we have

$$\begin{aligned}\widetilde{\sigma}_p &= \sum_{\substack{i,j=1\\ i\neq j}}^{m}\widetilde{K}(E_i,E_j)+2\sum_{i,j=1}^{m}\widetilde{K}(E_i,E_{m+j})+\sum_{\substack{i,j=1\\ i\neq j}}^{m}\widetilde{K}(E_{m+i},E_{m+j})\\ &= \sum_{\substack{i,j=1\\ i\neq j}}^{m}\widetilde{K}(E_i,E_j)+2\sum_{i=1}^{m}\widetilde{K}(E_i,E_{m+1})+2\sum_{i=1,j=2}^{m}\widetilde{K}(E_i,E_{m+j})\\ &\quad +2\sum_{i=2}^{m}\widetilde{K}(E_{m+i},E_{m+1})+\sum_{\substack{i,j=2\\ i\neq j}}^{m}\widetilde{K}(E_{m+i},E_{m+j}).\end{aligned}$$

Using Lemma 4, we have

$$\begin{aligned}\widetilde{\sigma}_p &= \sum_{\substack{i,j=1\\ i\neq j}}^{m}\Big(K(e_i,e_j)-\frac{3}{4}\|R(e_i,e_j)u\|^2-\frac{3\delta^2}{4}g(R(e_i,e_j)u,Fu)^2\Big)\\ &\quad+\sum_{i=1}^{m}\big(\frac{\delta^2\lambda}{2(\lambda-1)}\|R(u,Fu)e_i\|^2\big)\\ &\quad+\frac{1}{2}\sum_{i=1,j=2}^{m}\|R(u,e_j)e_i\|^2-3\delta^2\sum_{\substack{i,j=2\\ i\neq j}}^{m}g(e_i,Fe_j)^2\\ &\quad+2\sum_{i=2}^{m}\big(\frac{\delta^2(\lambda-1)}{\lambda}\|Fe_i\|^2-\frac{\delta^4(\lambda^2+\lambda+1)}{\lambda^2(\lambda-1)}\big(g(Fe_i,Fu)\big)^2\Big).\end{aligned}$$

In order to simplify this last expression, we use

$$\begin{aligned}\sum_{\substack{i,j=2\\ i\neq j}}^{M}g(e_i,Fe_j)^2 &= \sum_{\substack{i=1,j=2\\ i\neq j}}^{m}g(e_i,Fe_j)^2-\sum_{i=1}^{m}g(e_1,Fe_i)^2=\sum_{i=2}^{m}\|Fe_i\|^2-1,\\ \sum_{i=2}^{m}g(Fe_i,Fu)^2 &= \sum_{i=1}^{n}g(Fe_i,Fu)^2=\|Fu\|^2,\\ \sum_{i,j=1}^{m}\|R(u,e_j)e_i\|^2 &= \sum_{i,j=1}^{m}\|R(e_i,e_j)u\|^2.\end{aligned}$$

From the last equation (see, also [14, 24]), we get

$$\begin{aligned}\widetilde{\sigma}_p &= \sigma_x-\frac{1}{4}\sum_{i,j=1}^{m}\|R(e_i,e_j)u\|^2-\frac{\delta^2}{4}\sum_{i=1}^{m}\|R(u,Fu)e_i\|^2\\ &\quad-\frac{\delta^2(\lambda+2)}{\lambda}\sum_{i=2}^{m}\|Fe_i\|^2+\big(3\delta^2-\frac{2\delta^2(\lambda^2+\lambda+1)}{\lambda^2}\big).\end{aligned}$$

$\square$

From Theorem 4, we deduce the following corollary

Corollary 1. *Let (M, F, g) be a locally flat $F-$Kählerian manifold and (TM, g_{BS}) its tangent bundle equipped with the Berger type deformed Sasaki metric. If $\widetilde{\sigma}$ denote the scalar curvature of (TM, g_{BS}), then we have*

$$\widetilde{\sigma}_p = -\frac{\delta^2(\lambda+2)}{\lambda}\sum_{i=2}^{m}\|Fe_i\|^2 + \frac{\delta^2}{\lambda^2}\big(\lambda^2 - 2\lambda - 2\big),$$

where $p = (x, u) \in TM$ and $\lambda = 1 + \delta^2\|Fu\|^2$.

Theorem 5. *Let (M, F, g) be an $F-$Kählerian manifold of constant sectional curvature k and (TM, g_{BS}) its tangent bundle equipped with the Berger type deformed Sasaki metric. The scalar curvature $\widetilde{\sigma}$ of (TM, g_{BS}) is given by*

$$\begin{aligned}\widetilde{\sigma}_p &= m(m-1)k - \frac{k^2}{2}(m+\lambda-2)\|u\|^2 \\ &\quad - \frac{\delta^2(\lambda+2)}{\lambda}\sum_{i=2}^{m}\|Fe_i\|^2 + \frac{\delta^2}{\lambda^2}(\lambda^2 - 2\lambda - 2), \qquad (20)\end{aligned}$$

where $p = (x, u) \in TM$ and $\lambda = 1 + \delta^2\|Fu\|^2$.

Proof. Using the property of constant sectional curvature, for $X, Y, Z \in \Gamma(TM)$, we have

$$R(X, Y)Z = k\big(g(Y, Z)X - g(X, Z)Y\big)$$

then

$$\begin{aligned}\sum_{\substack{i,j=1\\ i\neq j}}^{m}\|R(e_i, e_j)u\|^2 &= k^2\sum_{\substack{i,j=1\\ i\neq j}}^{m}\|g(e_j, u)e_i - g(e_i, u)e_j\|^2 \\ &= k^2\sum_{\substack{i,j=1\\ i\neq j}}^{m} g(e_j, u)^2 + g(e_i, u)^2 \\ &= 2k^2(m-1)\|u\|^2 \qquad (21)\end{aligned}$$

and

$$\begin{aligned}\sum_{i=1}^{m}\|R(u, Fu)e_i\|^2 &= k^2\sum_{i=1}^{m}\|g(e_i, Fu)u - g(e_i, u)Fu\|^2 \\ &= k^2\sum_{i=1}^{m}\big(g(e_i, Fu)^2\|u\|^2 + g(e_i, u)^2\|Fu\|^2\big) \\ &= 2k^2\|u\|^2\|Fu\|^2 \qquad (22)\end{aligned}$$

□

Substituting formulas (21) and (22) in Theorem 4 , we obtain formula (20).

Theorem 6. *Let (M, F, g) be an $F-$Kählerian manifold of constant sectional curvature k and (TM, g_{BS}) its tangent bundle equipped with the Berger type deformed Sasaki metric. If F is invertible then the dimension of M is even ($m = 2q$) and the scalar curvature $\widetilde{\sigma}$ of (TM, g_{BS}) is given by*

$$\begin{aligned} \widetilde{\sigma}_p &= m(m-1)k - \frac{k^2}{2}(m+\lambda-2)\|u\|^2 \\ &\quad -\frac{\delta^2}{\lambda^2}\big[(m-2)(\lambda^2+2\lambda)+4\lambda+2\big], \end{aligned} \tag{23}$$

where $p = (x, u) \in TM$ and $\lambda = 1 + \delta^2\|u\|^2$.

Proof. Using formula (1), if F is invertible then $F^2 = -I$. So F is an almost complex structure and from formula (8), we obtain

$$|Fu| = |u|, \quad \forall u \in TM.$$

Applying the Theorem 5 the Theorem 6 follows. □

Remark 2. *(M, F, g) be an $F-$Kählerian manifold of constant sectional curvature k. If F is invertible then we have*

1. *If $k \leq 0$ then $\widetilde{\sigma}_p < 0$ for all $p \in TM$.*
2. *If $k \neq 0$ then $\widetilde{\sigma}_p < 0$ for all $p \in TM$ if and only if $m < 1 + \frac{3\delta^2}{k}$.*

Example 4.1. *Let $M = \mathbb{R}^3$, $F = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $g = dx^2 + dy^2 + dz^2$.*

$(\mathbb{R}^3, F, g)$ is an $F-$Kählerian manifold such as we have

$$\begin{aligned} g(X,Y) &= X_1Y_1 + X_2Y_2 + X_3Y_3, \\ g(FX,FY) &= X_1Y_1 + X_2Y_2, \\ g(FX,Y) &= -X_2Y_1 + X_1Y_2 = -g(X,FY). \end{aligned}$$

for all vector fields $X = X_1\partial_1 + X_2\partial_2 + X_3\partial_3$ and $Y = Y_1\partial_1 + Y_2\partial_2 + Y_3\partial_3$.

$(\mathbb{R}^3, F, g)$ is a flat $F-$Kählerian manifold, then we have

$$\begin{aligned} \widetilde{\sigma}_p &= -\frac{\delta^2(\lambda+2)}{\lambda}\big(\|Fe_2\|^2 + \|Fe_3\|^2\big) + \big(\frac{\delta^2(\lambda^2-2\lambda-2)}{\lambda^2}\big) \\ &\quad -\frac{\delta^2(\lambda+2)}{\lambda} + \frac{\delta^2(\lambda^2-2\lambda-2)}{\lambda^2} \\ &\quad -\frac{2\delta^2(2\lambda+1)}{\lambda^2} < 0 \end{aligned}$$

We conclude that tangent bundle $(T\mathbb{R}^3, g_{BS})$ equipped with the Berger type deformed Sasaki metric has negative scalar curvature.

Example 4.2. *Let* $M = \mathbb{R}^3$, $F = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$ *and* $g = dx^2 + dy^2 + dz^2$.

$(\mathbb{R}^3, F, g)$ *is a flat* $F-$*Kählerian manifold such as we have*

$$\widetilde{\sigma}_p = -\frac{\delta^2}{\lambda^2}(\lambda^2 + 6\lambda + 2) < 0$$

5 Harmonic vector field.

In the section, we will study some harmonicity problems on the tangent bundle equipped with the Berger type deformed Sasaki metric. Given a smooth map $\phi : (M^m, g) \to (N^n, h)$ between two Riemannian manifolds, then the second fundamental form of ϕ is defined by

$$(\nabla d\phi)(X, Y) = \nabla^{\phi}_{X} d\phi(Y) - d\phi(\nabla_X Y).$$

Here ∇ is the Riemannian connection on M and ∇^{ϕ} is the pull-back connection on the pull-back bundle $\phi^{-1}TN$, and

$$\tau(\phi) = trace_g \nabla d\phi$$

is the tension field of ϕ.

The energy functional of ϕ is defined by

$$E(\phi) = \int_K e(\phi) dv_g$$

such that K is any compact of M, where

$$e(\phi) = \frac{1}{2} trace_g h(d\phi, d\phi)$$

is the energy density of ϕ.

A map is called harmonic if it is a critical point of the energy functional E. For any smooth variation $\{\phi_t\}_{t\in I}$ of ϕ with $\phi_0 = \phi$ and $V = \frac{d}{dt}\phi_t\Big|_{t=0}$, we have

$$\frac{d}{dt}E(\phi_t)\Big|_{t=0} = -\int_K h(\tau(\phi), V) dv_g.$$

Then ϕ is harmonic if and only if $\tau(\phi) = 0$. We refer to [8, 9, 10, 15] for background on harmonic maps.

Lemma 5. *[6] A Killing vector field ζ on a Riemannian manifold (M, g) satisfies the following equations*

$$\begin{aligned} \nabla^2_{X\,Y}\zeta &= \nabla_X\nabla_Y\zeta - \nabla_{\nabla_X Y}\zeta = -R(\zeta, X)Y, && (24)\\ g(\nabla_X\zeta, Y) &= -g(\nabla_Y\zeta, X), && (25)\\ g(\nabla_X\zeta, X) &= 0, && (26) \end{aligned}$$

for all $X, Y \in \mathcal{H}(M)$.

If Q denote the Ricci operator, then we have

$$g(Q(X), Y) = Ric(X, Y), \tag{27}$$

for all $X, Y \in \mathcal{H}(M)$, where Ric is the Ricci tensor (see [6]).

Lemma 6. *If ζ is a Killing vector field on a Riemannian manifold (M, g), then we have*

$$Tr_g\nabla^2\zeta \;=\; -Q(\zeta) \tag{28}$$

Proof. From Lemma 5 (24), we obtain

$$\begin{aligned} Tr_g g(\nabla^2\zeta, X) &= -Tr_g g(R(\zeta, *)*, X)\\ &= -Ric(\zeta, X)\\ &= -g(Q(\zeta), X) \end{aligned}$$

for all $X \in \mathcal{H}(M)$. Which proves the relationship (28).

□

Lemma 7. *Let (M, g) be an Einstein manifold. Then we have*

$$\begin{aligned} Ric(X, Y) &= \mu\, g(X, Y),\\ Scal &= m\,\mu && (29)\\ Q(X) &= \mu X && (30) \end{aligned}$$

for all $X, Y \in \mathcal{H}(M)$, where μ is a some constant, Q is the Ricci operator and $Scal$ is the scalar curvature.

Proof. We have

1) $Scal = \sum_{i=1}^{m} Ric(E_i, E_i) = \sum_{i=1}^{m} \mu\, g(E_i, E_i) = m\,\mu$

2) $g(Q(X), Y) = Ric(X, Y) = \mu\, g(X, Y)$ for all $Y \in \mathcal{H}(M)$, then

$$Q(X) = \mu X$$

.

□

Let (M_{2k}, g) be a real space form $M^{2k}(c)$, that is mean, the curvature tensor is expressed as

$$R(X,Y)Z = c[g(Y,Z)X - g(X,Z)Y] \tag{31}$$

Lemma 8. *If (M,g) is a real space form, then (M_{2k}, g) is an Einstein manifold such that*

$$\begin{aligned} \mu &= (m-1)c, \\ Scal &= m(m-1)c \end{aligned}$$

Proof. Let $(E_i)_{i=1,..,m}$ be an orthonormal frame. From formula (31), we get

$$\begin{aligned} Ric(X,Y) &= \sum_{i=1}^{m} g(R(X,E_i)E_i, Y) \\ &= c\sum_{i=1}^{m}[g(g(E_i)E_i)X,Y) - g(g(X,E_i)E_i,Y)] \\ &= c[m\, g(X,Y) - g(X,Y)] \\ &= c(m-1)g(X,Y) \end{aligned}$$

Using Lemma 7, we obtain $\mu = (m-1)c$ and $Scal = m\mu = m(m-1)c$.

□

Lemma 9. *[16]Let (M,g) be a Riemannian manifold . If X, Y are vector fields and $(x,u) \in TM$ such that $X_x = u$, then we have*

$$d_x X(Y_x) = Y^H_{(x,u)} + (\nabla_Y X)^V_{(x,u)}.$$

Theorem 7. *Let (M,F,g) be an $F-$Kählerian manifold and g_{BS} be the Berger type deformed Sasaki metric on TM. If $\xi : (M,g,F) \to (TM, g_{BS})$ is a vector field on M, then the tension field of ξ is given by*

$$\begin{aligned} \tau(\xi)_x &= Tr_g\Big[R(\xi,\nabla_*\xi)* + \delta^2 g(\nabla_*\xi, F\xi)R(\xi,F\xi)*\Big]^H \\ &\quad + Tr_g\Big[\nabla^2_*\xi + 2\delta^2 g(\nabla_*\xi, F\xi)F(\nabla_*\xi) \\ &\quad - \frac{2\delta^4}{\lambda} g(F(\nabla_*\xi), F\xi)g(\nabla_*\xi, F\xi)F\xi\Big]^V. \end{aligned}$$

Proof. Let $\{E_i\}_{i=1}^{2k}$ be a local orthonormal frame on M. From Lemma 9 and

Theorem 1, at $x \in M$, we have

$$\begin{aligned}
& \overline{\nabla}_{d\xi(E_i)} d\xi(E_i) = \overline{\nabla}_{E_i^H + (\nabla_{E_i}\xi)^V} (E_i^H + (\nabla_{E_i}\xi)^V) \\
= \; & \overline{\nabla}_{E_i^H} E_i^H + \overline{\nabla}_{E_i^H} (\nabla_{E_i}\xi)^V + \overline{\nabla}_{(\nabla_{E_i}\xi)^V} E_i^H + \overline{\nabla}_{(\nabla_{E_i}\xi)^V} (\nabla_{E_i}\xi)^V \\
= \; & \big(\nabla_{E_i} E_i\big)^H + \big(\nabla^2_{E_i}\xi\big)^V \\
& + \frac{1}{2}\big[R(u, \nabla_{E_i}\xi)E_i + \delta^2 g(\nabla_{E_i}\xi, Fu) R(u, Fu)E_i\big]^H \\
& + \frac{1}{2}\big[R(u, \nabla_{E_i}\xi)E_i + \delta^2 g(\nabla_{E_i}\xi, Fu) R(u, Fu)E_i\big]^H \\
& + 2\delta^2 \big[g(\nabla_{E_i}\xi, Fu) F(\nabla_{E_i}\xi)\big]^V \\
& - \frac{2\delta^4}{\lambda}\big[g(F(\nabla_{E_i}\xi), Fu) g(\nabla_{E_i}\xi, Fu) Fu\big]^V \\
= \; & \big(\nabla_{E_i} E_i\big)^H + \big(\nabla^2_{E_i}\xi\big)^V + \big[R(u, \nabla_{E_i}\xi)E_i \\
& + \delta^2 g(\nabla_{E_i}\xi, Fu) R(u, Fu)E_i\big]^H + 2\delta^2 \big[g(\nabla_{E_i}\xi, Fu) F(\nabla_{E_i}\xi)\big]^V \\
& - \frac{2\delta^4}{\lambda}\big[g(F(\nabla_{E_i}\xi), Fu) g(\nabla_{E_i}\xi, Fu) Fu\big]^V.
\end{aligned}$$

where $u = \xi_x$. So

$$\begin{aligned}
& \overline{\nabla}_{d\xi(E_i)} d\xi(E_i) - d\xi(\nabla_{E_i} E_i) \\
= \; & \overline{\nabla}_{d\xi(E_i)} d\xi(E_i) - (\nabla_{E_i} E_i)^H - (\nabla_{\nabla_{E_i} E_i}\xi)^V \\
= \; & -(\nabla_{\nabla_{E_i} E_i}\xi)^V + \big(\nabla^2_{E_i}\xi\big)^V + \big[R(u, \nabla_{E_i}\xi)E_i \\
& + \delta^2 g(\nabla_{E_i}\xi, Fu) R(u, Fu)E_i\big]^H \\
& + 2\delta^2 \big[g(\nabla_{E_i}\xi, Fu) F(\nabla_{E_i}\xi)\big]^V \\
& - \frac{2\delta^4}{\lambda}\big[g(F(\nabla_{E_i}\xi), Fu) g(\nabla_{E_i}\xi, Fu) Fu\big]^V.
\end{aligned}$$

□

The Theorem 7 gives the following theorem.

Theorem 8. *Let (M, F, g) be an $F-$Kählerian manifold and g_{BS} be the Berger type deformed Sasaki metric on TM. The map $\xi : (M, F, g) \to (TM, g_{BS})$ is harmonic if and only if*

$$\begin{cases} 1) \quad Tr_g\Big[R(\xi, \nabla_*\xi) * + \delta^2 g(\nabla_*\xi, F\xi) R(\xi, F\xi) *\Big] = 0, \\ 2) \quad Tr_g\Big[\frac{2\delta^4}{\lambda} g(F(\nabla_*\xi), F\xi) g(\nabla_*\xi, F\xi) F\xi - \nabla^2_*\xi - 2\delta^2 g(\nabla_*\xi, F\xi) F(\nabla_*\xi)\Big] = 0. \end{cases}$$

Proposition 4. *Let (M, F, g) be an $F-$Kählerian manifold and g_{BS} be the Berger type deformed Sasaki metric on TM. $\xi : (M, F, g) \to (TM, g_{BS})$ is an isometric immersion if and only if $\nabla\xi = 0$.*

Proof. Let X, Y be vector fields. From Lemma 9 we have

$$\begin{aligned}
& g_{BS}(d\xi(X), d\xi(Y)) \\
= \; & g_{BS}(X^H + (\nabla_X \xi)^V, Y^H + (\nabla_Y \xi)^V) \\
= \; & g_{BS}(X^H, Y^H) + g_{BS}((\nabla_X \xi)^V, (\nabla_Y \xi)^V) \\
= \; & g(X,Y) + g(\nabla_X \xi, \nabla_Y \xi) + \delta^2 g(\nabla_X \xi, Fu) g(\nabla_Y \xi, Fu),
\end{aligned}$$

from which it follows that

$$g_{BS}(d\xi(X), d\xi(Y)) = g(X,Y).$$

Therefore, ξ is an isometric immersion if and only if

$$g(\nabla_X \xi, \nabla_Y \xi) + \delta^2 g(\nabla_X \xi, Fu) g(\nabla_Y \xi, Fu) = 0,$$

which is equivalent to $\nabla \xi = 0$. □

As a direct consequence of Theorem 8 and Proposition 4, we obtain the following theorem.

Theorem 9. *Let (M, F, g) be an $F-$Kählerian manifold and g_{BS} be the Berger type deformed Sasaki metric on TM. If $\xi : (M, F, g) \to (TM, g_{BS})$ is isometric immersion, then ξ is totally geodesic. Furthermore, ξ is harmonic.*

Theorem 10. *Let (M, F, g) be an $F-$Kählerian manifold and g_{BS} be the Berger type deformed Sasaki metric on TM. If $\xi : (M, F, g) \to (TM, g_{BS})$ is a Killing vector field, then ξ is harmonic if and only if*

$$\begin{cases} 1) \quad Tr_g R(\xi, \nabla_* \xi)* = \delta^2 R(\xi, F\xi) \nabla_{F\xi} \xi, \\ 2) \quad -Tr_g R(\xi, *)* = 2\delta^2 \nabla_{\nabla_{F\xi}\xi} F\xi. \end{cases}$$

Proof. Using formula (25), we obtain

$$\begin{aligned}
Tr_g g(F(\nabla_* \xi), F\xi) g(\nabla_* \xi, F\xi) &= Tr_g g(\nabla_{F\xi} F(\xi), *) g(\nabla_{F\xi} \xi, *) \\
&= Tr_g g(\nabla_{F\xi} F(\xi), g(\nabla_{F\xi} \xi, *)*) \\
&= g(\nabla_{F\xi} F(\xi), \nabla_{F\xi} \xi) \\
&= g(F(\nabla_{F\xi} \xi), \nabla_{F\xi} \xi) \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
Tr_g g(\nabla_* \xi, F\xi) F(\nabla_* \xi) &= -Tr_g g(\nabla_{F\xi} \xi, *) F(\nabla_* \xi) \\
&= -Tr_g F(\nabla_{g(\nabla_{F\xi}\xi, *)*} \xi) \\
&= -F(\nabla_{\nabla_{F\xi}\xi} \xi)
\end{aligned}$$

$$\begin{aligned}
Tr_g g(\nabla_* \xi, F\xi) R(\xi, F\xi)* &= -Tr_g g(\nabla_{F\xi} \xi, *) R(\xi, F\xi) * \\
&= -Tr_g R(\xi, F\xi) g(\nabla_{F\xi} \xi, *) * \\
&= -R(\xi, F\xi) \nabla_{F\xi} \xi.
\end{aligned}$$

By substituting the results below in formulas (1) and (2) of Theorem 8, Theorem 10 follows.

$\square$

From Theorem 10 and Lemma 8, we get the following theorem

Theorem 11. *Let (M, F, g) be an $F-$Kählerian manifold and g_{BS} be the Berger type deformed Sasaki metric on TM. If (M, g) is a space form (M, c) and $\xi : (M, F, g) \to (TM, g_{BS})$ is a harmonic Killing vector field,, then ξ is an eigenvector of $A_\xi(X) = \nabla_{\nabla_{F\xi}\xi} F\,X$ with respect to eigenvalue $q = \frac{(1-m)c}{2\delta^2}$.*

Acknowledgement

The authors would like to thank and express their special gratitude to Prof. Mustapha Djaa for his suggestion of this work as well as for helping him with some accounts for his helpful suggestions and valuable comments which helped to improve the paper.

References

[1] Abbassi, M.T.K. and Sarih, M., *On natural metrics on tangent bundles of Riemannian manifolds*, Arch. Math. **41** (2005), 71-92.

[2] Abbassi, M.T.K., *Note on the classification theorems of g-natural metrics on the tangent bundle of a Riemannian manifold $(M; g)$*, Comment. Math. Univ. Carolinae **45** (2004), no. 4, 591-596.

[3] Altunbas, M., Simsek, R. and Gezer, A., *A study concerning Berger type deformed Sasaki metric on the tangent bundle.* Zh. Mat. Fiz. Anal. Geom. **15** (2019), no. 4, 435-447.

[4] Balashchenko, V. V. and Samsonov, A. A., *Nearly Kähler and Hermitian $f-$Structures on Homogeneous $k-$Symmetric Spaces.* Dokl. Math. **81** (2010), no. 3, 386–389.

[5] Balashchenko, V. V., *Invariant $f-$structures in generalized Hermitian geometry, Contemporary geometry and related topics.* 5–27, Univ. Belgrade Fac. Math., Belgrade, 2006.

[6] Besse, A. I., *Einstein manifold*, Springer-Verlag Berlin, Heidelberg, 1987.

[7] Bures, J. and Vanzura, J., *Metric polynomial structures.* Kodai Math. Sem. Rep. **27** (1976), no. 3, 345–352.

[8] Eells, J. and Lemaire, L., *Selected topics in harmonic maps*, Conf. Board of the Math. Sci. A.M.S. **50** (1983) 85-95, 1983.

[9] Eells, J. and Lemaire, L., *Another report on harmonic maps,* Bull. London Math. Soc. **20** (1988), 385-524.

[10] Eells, J. and Sampson, J. H., *Harmonic mappings of Riemannian manifolds.* Amer. J. Math. **86** (1964) 109–160.

[11] Djaa, N.E.H., Gezer, A. and Djaa, M., *Magnetic fields on the tangent bundle over Kählerian manifolds.* Rep. Math. Phys. **91**, (2023), no. 2, 143-164.

[12] Djaa, N.E.H. and Zagane, A., *Mus-Sasaki metric and complex structures*, HSJG. **2** (2020), 1-11.

[13] Dombrowski, P., *On the geometry of the tangent bundle.* J. Reine und Angew. Math. **210** (1962), 73-88.

[14] Gudmundsson, S. and Kappos, E., *On the geometry of the tangent bundle with the Cheeger-Gromoll metric*, Tokyo J. Math. **25** (2002), no. 1, 75–83.

[15] Ishihara, T., *Harmonic sections of tangent bundles.* J. Math. Tokushima Univ. **13** (1979), 23-27.

[16] Konderak, J. J., *On harmonic vector fields.* Publ. Mat. **36** (1992), no. 1, 217–228.

[17] Opozda, B., *The $f-$sectional curvature of $f-$Kaehlerian manifolds.* Ann. Polon. Math. **43** (1983), no. 2, 141–150.

[18] Pusic, N., *On quarter-symmetric metric connections on a hyperbolic Kaehlerian space.* Publ. Inst. Math. (Beograd) (N.S.) **73** (87) (2003), 73–80.

[19] Vanzura, J., *Integrability conditions for polynomial structures.* Kodai Math. Sem. Rep. **27** 1976, no. 1-2, 42-50.

[20] Yampolsky, A., *On geodesics of tangent bundle with fiberwise deformed Sasaki metric over Kählerian manifold.* Zh. Mat. Fiz. Anal. Geom. **8** (2012), no. 2, 177-189.

[21] Yano, K. and Ishihara, S., *Tangent and cotangent bundles.* Marcel Dekker. INC. New York 1973.

[22] Yano, K. and Kon, M., *CR Submanifolds of Kahlerian and Sasakian manifolds*, Birkhauser, Boston. MA 1983. Nauka. Moscow. 1990.

[23] Zagane, A. and Djaa, M., *On geodesics of warped Sasaki metric.* Math. Sci. Appl. E-Notes **5** (2017), no. 1, 85–92.

[24] Zagane, A. and Djaa, M., *Geometry of Mus-Sasaki metric*, Commun. Math. **26** (2018), 113–126.