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ALMOST η-RICCI SOLITONS ON TWO CLASSES OF ALMOST KENMOTSU MANIFOLDS

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Abstract

The object of the present paper is to characterize two classes of almost Kenmotsu manifolds admitting almost η-Ricci solitons. In this context we have shown that in a (k,μ) and $(k,\mu)'$-almost Kenmotsu manifold admitting an almost η-Ricci soliton the curvature conditions (i) the manifold is Einstein, (ii) the manifold is Ricci symmetric ($\nabla S = 0$), (iii) the manifold is Ricci semisymmetric ($R \cdot S = 0$) and (iv) the manifold is projective Ricci semisymmetric ($P \cdot S = 0$) are equivalent. Also we have shown that the curvature condition $Q \cdot P = 0$ in a (k,μ)-almost Kenmotsu manifold admitting an almost η-Ricci soliton holds if and only if the manifold is locally isometric to the hyperbolic space $\mathbb{H}^{2n+1}(-1)$ and if a $(k,\mu)'$-almost Kenmotsu manifold admitting an almost η-Ricci soliton satisfies the curvature condition $Q \cdot R = 0$, then it is locally isometric to the Riemannian product $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$.

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1 Introduction

In the present time, the study of Ricci solitons and it's generalizations are very interesting topic in differential geometry and mathematical physics. The notion of Ricci solitons was introduced by Hamilton [12] as a natural generalization of Einstein metrics, being generalized fixed points of Hamilton's Ricci flow

$$\frac{\partial}{\partial t} g = -2S.$$

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The Ricci flow is one type of nonlinear diffusion equation analogous to the heat equation for metrics. Under the Ricci flow, a metric can be improved to evolve into a more canonical one by smoothing out its irregularities, depending on the Ricci curvature of the manifold.
A Ricci soliton (g, V, λ) is defined on a Riemannian manifold (M, g) by

$$\pounds_V g + 2S + 2\lambda g = 0, \tag{1}$$

where $\pounds_V g$ denotes the Lie derivative of the Riemannian metric g in the direction of the vector field V, S is the Ricci tensor associated to g and λ is a constant. The Ricci soliton is said to be shrinking, steady or expanding according as λ is negative, zero or positive, respectively.
Recently, Pigola et al. [16] introduced a modified class of Ricci solitons by considering the soliton constant λ as a smooth function and then (g, V, λ) is called an almost Ricci soliton. In 2016, Wang [21] has studied gradient Ricci almost solitons on two classes of almost Kenmotsu manifolds. Also Wang and Liu [20] has studied Ricci solitons on three-dimensional η-Einstein almost Kenmotsu manifolds. In 2016, Mandal and De [14] also studied Ricci solitons on almost Kenmotsu manifolds.

As a generalizations of the Ricci solitons, the notion of η-Ricci solitions was introduced by Cho and Kimura [8]. An η-Ricci soliton is a tuple (g, V, λ, α) satisfying the equation

$$\pounds_V g + 2S + 2\lambda g + 2\alpha\eta \otimes \eta = 0, \tag{2}$$

where λ and α are constants. In particular, if $\alpha = 0$ then the notion of η-Ricci soliton reduces to the Ricci soliton. Different aspects of η-Ricci solitons have been studied by many authors (see [4], [5], [7]).
The η-Ricci soliton can be naturally generalized to almost η-Ricci soliton by considering the soliton constants λ and α as smooth functions. Almost η-Ricci solitons in $(LCS)_n$-manifolds have been studied by Blaga [6].
The projective curvature tensor P on a Riemannian manifold M^{2n+1} is defined by

$$P(X, Y)Z = R(X, Y)Z - \frac{1}{2n}[S(Y, Z)X - S(X, Z)Y], \tag{3}$$

for any vector fields $X,\ Y,\ Z$ on M^{2n+1}, where R is the Riemannian curvature tensor of type (1,3) and S is the Ricci tensor of type (0,2).
A Riemannian manifold M^{2n+1} is said to be Ricci semisymmetric if $R \cdot S = 0$ and projective Ricci semisymmetric if $P \cdot S = 0$.
An example of a curvature condition of semisymmetry type is $Q \cdot R = 0$, where Q is the Ricci operator of type (1,1) and is defined by $S(X, Y) = g(QX, Y)$.
A natural extension of such curvature conditions form curvature conditions of pseudosymmetry type. The curvature condition $Q \cdot R = 0$ have been studied by Verstraelen et al. in [17].

Motivated by the above studies, in the present paper, we characterize two classes of almost Kenmotsu manifolds with certain curvature conditions and proved some equivalent conditions under η-Ricci soliton.

2 Almost Kenmotsu manifolds with nullity distributions

A differentiable $(2n+1)$-dimensional manifold M^{2n+1} is said to have a (ϕ, ξ, η)-structure or an almost contact structure, if it admits a $(1,1)$ tensor field ϕ, a characteristic vector field ξ and a 1-form η satisfying ([2], [3]),

$$\phi^2 = -I + \eta \otimes \xi, \ \ \eta(\xi) = 1, \tag{4}$$

where I denote the identity endomorphism. Here also $\phi\xi = 0$ and $\eta \circ \phi = 0$; both can be derived from (4) easily.
If a manifold M with a (ϕ, ξ, η)-structure admits a Riemannian metric g such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y),$$

for any vector fields X, Y on M^{2n+1}, then M is said to be an almost contact metric manifold. The fundamental 2-form Φ on an almost contact metric manifold is defined by $\Phi(X, Y) = g(X, \phi Y)$ for any X, Y on M^{2n+1}. The condition for an almost contact metric manifold being normal is equivalent to vanishing of the $(1,2)$-type torsion tensor N_ϕ, defined by $N_\phi = [\phi, \phi] + 2d\eta \otimes \xi$, where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ [2]. Recently in ([9],[10],[11],[15]), almost contact metric manifold such that η is closed and $d\Phi = 2\eta \wedge \Phi$ are studied and they are called almost Kenmotsu manifolds. Obviously, a normal almost Kenmotsu manifold is a Kenmotsu manifold. Also Kenmotsu manifolds can be characterized by $(\nabla_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X$, for any vector fields X,Y on M^{2n+1}. It is well known [13] that a Kenmotsu manifold M^{2n+1} is locally a warped product $I \times_f N^{2n}$ where N^{2n} is a Kähler manifold, I is an open interval with coordinate t and the warping function f, defined by $f = ce^t$ for some positive constant c. Let us denote the distribution orthogonal to ξ by $\mathcal{D}$ and defined by $\mathcal{D} = Ker(\eta) = Im(\phi)$. In an almost Kenmotsu manifold, since η is closed, $\mathcal{D}$ is an integrable distribution.
Let M^{2n+1} be an almost Kenmotsu manifold. We denote by $h = \frac{1}{2}\pounds_\xi \phi$ and $l = R(\cdot, \xi)\xi$ on M^{2n+1}. The tensor fields l and h are symmetric operators and satisfy the following relations [15]:

$$h\xi = 0, \ l\xi = 0, \ tr(h) = 0, \ tr(h\phi) = 0, \ h\phi + \phi h = 0, \tag{5}$$

$$\nabla_X \xi = X - \eta(X)\xi - \phi h X (\Rightarrow \nabla_\xi \xi = 0), \tag{6}$$

$$\phi l \phi - l = 2(h^2 - \phi^2), \tag{7}$$

$$R(X, Y)\xi = \eta(X)(Y - \phi h Y) - \eta(Y)(X - \phi h X) + (\nabla_Y \phi h)X - (\nabla_X \phi h)Y, \tag{8}$$

for any vector fields X, Y on M^{2n+1}. The $(1,1)$-type symmetric tensor field $h' = h \circ \phi$ is anti-commuting with ϕ and $h'\xi = 0$. Also it is clear that ([9], [19])

$$h = 0 \Leftrightarrow h' = 0, \quad h'^2 = (k+1)\phi^2 (\Leftrightarrow h^2 = (k+1)\phi^2). \tag{9}$$

The notion of the (k,μ)-nullity distribution on a contact metric manifold (M^{2n+1}, ϕ, ξ, η, g) was introduced by Blair, Koufogiorgos and Papantoniou [1], which is defined for any $p \in M^{2n+1}$ and $k, \mu \in \mathbb{R}$ as follows:

$$\begin{aligned} N_p(k,\mu) = \{Z \in T_p(M) : R(X,Y)Z &= k[g(Y,Z)X - g(X,Z)Y] \\ &+ \mu[g(Y,Z)hX - g(X,Z)hY]\}, \end{aligned} \tag{10}$$

for any $X, Y \in T_p(M^{2n+1})$, where $T_p(M^{2n+1})$ denotes the tangent space of M^{2n+1} at any point $p \in M^{2n+1}$, R denotes the Riemannian curvature tensor of type $(1,3)$, $h = \frac{1}{2}\pounds_\xi \phi$ and $\pounds$ denotes the Lie differentiation.
In [9], Dileo and Pastore introduced the notion of $(k,\mu)'$-nullity distribution, on an almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$, which is defined for any $p \in M^{2n+1}$ and $k, \mu \in \mathbb{R}$ as follows:

$$\begin{aligned} N_p(k,\mu)' = \{Z \in T_p(M^{2n+1}) : R(X,Y)Z &= k[g(Y,Z)X - g(X,Z)Y] \\ &+ \mu[g(Y,Z)h'X - g(X,Z)h'Y]\}, \end{aligned} \tag{11}$$

where $h' = h \circ \phi$.

3 (k,μ)-almost Kenmotsu manifolds

In this section we study almost η-Ricci solitons in almost Kenmotsu manifolds with ξ belonging to the (k,μ)-nullity distribution.
From (10) we obtain

$$R(X,Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)hX - \eta(X)hY], \tag{12}$$

where $k, \mu \in \mathbb{R}$. Before proving our main results in this section we first state the following:

Lemma 1. *[9] Let M^{2n+1} be an almost Kenmotsu manifold of dimension $(2n+1)$. Suppose that the characteristic vector field ξ belonging to the (k,μ)-nullity distribution. Then $k = -1$, $h = 0$ and M^{2n+1} is locally a wrapped product of an open interval and an almost Kähler manifold.*

In view of Lemma 1 it follows from (12) that the following relations

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{13}$$

$$R(\xi,X)Y = -g(X,Y)\xi + \eta(Y)X, \tag{14}$$

$$S(X,\xi) = -2n\eta(X) \tag{15}$$

and

$$Q\xi = -2n\xi \tag{16}$$

hold for any vector fields X, Y on M^{2n+1}. Again in [9], Dileo and Pastore prove that in an almost Kenmotsu manifold with ξ belonging to the (k,μ)-nullity distribution the sectional curvature $K(X,\xi) = -1$. From this we get in an almost Kenmotsu manifold with ξ belonging to the (k,μ)-nullity distribution the scalar curvature $r = -2n(2n+1)$.

Suppose an almost Kenmotsu manifold M^{2n+1} with ξ belonging to the (k,μ)-nullity distribution admits an almost η-Ricci soliton defined by (2), where λ and α are smooth functions on M^{2n+1}. Then from (2) we have

$$(\pounds_\xi g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) + 2\alpha\eta(X)\eta(Y) = 0. \tag{17}$$

Now

$$(\pounds_\xi g)(X,Y) = g(\nabla_X\xi, Y) + g(X, \nabla_Y\xi). \tag{18}$$

Using (6) and Lemma 1 it follows from the previous equation that

$$(\pounds_\xi g)(X,Y) = 2g(X,Y) - 2\eta(X)\eta(Y). \tag{19}$$

Using (19) in (17) we obtain

$$S(X,Y) = -(\lambda+1)g(X,Y) - (\alpha-1)\eta(X)\eta(Y). \tag{20}$$

Now setting $X = Y = \xi$ in the foregoing equation and using (15) we have

$$\lambda + \alpha = 2n. \tag{21}$$

Hence (20) reduces to

$$S(X,Y) = -(2n+1-\alpha)g(X,Y) - (\alpha-1)\eta(X)\eta(Y). \tag{22}$$

Hence, we conclude that M^{2n+1} is an η-Einstein manifold. Thus we state:

Theorem 1. *A (k,μ)-almost Kenmotsu manifold admitting an almost η-Ricci soliton is an η-Einstein manifold.*

Proposition 1. *A (k,μ)-almost Kenmotsu manifold admitting an almost η-Ricci soliton is Ricci semisymmetric if and only if it is an Einstein manifold. Also the almost η-Ricci soliton is expanding.*

Proof: Let us first consider that the manifold M^{2n+1} is Ricci semisymmetric. Then we have

$$(R(X,Y)\cdot S)(U,V) = 0,$$

which implies that

$$S(R(X,Y)U,V) + S(U,R(X,Y)V) = 0. \tag{23}$$

Using (22) in the foregoing equation yields

$$\begin{aligned}&(\alpha-1)[\{g(X,U)\eta(Y) - g(Y,U)\eta(X)\}\eta(V)\\&+\eta(U)\{g(X,V)\eta(Y) - g(Y,V)\eta(X)\}] = 0.\end{aligned} \tag{24}$$

Setting $U = Y = \xi$ in (24) we get

$$(\alpha-1)[g(X,V) - \eta(X)\eta(V)] = 0$$

and this implies

$$(\alpha-1)g(\phi X,\phi V) = 0,$$

which implies that $\alpha = 1$. Hence from (21), $\lambda = 2n-1 > 0$ and therefore the almost η-Ricci soliton is expanding.
Consequently, we have from (22)

$$S(X,Y) = -2ng(X,Y). \tag{25}$$

This implies that the manifold is an Einstein manifold.
The converse is obviously true. This completes the proof.

Proposition 2. *A (k,μ)-almost Kenmotsu manifold admitting an almost η-Ricci soliton is projective Ricci semisymmetric if and only if it is an Einstein manifold. Here the almost η-Ricci soliton is expanding.*

Proof: Let the manifold M^{2n+1} is projective Ricci semisymmetric. Then we have

$$(P(X,Y)\cdot S)(W,Z) = 0,$$

which implies

$$S(P(X,Y)W,Z) + S(W,P(X,Y)Z) = 0. \tag{26}$$

Making use of (22) in (26) we get

$$\begin{aligned}&(2n+1-\alpha)g(P(X,Y)W,Z) + (\alpha-1)\eta(P(X,Y)W)\eta(Z)\\&+(2n+1-\alpha)g(P(X,Y)Z,W) + (\alpha-1)\eta(P(X,Y)Z)\eta(W) = 0.\end{aligned} \tag{27}$$

Putting $X = \xi$ in the foregoing equation we obtain

$$\begin{aligned}&(2n+1-\alpha)g(P(\xi,Y)W,Z) + (\alpha-1)\eta(P(\xi,Y)W)\eta(Z)\\&+(2n+1-\alpha)g(P(\xi,Y)Z,W) + (\alpha-1)\eta(P(\xi,Y)Z)\eta(W) = 0.\end{aligned} \tag{28}$$

We now compute $P(\xi, Y)Z$ separately by using (14) and (22), we have

$$P(\xi, Y)Z = -\frac{\alpha - 1}{2n} g(Y, Z)\xi + \frac{\alpha - 1}{2n} \eta(Y)\eta(Z)\xi. \tag{29}$$

Using (29) in (28) we get after simplifying

$$(\alpha - 1)[g(Y, W)\eta(Z) + g(Y, Z)\eta(W) - 2\eta(Y)\eta(Z)\eta(W)] = 0. \tag{30}$$

Setting $W = \xi$ in (30) we obtain

$$(\alpha - 1)g(\phi Y, \phi Z) = 0,$$

which implies $\alpha = 1$. This implies $\lambda = 2n-1 > 0$ and therefore the almost η-Ricci soliton is expanding.
Since $\alpha = 1$, we have from (22)

$$S(X, Y) = -2ng(X, Y), \tag{31}$$

which implies that the manifold is Einstein.
Converse part is obvious. This completes the proof.

Theorem 2. *In a (k, μ)-almost Kenmotsu manifold admitting an almost η-Ricci soliton the followings are equivalent:*

(a) The manifold M^{2n+1} is Einstein.

(b) The manifold M^{2n+1} is Ricci symmetric ($\nabla S = 0$).

(c) The manifold M^{2n+1} is Ricci semisymmetric ($R \cdot S = 0$).

(d) The manifold M^{2n+1} is projective Ricci semisymmetric ($P \cdot S = 0$).

Proof: (a) $\Rightarrow$ (b) is obvious. Since, $\nabla S = 0 \Rightarrow R \cdot S = 0$ we have (b) $\Rightarrow$ (c). Proposition 1 and Proposition 2 shows that (c) $\Leftrightarrow$ (a) and (d) $\Leftrightarrow$ (a), respectively. Therefore, we get (c) $\Leftrightarrow$ (d). This completes the proof.

Theorem 3. *A (k, μ)-almost Kenmotsu manifold admitting an almost η-Ricci soliton satisfies the curvature condition $Q \cdot P = 0$ if and only if the manifold is locally isometric to the hyperbolic space $\mathbb{H}^{2n+1}(-1)$. In such a case, the almost η-Ricci soliton is expanding.*

Proof: Let us first suppose that the curvature condition $Q \cdot P = 0$ holds on M^{2n+1}. Then we have

$$Q(P(X, Y)Z) - P(QX, Y)Z - P(X, QY)Z - P(X, Y)QZ = 0, \tag{32}$$

for all vector fields $X,\ Y,\ Z$ on M^{2n+1}.
From (22) we get

$$QX = -(2n + 1 - \alpha)X - (\alpha - 1)\eta(X)\xi. \tag{33}$$

With the help of (33) we can write from (32)

$$\begin{aligned}&-(2n+1-\alpha)P(X,Y)Z-(\alpha-1)\eta(P(X,Y)Z)\xi\\&-P(-(2n+1-\alpha)X-(\alpha-1)\eta(X)\xi,Y)Z\\&-P(X,-(2n+1-\alpha)Y-(\alpha-1)\eta(Y)\xi)Z\\&-P(X,Y)(-(2n+1-\alpha)Z-(\alpha-1)\eta(Z)\xi)=0.\end{aligned}\tag{34}$$

Simplifying the above equation and using (3) we have

$$\begin{aligned}&-2(2n+1-\alpha)P(X,Y)Z+(\alpha-1)[\eta(R(X,Y)Z-\frac{1}{2n}[S(Y,Z)X\\&-S(X,Z)Y])\xi-\eta(X)P(\xi,Y)Z-\eta(Y)P(X,\xi)Z-\eta(Z)P(X,Y)\xi]=0.\end{aligned}\tag{35}$$

With the help of (13), (14) and (15), we obtain the expressions of the terms $P(\xi,Y)Z$, $P(X,\xi)Z$ and $P(X,Y)\xi$ separately as given below

$$P(\xi,Y)Z=-g(Y,Z)\xi-\frac{1}{2n}S(Y,Z)\xi,\tag{36}$$

$$P(X,\xi)Z=g(X,Z)\xi+\frac{1}{2n}S(X,Z)\xi,\tag{37}$$

and

$$P(X,Y)\xi=0.\tag{38}$$

Substituting (36)-(38) in (35) we obtain

$$\begin{aligned}&-2(2n+1-\alpha)P(X,Y)Z+(\alpha-1)[\eta(R(X,Y)Z)\xi+g(Y,Z)\eta(X)\xi\\&-g(X,Z)\eta(Y)\xi]=0.\end{aligned}\tag{39}$$

Now from (13) we have

$$\eta(R(X,Y)Z)=g(X,Z)\eta(Y)-g(Y,Z)\eta(X).\tag{40}$$

Using (40) in (39) we get

$$(2n+1-\alpha)P(X,Y)Z=0,\tag{41}$$

which implies that either $\alpha=2n+1$ or $P(X,Y)Z=0$.
Case 1: If $\alpha=2n+1$, then from (22) we have

$$S(X,Y)=-2n\eta(X)\eta(Y),$$

which implies $r=-2n$. This is a contradiction to the fact that in an almost Kenmotsu manifold with ξ belongs to the (k,μ)-nullity distributions, the scalar

curvature $r = -2n(2n+1)$.
Case 2: If $P(X,Y)Z = 0$, then from (3) we have

$$R(X,Y)Z = \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y]. \tag{42}$$

Setting $X = \xi$ in (42) and using (14), (15) and (22) we obtain

$$(\alpha - 1)g(\phi X, \phi Y) = 0,$$

which implies $\alpha = 1$ and hence, from (21), $\lambda = 2n - 1 > 0$. This shows that the almost η-Ricci soliton is expanding.
Therefore, from (22) we obtain

$$S(X,Y) = -2ng(X,Y). \tag{43}$$

Using (43) in (42) we obtain

$$R(X,Y)Z = -[g(Y,Z)X - g(X,Z)Y].$$

This shows that the manifold M^{2n+1} is locally isometric to the hyperbolic space $\mathbb{H}^{2n+1}(-1)$.
Conversely, if the manifold M^{2n+1} is locally isometric to the hyperbolic space $\mathbb{H}^{2n+1}(-1)$, then M^{2n+1} is projectively flat (see [22]) and hence, the curvature condition $Q \cdot P = 0$ holds on M^{2n+1}. This completes the proof of our theorem.

4 $(k,\mu)'$-almost Kenmotsu manifolds

In this section we study almost η-Ricci solitons in $(k,\mu)'$-almost Kenmotsu manifolds. Let $X \in \mathcal{D}$ be the eigen vector of h' corresponding to the eigen value δ. Then from (8) it is clear that $\delta^2 = -(k+1)$, a constant. Therefore $k \leq -1$ and $\delta = \pm\sqrt{-k-1}$. We denote by $[\delta]'$ and $[-\delta]'$ the corresponding eigen spaces related to the non-zero eigen value δ and $-\delta$ of h', respectively. Throughout this section we consider $h' \neq 0$. Before presenting our main theorems we recall some results:

Lemma 2. *(Prop. 4.1 and Prop. 4.3 of [9]) Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold such that ξ belongs to the $(k,\mu)'$-nullity distribution and $h' \neq 0$. Then $k < -1$, $\mu = -2$ and Spec$(h') = \{0, \delta, -\delta\}$, with 0 as simple eigen value and $\delta = \sqrt{-k-1}$. The distributions $[\xi] \oplus [\delta]'$ and $[\xi] \oplus [-\delta]'$ are integrable with totally geodesic leaves. The distributions $[\delta]'$ and $[-\delta]'$ are integrable with totally umbilical leaves. Furthermore, the sectional curvature are given by the following:*

(a) $K(X,\xi) = k - 2\delta$ *if* $X \in [\delta]'$ *and*
$K(X,\xi) = k + 2\delta$ *if* $X \in [-\delta]'$,

(b) $K(X,Y) = k - 2\delta$ *if* $X, Y \in [\delta]'$;
$K(X,Y) = k + 2\delta$ *if* $X, Y \in [-\delta]'$ *and*
$K(X,Y) = -(k+2)$ *if* $X \in [\delta]'$, $Y \in [-\delta]'$,

(c) M^{2n+1} has constant negative scalar curvature $r = 2n(k - 2n)$.

Lemma 3. *(Lemma 3 of [18]) Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold with ξ belonging to the $(k, \mu)'$-nullity distribution. If $h' \neq 0$, then the Ricci operator Q of M^{2n+1} is given by*

$$Q = -2nid + 2n(k+1)\eta \otimes \xi - 2nh'. \tag{44}$$

Moreover, the scalar curvature of M^{2n+1} is $2n(k - 2n)$.

Lemma 4. *(Prop. 3.1 of [14]) If in an η-Einstein almost Kenmotsu manifold with the characteristic vector field ξ belonging to the $(k, \mu)'$-nullity distributions and $h' \neq 0$, any one of the associated scalars a or b is constant, then the manifold is an Einstein manifold.*

Lemma 5. *(Prop. 4.2 of [9]) Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be an almost Kenmotsu manifold such that $h' \neq 0$ and ξ belonging to the $(k, -2)'$-nullity distribution. Then for any $X_\delta, Y_\delta, Z_\delta \in [\delta]'$ and $X_{-\delta}, Y_{-\delta}, Z_{-\delta} \in [-\delta]'$, the Riemann curvature tensor satisfies:*

$$R(X_\delta, Y_\delta)Z_{-\delta} = 0,$$

$$R(X_{-\delta}, Y_{-\delta})Z_\delta = 0,$$

$$R(X_\delta, Y_{-\delta})Z_\delta = (k+2)g(X_\delta, Z_\delta)Y_{-\delta},$$

$$R(X_\delta, Y_{-\delta})Z_{-\delta} = -(k+2)g(Y_{-\delta}, Z_{-\delta})X_\delta,$$

$$R(X_\delta, Y_\delta)Z_\delta = (k - 2\delta)[g(Y_\delta, Z_\delta)X_\delta - g(X_\delta, Z_\delta)Y_\delta],$$

$$R(X_{-\delta}, Y_{-\delta})Z_{-\delta} = (k + 2\delta)[g(Y_{-\delta}, Z_{-\delta})X_{-\delta} - g(X_{-\delta}, Z_{-\delta})Y_{-\delta}].$$

From (11), we have

$$R(X, Y)\xi = k[\eta(Y)X - \eta(X)Y] + \mu[\eta(Y)h'X - \eta(X)h'Y], \tag{45}$$

where $k, \mu \in \mathbb{R}$. Also we get from (45)

$$R(\xi, X)Y = k[g(X, Y)\xi - \eta(Y)X] + \mu[g(h'X, Y)\xi - \eta(Y)h'X]. \tag{46}$$

Contracting X in (45), we have

$$S(Y, \xi) = 2nk\eta(Y). \tag{47}$$

Suppose a $(k, \mu)'$-almost Kenmotsu manifold M^{2n+1} admits an almost η-Ricci soliton given by (2), where λ and α are smooth functions on M^{2n+1}. Then from (2) we have

$$(\pounds_\xi g)(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) + 2\alpha\eta(X)\eta(Y) = 0. \tag{48}$$

Now using (6) and (18) we can write

$$(\pounds_\xi g)(X,Y) = 2g(X,Y) - 2\eta(X)\eta(Y) - 2g(\phi hX, Y). \tag{49}$$

Again from (44) we have

$$g(\phi hX, Y) = \frac{1}{2n}S(X,Y) + g(X,Y) - (k+1)\eta(X)\eta(Y). \tag{50}$$

Using (49) and (50) in (48) we obtain

$$S(X,Y) = -\frac{2n\lambda}{2n-1}g(X,Y) - \frac{2n(\alpha+k)}{2n-1}\eta(X)\eta(Y). \tag{51}$$

Setting $X = Y = \xi$ in the foregoing equation and using (47) we get

$$\lambda + \alpha = -2nk. \tag{52}$$

With the help of (52) we can write (51) as

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{53}$$

where $a = \frac{2n(\alpha+2nk)}{2n-1}$ and $b = -\frac{2n(\alpha+k)}{2n-1}$.

Hence, we conclude that M^{2n+1} is an η-Einstein manifold. Thus we arrive to the following:

Theorem 4. *A $(k, \mu)'$-almost Kenmotsu manifold admitting an almost η-Ricci soliton is an η-Einstein manifold.*

Proposition 3. *A $(k, \mu)'$-almost Kenmotsu manifold admitting an almost η-Ricci soliton is Ricci semisymmetric if and only if it is an Einstein manifold. Also the almost η-Ricci soliton is expanding, provided $-2 < k < -1$.*

Proof: Let us first suppose that the manifold M^{2n+1} is Ricci semisymmetric. Then we have

$$(R(X,Y) \cdot S)(U,V) = 0,$$

which implies that

$$S(R(X,Y)U,V) + S(U, R(X,Y)V) = 0. \tag{54}$$

With the help of (53) we have from (54)

$$b[\eta(R(X,Y)U)\eta(V) + \eta(R(X,Y)V)\eta(U)] = 0.$$

Putting $V = \xi$ in the previous equation we get

$$b\eta(R(X,Y)U) = 0.$$

Setting $X = \xi$ in the foregoing equation and using (46) we obtain

$$bk[g(Y,U) - \eta(Y)\eta(U)] + b\mu g(h'Y,U) = 0. \tag{55}$$

Using (50) and (53) in (55) we get

$$b[(k - \mu - \frac{a\mu}{2n})g(Y,U) - (k - \mu(k+1) + \frac{b\mu}{2n})\eta(Y)\eta(U)] = 0. \tag{56}$$

Now using $\mu = -2$, the values of a and b as given in (53), we can show by an easy computation that the value of both $(k - \mu - \frac{a\mu}{2n})$ and $(k - \mu(k+1) + \frac{b\mu}{2n})$ is numerically equal to $\frac{6nk-k+4n-2+2\alpha}{2n-1}$. Hence, equation (56) can be written as

$$b(\frac{6nk - k + 4n - 2 + 2\alpha}{2n-1})g(\phi X, \phi Y) = 0, \tag{57}$$

which implies that either $b = 0$ or $\alpha = -3nk + \frac{k}{2} - 2n + 1$.
Case 1: If $b = 0$, then $\alpha = -k$ and this implies $\lambda = -(2n-1)k > 0$ as $k < -1$ (see Lemma 2), which shows that the almost η-Ricci soliton is expanding.
Therefore, from (53) we have

$$S(X,Y) = 2nkg(X,Y),$$

which shows that the manifold is Einstein.
Case 2: If $\alpha = -3nk + \frac{k}{2} - 2n + 1$, then $\lambda = 2n + nk - \frac{k}{2} - 1$, which is greater than zero when $k > -2$. Thus the almost η-Ricci soliton will be expanding when $k > -2$.
Now we see that a and b in (53) becomes constant as α and λ are constants in this case. Hence, Lemma 4 implies that the manifold is an Einstein manifold.

Combining Case 1 and Case 2, the almost η-Ricci soliton will be expanding when $-2 < k < -1$.
The converse part is obvious. This completes the proof.

Proposition 4. *A $(k,\mu)'$-almost Kenmotsu manifold admitting an almost η-Ricci soliton is projective Ricci semisymmetric if and only if it is an Einstein manifold. Here the almost η-Ricci soliton is expanding, provided $-2 < k < -1$.*

Proof: Let us first suppose that the manifold M^{2n+1} is projective Ricci semisymmetric. Then we have

$$(P(X,Y) \cdot S)(U,V) = 0,$$

which implies that

$$S(P(X,Y)U,V)+S(U,P(X,Y)V)=0. \tag{58}$$

Using (53), the foregoing equation reduces to

$$\begin{aligned}&ag(P(X,Y)U,V)+b\eta(P(X,Y)U)\eta(V)\\&+ag(U,P(X,Y)V)+b\eta(U)\eta(P(X,Y)V)=0.\end{aligned} \tag{59}$$

Setting $V=\xi$ in (59) we obtain

$$\begin{aligned}&ag(P(X,Y)U,\xi)+b\eta(P(X,Y)U)\\&+ag(U,P(X,Y)\xi)+b\eta(U)\eta(P(X,Y)\xi)=0.\end{aligned} \tag{60}$$

Using (45) and (47) we have from (3)

$$P(X,Y)\xi=-2[\eta(Y)h'X-\eta(X)h'Y]. \tag{61}$$

Using (61) in (60) we get

$$(a+b)\eta(P(X,Y)U)-2a[\eta(Y)g(h'X,U)-\eta(X)g(h'Y,U)]=0. \tag{62}$$

Putting $X=\xi$ in the above equation, it reduces to

$$(a+b)\eta(P(\xi,Y)U)=0. \tag{63}$$

Now

$$a+b=\frac{2n(\alpha+2nk)}{2n-1}-\frac{2n(\alpha+k)}{2n-1}=2nk. \tag{64}$$

With the help of (46), (47) and (50), from (3) we obtain

$$\begin{aligned}P(\xi,Y)U\ =\ &kg(Y,U)\xi+2[\frac{1}{2n}S(Y,U)+g(Y,U)-(k+1)\eta(Y)\eta(U)]\xi\\&+2\eta(U)h'Y-\frac{1}{2n}S(Y,U)\xi.\end{aligned} \tag{65}$$

Substituting (64) and (65) in (63) we get

$$2nk[(k+2)g(Y,U)+\frac{1}{2n}S(Y,U)-2(k+1)\eta(Y)\eta(U)]=0. \tag{66}$$

Making Use of (53) in (66) we have

$$(\frac{a}{2n}+k+2)g(Y,U)+(\frac{b}{2n}-2k-2)\eta(Y)\eta(U)=0. \tag{67}$$

Now it is easy to calculate that

$$(\frac{a}{2n}+k+2)=\frac{\alpha+4nk-k+4n-2}{2n-1}=-(\frac{b}{2n}-2k-2)$$

Hence from (67) it follows that

$$\frac{\alpha+4nk-k+4n-2}{2n-1}g(\phi Y,\phi U)=0, \tag{68}$$

which implies $\alpha=-4nk+k-4n+2$ and this implies $\lambda=(2n-1)k+4n-2$, which is greater than zero when $k>-2$. Thus the almost η-Ricci soliton will be expanding when $-2<k<-1$.
Now we see that a and b in (53) becomes constant as α and λ are constants here. Hence, Lemma 4 implies that the manifold is an Einstein manifold.
The converse part is obvious. This completes the proof.

Theorem 5. *In a $(k,\mu)'$-almost Kenmotsu manifold admitting an almost η-Ricci soliton the followings are equivalent:*

(a) The manifold M^{2n+1} is Einstein.

(b) The manifold M^{2n+1} is Ricci symmetric ($\nabla S=0$).

(c) The manifold M^{2n+1} is Ricci semisymmetric ($R\cdot S=0$).

(d) The manifold M^{2n+1} is projective Ricci semisymmetric ($P\cdot S=0$).

Proof: (a) $\Rightarrow$ (b) is obvious. Since, $\nabla S=0\Rightarrow R\cdot S=0$ we have (b) $\Rightarrow$ (c). Proposition 3 and Proposition 4 shows that (c) $\Leftrightarrow$ (a) and (d) $\Leftrightarrow$ (a), respectively. Therefore, we get (c) $\Leftrightarrow$ (d). This completes the proof.

Theorem 6. *If a $(k,\mu)'$-almost Kenmotsu manifold admitting an almost η-Ricci soliton satisfies the curvature condition $Q\cdot R=0$ then the manifold is locally isometric to the Riemannian product of an $(n+1)$-dimensional manifold of constant sectional curvature -4 and a flat n-dimensional manifold.*

Proof: Let us first suppose that the curvature condition $Q\cdot R=0$ holds on M^{2n+1}. Then we have

$$Q(R(X,Y)Z)-R(QX,Y)Z-R(X,QY)Z-R(X,Y)QZ=0, \tag{69}$$

for all vector fields $X,\ Y,\ Z$ on M^{2n+1}.
Using (53), from equation (69) it follows that

$$\begin{aligned}&b\eta(R(X,Y)Z)\xi-2aR(X,Y)Z-b\eta(X)R(\xi,Y)Z\\&-b\eta(Y)R(X,\xi)Z-b\eta(Z)R(X,Y)\xi=0.\end{aligned} \tag{70}$$

Setting $Z=\xi$ in the foregoing equation and using (45) and (46) we obtain

$$\begin{aligned}&(2a+b)[k\{\eta(Y)X-\eta(X)Y\}-2\{\eta(Y)h'X-\eta(X)h'Y\}]\\&+b\eta(X)[k\{\eta(Y)\xi-Y\}+2h'Y]+b\eta(Y)[k\{X-\eta(X)\xi\}-2h'X]=0.\end{aligned} \tag{71}$$

Now letting $X,Y\in[\delta]'$ we have from (71)

$$2(a+b)(k-2\delta)(\eta(Y)X-\eta(X)Y)=0.$$

Since, $a+b=2nk$, the above equation gives $k=2\delta$.
Again from Lemma 2 we have $\delta^2=-k-1$. Using $k=2\delta$ in $\delta^2=-k-1$ we obtain $\delta=-1$ and hence $k=-2$. Then from Lemma 5 we have

$$R(X_\delta, Y_\delta)Z_\delta = 0,$$

and

$$R(X_{-\delta}, Y_{-\delta})Z_{-\delta} = -4[g(Y_{-\delta}, Z_{-\delta})X_{-\delta} - g(X_{-\delta}, Z_{-\delta})Y_{-\delta}],$$

for any $X_\delta, Y_\delta, Z_\delta \in [\delta]'$ and $X_{-\delta}, Y_{-\delta}, Z_{-\delta} \in [-\delta]'$. Also noticing $\mu=-2$ it follows from Lemma 4.1 that $K(X,\xi)=-4$ for any $X \in [-\delta]'$ and $K(X,\xi)=0$ for any $X \in [\delta]'$. Again from Lemma 4.1 we see that $K(X,Y)=-4$ for any $X, Y \in [-\delta]'$ and $K(X,Y)=0$ for any $X, Y \in [\delta]'$. As is shown in [9] that the distribution $[\xi] \oplus [\delta]'$ is integrable with totally geodesic leaves and the distribution $[-\delta]'$ is integrable with totally umbilical leaves by $H=-(1-\delta)\xi$, where H is the mean curvature tensor field for the leaves of $[-\delta]'$ immersed in M^{2n+1}. Here $\delta=-1$, then the two orthogonal distributions $[\xi] \oplus [\delta]'$ and $[-\delta]'$ are both integrable with totally geodesic leaves immersed in M^{2n+1}. Then we can say that M^{2n+1} is locally isometric to $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$. This completes the proof of our theorem.

5 Example of an Einstein $(k,\mu)'$-almost Kenmotsu manifold

Let us consider $M=\{(x,y,z,u,v) \in \mathbb{R}^5\}$, where (x,y,z,u,v) are the standard coordinates in $\mathbb{R}^5$. Let $\xi, e_2, e_3, e_4, e_5 \in \mathbb{R}^5$ be five vector fields , which satisfies [9]

$$[\xi, e_2] = -2e_2,\ [\xi, e_3] = -2e_3,\ [\xi, e_4] = 0,\ [\xi, e_5] = 0,$$

$[e_i, e_j]=0$, where $i, j = 2, 3, 4, 5$.
Let g be the Riemannian metric given by

$$g(\xi,\xi) = g(e_2,e_2) = g(e_3,e_3) = g(e_4,e_4) = g(e_5,e_5) = 1$$

and $g(\xi, e_i) = g(e_i, e_j) = 0$ for $i \neq j$; $i, j = 2, 3, 4, 5$.
Let η be the 1-form defined by $\eta(Z) = g(Z, \xi)$, for any vector field Z on M.
Let ϕ be the $(1,1)$-tensor field defined by

$$\phi(\xi) = 0,\ \phi(e_2) = e_4,\ \phi(e_3) = e_5,\ \phi(e_4) = -e_2,\ \phi(e_5) = -e_3.$$

Since, ϕ and g are linear, we have
$\eta(\xi) = 1,\ \phi^2(Z) = -Z + \eta(Z)\xi,\ g(\phi Z, \phi U) = g(Z,U) - \eta(Z)\eta(U),$
for any $Z, U \in T(M)$.
Moreover, $h'\xi = 0,\ h'e_2 = e_2,\ h'e_3 = e_3,\ h'e_4 = -e_4,\ h'e_5 = -e_5.$

The Riemannian connection ∇ of the metric tensor g is given by Koszul's formula which is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ &\quad -g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]). \end{aligned}$$

Using Koszul's formula we get the following:

$$\nabla_\xi \xi = 0,\ \nabla_\xi e_2 = 0,\ \nabla_\xi e_3 = 0,\ \nabla_\xi e_4 = 0,\ \nabla_\xi e_5 = \xi,$$

$$\nabla_{e_2} \xi = 2e_2,\ \nabla_{e_2} e_2 = -2\xi,\ \nabla_{e_2} e_3 = 0,\ \nabla_{e_2} e_4 = 0,\ \nabla_{e_2} e_5 = 0,$$

$$\nabla_{e_3} \xi = 2e_3,\ \nabla_{e_3} e_2 = 0,\ \nabla_{e_3} e_3 = -2\xi,\ \nabla_{e_3} e_4 = 0,\ \nabla_{e_3} e_5 = 0,$$

$$\nabla_{e_4} \xi = 0,\ \nabla_{e_4} e_2 = 0,\ \nabla_{e_4} e_3 = 0,\ \nabla_{e_4} e_4 = 0,\ \nabla_{e_4} e_5 = 0,$$

$$\nabla_{e_5} \xi = 0,\ \nabla_{e_5} e_2 = 0,\ \nabla_{e_5} e_3 = 0,\ \nabla_{e_5} e_4 = 0,\ \nabla_{e_5} e_5 = 0.$$

In view of the above relations we have

$$\nabla_X \xi = -\phi^2 X + h' X,$$

for any $X \in T(M)$. Therefore, the structure (ϕ, ξ, η, g) is an almost contact metric structure such that $d\eta = 0$ and $d\Phi = 2\eta \wedge \Phi$, so that M is an almost Kenmotsu manifold.
By the above results, we can easily obtain the components of the curvature tensor R as follows:

$$R(\xi, e_2)\xi = 4e_2,\ R(\xi, e_2)e_2 = -4\xi,\ R(\xi, e_3)\xi = 4e_3,\ R(\xi, e_3)e_3 = -4\xi,$$

$$R(\xi, e_4)\xi = R(\xi, e_4)e_4 = R(\xi, e_5)\xi = R(\xi, e_5)e_5 = 0,$$

$$R(e_2, e_3)e_2 = 4e_3,\ R(e_2, e_3)e_3 = -4e_2,\ R(e_2, e_4)e_2 = R(e_2, e_4)e_4 = 0,$$

$$R(e_2, e_5)e_2 = R(e_2, e_5)e_5 = R(e_3, e_4)e_3 = R(e_3, e_4)e_4 = 0,$$

$$R(e_3, e_5)e_3 = R(e_3, e_5)e_5 = R(e_4, e_5)e_4 = R(e_4, e_5)e_5 = 0.$$

With the help of the expressions of the curvature tensor we conclude that the characteristic vector field ξ belongs to the $(k, \mu)'$-nullity distribution, with $k = -2$ and $\mu = -2$.

Using the expressions of the curvature tensor R we have

$$R(X,Y)Z = -4[g(Y,Z)X - g(X,Z)Y].$$

From the above equation we obtain

$$S(Y,Z) = -16g(Y,Z), \text{ which implies } r = -80.$$

Hence, M is an Einstein manifold. Therefore, M is Ricci symmetric, Ricci semisymmetric and projective Ricci semisymmetric.

Now, it is easy to see that

$$(\pounds_\xi g)(\xi,\xi) = (\pounds_\xi g)(e_4,e_4) = (\pounds_\xi g)(e_5,e_5) = 0,$$

$$(\pounds_\xi g)(e_2,e_2) = (\pounds_\xi g)(e_3,e_3) = 4.$$

Tracing (48) yields

$$5\lambda + \alpha = 76.$$

Again from (52), we obtain

$$\lambda + \alpha = 8.$$

Solving the above two equation, we have $\lambda = 17$ and $\alpha = -9$.
Thus (g,ξ,λ,α) is an almost η-Ricci soliton on M, where $\lambda = 17$ and $\alpha = -9$. Thus Theorem 5 is verified.

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