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CHARACTERIZATION OF SOLUTIONS OF CERTAIN CLASS OF LINEAR AND NON-LINEAR SHIFT EQUATIONS UNDER SHARED VALUES

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Abstract

In this paper we have considered the generalized form of Pielou Logistic Equation and Riccati Difference equation [6] and characterize the solution of that equation in terms of shared value problem. We have improved and extended the result of Li-Chen in [8].

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1 Introduction and some definitions

Population dynamics is the branch of life science that studies the knowledge concerning the sizes of populations and the factors involved for maintenance, decline or expansion of the same according to the progression of time. Traditionally, to study the dynamics of the size of a population with the help of mathematical modeling, researchers used the continuous deterministic methods based on differential equations. In this respect, we can recall the well-known Verhulst-Pearl equation

$$x'(t) = x(t)[a - bx(t)] \quad (a, b > 0),$$

which is used mainly for continuous model of growth of a population. But if the data are available for discrete times only, then a difference rather shift equation instead of a differential equation will be more effective. In this regard to study the

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discrete versions of some population models, the following shift equation known as Pielou Logistic equation

$$y(z+1) = \frac{P_1(z)y(z)}{P_2(z) + P_3(z)y(z)}, \tag{1.1}$$

where $P_1(z)$, $P_2(z)$ and $P_3(z)$ are non-zero polynomials, have been immensely used by the researchers.

Now for a non-constant meromorphic function f in the open complex plane $\mathbb{C}$, we define its difference operator by $\Delta f = f(z+c) - f(c)$, where c is a non-zero constant. In this respect we also want to recall the important difference equation introduced by Ishizaki [6] which is the difference analogous form of Riccati differential equation, i.e.,

$$\Delta f(z) + \frac{f^2(z) + A(z)}{f(z) - 1} = 0,$$

where $A(z)$ is a meromorphic function, which can also be represented as

$$f(z+1) = \frac{A(z) + f(z)}{1 - f(z)}. \tag{1.2}$$

In view of (1.2) we can term the equation as Riccati shift equation rather than difference equation. Ishizaki [6] characterized the solution and have been able to find the possible form of (1.2).

Considering (1.1) and (1.2) it will be natural to find the possible structural relationship between them and to characterize the solutions. To this end, we introduce the following shift equation which includes both (1.1) and partially (1.2),

$$A(z)w(z)w(z+1) + B(z)w(z+1) + C(z)w(z) = D(z), \tag{1.3}$$

where $A(z)$, $B(z)$, $C(z)$ and $D(z)$ are rational functions for all $z \in \mathbb{C}$.

So clearly if $D(z) \equiv 0$, we get the generalized homogeneous Riccati shift equation

$$A(z)w(z)w(z+1) + B(z)w(z+1) + C(z)w(z) = 0. \tag{1.4}$$

Also for $A(z) \equiv 0$ the equation (1.3) becomes

$$B(z)w(z+1) + C(z)w(z) = D(z). \tag{1.5}$$

In this paper we will characterize the solution of (1.3) in terms of shared values. In this regard, Nevanlinna value distribution theory will be rendered as an important tool. For the sake of subsequent discussion we are now going to demonstrate the notion of value sharing in a precise way.

Throughout the paper we consider f and g as meromorphic functions defined in the open complex plane $\mathbb{C}$.

Let $a \in \mathbb{C}$, we denote by $E(a; f)$, the collection of the zeros of $f - a$, where a zero is counted according to its multiplicity. In addition to this, when $a = \infty$, the above definition implies that we are considering the poles. In the same manner, by $\overline{E}(a; f)$, we denote the collection of the distinct zeros or poles of $f - a$ according as $a \in \mathbb{C}$ or $a = \infty$ respectively. If $E(a; f) = E(a; g)$ we say that f and g share the value a CM (counting multiplicities) and if $\overline{E}(a; f) = \overline{E}(a; g)$, then we say that f and g share the value a IM (ignoring multiplicities).

Though the standard definitions and notations of the value distribution theory are available in [1, 7, 11], we explain some definitions and notations which are used in the paper.

The term $N(r; a; f)$ ($\overline{N}(r; a; f)$) denotes the counting function (reduced counting function) of a-points of meromorphic function f in $\mid z \mid \leq r$ and $m(r, f)$ is called the proximity function of f which is the average of the positive logarithm of $\mid f(z) \mid$ on the circle $\mid z \mid = r$. Also the term $T(r, f) = N(r, \infty; f) + m(r, f)$ denotes the Nevanlinna characteristic function of the non-constant meromorphic function.

Usually, $S(r, f)$ denotes any quantity satisfying $S(r, f) = o(T(r, f))$ for all r outside of a possible exceptional set of finite linear measure.

We use the notation $\rho(f)$ to denote the order of growth of f as follows:

$$\rho(f) = \limsup_{r \to \infty} \frac{\log(T(r, f))}{\log r}.$$

2 Auxiliary and main results

The investigations of meromorphic functions sharing values and possible relationships between them is an important feature of Nevanlinna theory. The following is the famous Nevanlinna's 5 IM (4 CM) theorem.

Theorem A. [10] *Let $f(z)$ and $g(z)$ be two non-constant meromorphic functions. If $f(z)$ and $g(z)$ share five values IM (four values CM, respectively) in the extended complex plane, then $f(z) \equiv g(z)$ ($f(z) \equiv T(g(z))$, where T is a Möbius transformation, respectively).*

With respect to the Nevanlinna's 5 IM (4 CM) theorems, recently a new section have been emerged out, where researchers mainly focused to characterize the meromorphic solutions of several kinds of shift equations under the aegis of sharing values.

In 2016, the first kind of such result was obtained by Cui-Chen [3]. Actually, Cui-Chen [3] considered meromorphic solution of the shift equation (1.5) having polynomial coefficients sharing values with a meromorphic function.

Theorem B. [3] *Let $f(z)$ be a finite order transcendental meromorphic solution of the equation (1.5), where $B(z)$ and $C(z)$ are non-zero polynomials such that $B(z) + C(z) \not\equiv 0$ and $D(z) \equiv 0$. If a meromorphic function $g(z)$ share 0, 1, ∞ CM with $f(z)$, then either $f(z) \equiv g(z)$ or $f(z)g(z) \equiv 1$.*

In the next year, Cui-Chen [4] extended *Theorem B* for non-homogeneous components and exhaustively studied the nature of the solutions as follows:

Theorem C. [4] *Let $f(z)$ be a finite order transcendental meromorphic solution of the equation (1.5) where $B(z)$, $C(z)$ and $D(z)$ are non-zero polynomials such that $B(z) + C(z) \not\equiv 0$. If a meromorphic function $g(z)$ share 0, 1, ∞ CM with $f(z)$, then one of the following cases holds:*

(i) $f(z) \equiv g(z)$;

(ii) $f(z) + g(z) = f(z)g(z)$;

(iii) there exist a polynomial $\beta(z) = az + b_0$ and a constant a_0 satisfying $e^{a_0} \neq e^{b_0}$ such that

$$f(z) = \frac{1 - e^{\beta(z)}}{e^{\beta(z)}(e^{a_0 - b_0} - 1)}, \qquad g(z) = \frac{1 - e^{\beta(z)}}{1 - e^{b_0 - a_0}},$$

where $a_0 \neq 0$, b_0 are constants.

In 2019, Li-Chen [8] reduced the number of sharing values from 3 to 2, provided $g(z)$ is a solution of (1.5) and obtained the following two theorems.

Theorem D. [8] *Let $f(z)$and $g(z)$ be two finite order transcendental meromorphic solutions of equation (1.5), where $D(z) \equiv 0$. Suppose that $f(z)$ and $g(z)$ share 0, ∞ CM. Then $f(z) \equiv e^{2k_0\pi iz + a_0} g(z)$ for some integer k_0 and constant a_0. What is more, $f(z) \equiv g(z)$ provided that one of the following cases holds:*

(i) there exist two points z_1, z_2 such that $f(z_j) = g(z_j) \neq 0$ $(j = 1, 2)$ and $z_1 - z_2 \notin \mathbb{Q}$;

(ii) $f(z) - g(z)$ has a zero z_3 of multiplicity ≥ 2 such that $f(z_3) = g(z_3) \neq 0$.

Theorem E. [8] *Let $f(z)$ and $g(z)$ be two finite order transcendental meromorphic solutions of equation (1.5), where $D(z) \not\equiv 0$. Suppose that $f(z)$ and $g(z)$ share 0, ∞ CM. Then either $f(z) \equiv g(z)$ or*

$$f(z) = \frac{D(z)}{2C(z)}(e^{a_1 z + a_0} + 1), \qquad g(z) = \frac{D(z)}{2C(z)}(e^{-a_1 z - a_0} + 1),$$

where a_1, a_0 are constants such that $e^{-a_1} = e^{a_1} = -1$ and the coefficients of (1.5) satisfy $B(z)D(z+1) \equiv D(z)C(z+1)$.

It is to be noted that recently in 2020, Li-Chen [9] investigated on meromorphic solutions of (1.4) sharing 1 and ∞ CM. However, in this paper we will confine our investigations for two value sharing corresponding to 0 and ∞.

Now in view of the above theorems the following questions are inevitable:

Question 2.1. *What can be said about the possible relationship between a meromorphic function and a solution of the equation (1.3) if they share three values 0, 1 and ∞ or even two values 0 and ∞?*

Question 2.2. *What can be said about the possible relationship between the solutions of the equations (1.4) and (1.5) if they share 0 and ∞?*

The main aim of writing this paper is to investigate the possible answer of the above questions. To this end we present the following theorems as the main results of the paper.

Theorem 1. *Let $f(z)$ be a finite order transcendental meromorphic solution of equation (1.3). If a meromorphic function $g(z)$ share 0, 1, ∞ CM with $f(z)$ and*

$$\lim_{z\to\infty} D(z) \neq 0, \tag{2.1}$$

$$\lim_{z\to\infty} \frac{A(z) \pm B(z) \pm C(z)}{D(z)} \neq 1, \tag{2.2}$$

then $f(z) \equiv g(z)$.

Theorem 2. *Let $f(z)$ and $g(z)$ be two finite order transcendental meromorphic solutions of equation (1.3), such that $A(z)$, $B(z)$, $C(z)$, $D(z)$ are non-zero. Suppose that $f(z)$, $g(z)$ share 0, ∞ CM. If any one of the following conditions hold:*

(*a*) $A(z)D(z) + C(z)B(z) = 0$;
(*b*) $A(z)D(z) + C(z)B(z) \neq 0$ *and*
(*i*) $g(z)$ *has infinitely many poles of multiplicity* ≥ 2, *or*,
(*ii*) $\rho(g)$ *is not an integer and* $g(z)$ *has atmost finitely many poles*;

then $f(z) \equiv g(z)$.

Theorem 3. *Let $f(z)$ and $g(z)$ be two finite order transcendental meromorphic solutions of equation (1.4). Suppose that $f(z)$, $g(z)$ share 0, ∞ CM. Then either $f(z) \equiv g(z)$ or*

$$f(z) = \frac{B(z)e^{a_1 z + a_0}[e^{a_1} - 1]}{A(z)[1 - e^{a_1(z+1)+a_0}]}, \qquad g(z) = \frac{B(z)[e^{a_1} - 1]}{A(z)[1 - e^{a_1(z+1)+a_0}]},$$

where a_0, $a_1(\neq 0)$ are constants such that $e^{2a_1} = 1$ and the coefficients of (1.4) satisfy $A^2(z)B^2(z+1) = A^2(z+1)C^2(z)$.

Theorem 4. *Let $f(z)$ and $g(z)$ be two finite order transcendental meromorphic solutions of the equations (1.4) and (1.5) respectively. Suppose that $f(z)$, $g(z)$ share 0, ∞ CM. Then if*

(i) $D(z) \equiv 0$ and

$$\lim_{z\to\infty} \frac{A(z)B(z+1)}{A(z+1)C(z)} \neq 1, \tag{2.3}$$

or, (ii) $D(z) \not\equiv 0$ and f is of non-integer finite order,

then $f(z) \equiv g(z)$.

The following example shows that in *Theorem 3*, when the condition $A^2(z)B^2(z+1) = A^2(z+1)C^2(z)$ is not satisfied the conclusion cease to hold.

Example 2.1. *Let* $f(z) = \frac{1}{1+e^{\pi ikz}}$ *and* $g(z) = \frac{1}{1+e^{-\pi ikz}}$, *where* $\alpha = e^{\pi ik}$, k *being a constant. Now* $f(z)$ *satisfy the equation (1.4) if* $A(z) + B(z) + C(z) = 0$ *and* $\alpha = -\frac{B(z)}{C(z)}$. *Also* $g(z)$ *satisfy the equation (1.4) if* $A(z) + B(z) + C(z) = 0$ *and* $\alpha = -\frac{C(z)}{B(z)}$. *But from* $\alpha = -\frac{B(z)}{C(z)} = -\frac{C(z)}{B(z)}$, *we obtain* $A^2(z)B^2(z+1) = A^2(z)C^2(z+1) \neq A^2(z+1)C^2(z)$.

3 Lemmas

Lemma 1. *[2, 5] Let* $f(z)$ *be a meromorphic function of finite order* $\rho(f) = \rho$, ϵ *be a positive constant,* η_1 *and* η_2 *be two distinct complex constants. Then*

$$m\left(r, \frac{f(z+\eta_1)}{f(z+\eta_2)}\right) = O(r^{\rho-1+\epsilon}) = o(T(r,f)).$$

Lemma 2. *[1, 7] Let* $f(z)$ *be a transcendental meromorphic solution of the equation*

$$f^n P(z,f) = Q(z,f),$$

where $P(z,f)$ *and* $Q(z,f)$ *are polynomials in* f *and its derivatives with meromorphic coefficients, say* $\{a_\lambda \mid \lambda \in I\}$, *such that* $m(r, a_\lambda) = S(r,f)$ *for all* $\lambda \in I$. *If the total degree of* $Q(z,f)$ *as a polynomial in* f *and its derivatives is* $\leq n$, *then*

$$m(r, P(z,f)) = S(r,f).$$

4 Proofs of the theorems

Proof of Theorem 1. Since $f(z)$ is a finite order transcendental meromorphic function and $f(z)$, $g(z)$ share 0, 1, ∞ CM for all $z \in \mathbb{C}$, we have

$$\frac{g(z)}{f(z)} = e^{P(z)}, \tag{4.1}$$

$$\frac{g(z)-1}{f(z)-1} = e^{Q(z)}, \tag{4.2}$$

where $P(z)$, $Q(z)$ are polynomials such that $\deg P(z) = n$ and $\deg Q(z) = m$.

If $e^{P(z)} \equiv e^{Q(z)}$, then from (4.1) and (4.2) we get, $f(z) \equiv g(z)$.

Let if possible, $e^{P(z)} \not\equiv e^{Q(z)}$. Then clearly from (4.1) and (4.2), we see that $e^{P(z)} \not\equiv 1$ and $e^{Q(z)} \not\equiv 1$ and thus we obtain

$$f(z) = \frac{1-e^{Q(z)}}{e^{P(z)} - e^{Q(z)}}. \tag{4.3}$$

Also since $f(z)$ is a meromorphic solution of (1.3), so using (4.3), we get

$$A(z)\frac{1-e^{Q(z)}}{e^{P(z)}-e^{Q(z)}}\cdot\frac{1-e^{Q(z+1)}}{e^{P(z+1)}-e^{Q(z+1)}}+B(z)\frac{1-e^{Q(z+1)}}{e^{P(z+1)}-e^{Q(z+1)}} \tag{4.4}$$
$$+C(z)\frac{1-e^{Q(z)}}{e^{P(z)}-e^{Q(z)}}=D(z).$$

Let us consider

$$P(z)=a_nz^n+a_{n-1}z^{n-1}+\ldots+a_1z+a_0 \tag{4.5}$$

and

$$Q(z)=b_mz^m+b_{m-1}z^{m-1}+\ldots+b_1z+b_0, \tag{4.6}$$

where $a_nb_m\neq 0$. Denote $a_n=r_1e^{i\theta_1}$, $b_m=r_2e^{i\theta_2}$, where $\theta_1,\theta_2\in[-\pi,\pi)$.

Now if $n>m$, then there exist some $\theta=\theta_3$ such that for $z=re^{i\theta_3}$ we get $\theta_1+n\theta_3=0$ and

$$P(re^{i\theta_3}+j)=r_1r^n(1+o(1)),\quad Q(re^{i\theta_3}+j)=o(r^n),$$

as $r\to\infty$, where $j=0,1$. Thus from (4.4) we obtain,

$$\lim_{r\to\infty}D(re^{i\theta_3})=0.$$

As $D(z)$ is a rational function, for all $\theta\in[-\pi,\pi)$, we can obtain

$$\lim_{r\to\infty}D(re^{i\theta})=0,$$

which contradicts (2.1).

Thus $n\not> m$. Similarly we can deduce that $n<m$ is not possible. Hence we can conclude $n=m$.

Now we consider the following cases.

Case 1. Let $\theta_1=\theta_2$.

If $r_1>r_2$, then there exist some $\theta=\theta_4$ such that for $z=re^{i\theta_4}$ we get $\theta_1+n\theta_4=0$. Thus we have

$$a_nz^n=r_1r^ne^{i(\theta_1+n\theta_4)}=r_1r^n>b_nz^n=r_2r^ne^{i(\theta_1+n\theta_4)}=r_2r^n$$

and also

$$P(re^{i\theta_4}+j)=r_1r^n(1+o(1)),\quad Q(re^{i\theta_4}+j)=r_2r^n(1+o(1)),$$

as $r\to\infty$, where $j=0,1$. Hence from (4.4) we obtain,

$$\lim_{r\to\infty}D(re^{i\theta_4})=0.$$

As $D(z)$ is a rational function, for all $\theta\in[-\pi,\pi)$, we can obtain

$$\lim_{r\to\infty}D(re^{i\theta})=0,$$

which contradicts (2.1) and thus $r_1 \not> r_2$.

Again when $r_1 < r_2$, there exists some $\theta = \theta_5$ such that for $z = re^{i\theta_5}$ we get $\theta_1 + n\theta_5 = 0$. Thus we obtain

$$a_n z^n = r_1 r^n e^{i(\theta_1 + n\theta_5)} = r_1 r^n < b_n z^n = r_2 r^n e^{i(\theta_1 + n\theta_5)} = r_2 r^n$$

and also

$$P(re^{i\theta_5} + j) = r_1 r^n (1 + o(1)), \quad Q(re^{i\theta_5} + j) = r_2 r^n (1 + o(1)),$$

as $r \to \infty$, where $j = 0, 1$. Thus from (4.4), we obtain

$$\lim_{r \to \infty} \frac{A(re^{i\theta_5}) + B(re^{i\theta_5}) + C(re^{i\theta_5})}{D(re^{i\theta_5})} = 1.$$

As $D(z)$ is a rational function, for all $\theta \in [-\pi, \pi)$, we can obtain

$$\lim_{r \to \infty} \frac{A(re^{i\theta}) + B(re^{i\theta}) + C(re^{i\theta})}{D(re^{i\theta})} = 1,$$

which contradicts (2.2) and hence $r_1 \not< r_2$.

Thus $r_1 = r_2$. Since, $e^{P(z)} \neq e^{Q(z)}$, for all $z = re^{i\theta_6}$ such that $\theta_1 + n\theta_6 = 0$, we obtain

$$e^{Q(re^{i\theta_6} + j)} = e^{r_1 r^n (1 + o(1))}, \; e^{P(re^{i\theta_6} + j)} - e^{Q(re^{i\theta_6} + j)} = e^{r_1 r^n (1 + o(1))},$$

as $r \to \infty$, where $j = 0, 1$. Thus from (4.4), we deduce that

$$\lim_{r \to \infty} \frac{A(re^{i\theta_6}) - B(re^{i\theta_6}) - C(re^{i\theta_6})}{D(re^{i\theta_6})} = 1.$$

As $D(z)$ is a rational function, for all $\theta \in [-\pi, \pi)$, we can obtain

$$\lim_{r \to \infty} \frac{A(re^{i\theta}) - B(re^{i\theta}) - C(re^{i\theta})}{D(re^{i\theta})} = 1,$$

which again contradicts (2.2) and thus $r_1 \neq r_2$.

Case 2. Let $\theta_1 \neq \theta_2$. Thus if $\theta_2 - \theta_1 = \alpha$, then either $\alpha > 0$ or $\alpha < 0$. Now clearly there exists some $\theta = \theta_7$ such that for $z = re^{i\theta_7}$ we get $\theta_1 + n\theta_7 = 0$ and so $\theta_2 + n\theta_7 = \alpha$.

Hence we obtain

$$P(re^{i\theta_7} + j) = r_1 r^n (1 + o(1)), \quad Q(re^{i\theta_7} + j) = r_2 r^n (\cos\alpha + i \sin\alpha)(1 + o(1)),$$

as $r \to \infty$, where $j = 0, 1$.

Thus from (4.4), we deduce that

$$\lim_{r\to\infty}\frac{1}{D(re^{i\theta_7})}\left[A(re^{i\theta_7})\frac{1-e^{Q(re^{i\theta_7})}}{e^{P(re^{i\theta_7})}-e^{Q(re^{i\theta_7})}}.\frac{1-e^{Q(re^{i\theta_7}+1)}}{e^{P(re^{i\theta_7}+1)}-e^{Q(re^{i\theta_7}+1)}}\right.$$
$$\left.+B(re^{i\theta_7})\frac{1-e^{Q(re^{i\theta_7}+1)}}{e^{P(re^{i\theta_7}+1)}-e^{Q(re^{i\theta_7}+1)}}+C(re^{i\theta_7})\frac{1-e^{Q(re^{i\theta_7})}}{e^{P(re^{i\theta_7})}-e^{Q(re^{i\theta_7})}}\right]=1$$
$$or,\quad \lim_{r\to\infty}\frac{1}{D(re^{i\theta_7})}\left[A(re^{i\theta_7})\frac{1-e^{(\cos\alpha+i\sin\alpha)r_2r^n(1+o(1))}}{e^{r_1r^n(1+o(1))}-e^{(\cos\alpha+i\sin\alpha)r_2r^n(1+o(1))}}\right.$$
$$.\frac{1-e^{(\cos\alpha+i\sin\alpha)r_2r^n(1+o(1))}}{e^{r_1r^n(1+o(1))}-e^{(\cos\alpha+i\sin\alpha)r_2r^n(1+o(1))}}$$
$$+B(re^{i\theta_7})\frac{1-e^{(\cos\alpha+i\sin\alpha)r_2r^n(1+o(1))}}{e^{r_1r^n(1+o(1))}-e^{(\cos\alpha+i\sin\alpha)r_2r^n(1+o(1))}}$$
$$\left.+C(re^{i\theta_7})\frac{1-e^{(\cos\alpha+i\sin\alpha)r_2r^n(1+o(1))}}{e^{r_1r^n(1+o(1))}-e^{(\cos\alpha+i\sin\alpha)r_2r^n(1+o(1))}}\right]=1.$$

Subcase 2.1. Let $\alpha>0$.
Subcase 2.1.1. Let $\cos\alpha>0$ and $\sin\alpha>0$.
Then from (4.7) we see that

$$\lim_{r\to\infty}\quad\frac{1}{D(re^{i\theta_7})}\left[A(re^{i\theta_7})\frac{e^{-r_2r^n(\cos\alpha+i\sin\alpha)(1+o(1))}-1}{e^{(r_1-r_2\cos\alpha)r^n(1+o(1))}.e^{-ir_2r^n\sin\alpha(1+o(1))}-1}\right.$$
$$.\frac{e^{-r_2r^n(\cos\alpha+i\sin\alpha)(1+o(1))}-1}{e^{(r_1-r_2\cos\alpha)r^n(1+o(1))}.e^{-ir_2r^n\sin\alpha(1+o(1))}-1}$$
$$+B(re^{i\theta_7})\frac{e^{-r_2r^n(\cos\alpha+i\sin\alpha)(1+o(1))}-1}{e^{(r_1-r_2\cos\alpha)r^n(1+o(1))}.e^{-ir_2r^n\sin\alpha(1+o(1))}-1}$$
$$\left.+C(re^{i\theta_7})\frac{e^{-r_2r^n(\cos\alpha+i\sin\alpha)(1+o(1))}-1}{e^{(r_1-r_2\cos\alpha)r^n(1+o(1))}.e^{-ir_2r^n\sin\alpha(1+o(1))}-1}\right]=1.$$

This limit exist only if $\cos\alpha\geq\frac{r_1}{r_2}$ and then

$$\lim_{r\to\infty}\frac{A(re^{i\theta_7})+B(re^{i\theta_7})+C(re^{i\theta_7})}{D(re^{i\theta_7})}=1.$$

As $D(z)$ is a rational function, for all $\theta\in[-\pi,\pi)$, we can obtain

$$\lim_{r\to\infty}\frac{A(re^{i\theta})+B(re^{i\theta})+C(re^{i\theta})}{D(re^{i\theta})}=1,$$

which contradicts (2.2).
Subcase 2.1.2. Let $\cos\alpha<0$ and $\sin\alpha>0$.
Then from (4.7) we observe that the limit does not exist under no situation.
Subcase 2.1.3. Let $\cos\alpha<0$ and $\sin\alpha<0$.
Then from (4.7) we clearly obtain

$$\lim_{r\to\infty}D(re^{i\theta_7})=0.$$

As $D(z)$ is a rational function, for all $\theta \in [-\pi, \pi)$, we can obtain

$$\lim_{r\to\infty} D(re^{i\theta}) = 0,$$

which contradicts (2.1).

Subcase 2.1.4. Let $\cos\alpha > 0$ and $\sin\alpha < 0$.

Then from (4.7) we see that

$$\begin{aligned}\lim_{r\to\infty} \quad & \frac{1}{D(re^{i\theta_7})}\Bigg[A(re^{i\theta_7})\frac{e^{-r_2r^n\cos\alpha(1+o(1))} - e^{ir_2r^n\sin\alpha(1+o(1))}}{e^{(r_1-r_2\cos\alpha)r^n(1+o(1))} - e^{ir_2r^n\sin\alpha(1+o(1))}}\\ & .\frac{e^{-r_2r^n\cos\alpha(1+o(1))} - e^{ir_2r^n\sin\alpha(1+o(1))}}{e^{(r_1-r_2\cos\alpha)r^n(1+o(1))} - e^{ir_2r^n\sin\alpha(1+o(1))}}\\ & +B(re^{i\theta_7})\frac{e^{-r_2r^n\cos\alpha(1+o(1))} - e^{ir_2r^n\sin\alpha(1+o(1))}}{e^{(r_1-r_2\cos\alpha)r^n(1+o(1))} - e^{ir_2r^n\sin\alpha(1+o(1))}}\\ & +C(re^{i\theta_7})\frac{e^{-r_2r^n\cos\alpha(1+o(1))} - e^{ir_2r^n\sin\alpha(1+o(1))}}{e^{(r_1-r_2\cos\alpha)r^n(1+o(1))} - e^{ir_2r^n\sin\alpha(1+o(1))}}\Bigg] = 1.\end{aligned}$$

This limit exist only if $\cos\alpha \le \frac{r_1}{r_2}$ and then

$$\lim_{r\to\infty} D(re^{i\theta_7}) = 0.$$

As $D(z)$ is a rational function, for all $\theta \in [-\pi, \pi)$, we can obtain

$$\lim_{r\to\infty} D(re^{i\theta}) = 0,$$

which contradicts (2.1).

Subcase 2.2. Let $\alpha < 0$. Then clearly $\cos\alpha > 0$ and $\sin\alpha < 0$ is the only possibility. But from *Subcase 2.1.4.* above we arrive at a contradiction.

This proves the theorem. □

Proof of Theorem 2. Since $f(z)$, $g(z)$ are finite order transcendental meromorphic solutions of the equation (1.3) for all z in $\mathbb{C}$ and share 0, ∞ CM, we have

$$\frac{f(z)}{g(z)} = e^{P(z)}, \tag{4.7}$$

where $P(z)$ is a polynomial such that $\deg P(z) \le \max\{\rho(f), \rho(g)\}$.

If $e^{P(z)} = 1$, then clearly from (4.7), we obtain $f(z) = g(z)$.

Let $e^{P(z)} \ne 1$, then $e^{P(z+1)} \ne 1$.

If $e^{P(z)} = 1$, then clearly from (4.7), we obtain $f(z) = g(z)$.

Let $e^{P(z)} \ne 1$, then $e^{P(z+1)} \ne 1$.

So from (1.3) and (4.7), we get

$$\begin{aligned}&A(z)e^{P(z)}g(z).e^{P(z+1)}g(z+1) + B(z)e^{P(z+1)}g(z+1)\\ &+C(z)e^{P(z)}g(z) = D(z).\end{aligned} \tag{4.8}$$

Also since $g(z)$ is a solution of (1.3), so

$$A(z)g(z)g(z+1)+B(z)g(z+1)+C(z)g(z)=D(z). \tag{4.9}$$

From (4.8) and (4.9), we obtain

$$\begin{aligned}&A(z)C(z)e^{P(z)}\Big\{e^{P(z+1)}-1\Big\}g^2(z)+\Big[A(z)D(z)\Big\{1-e^{P(z+1)+P(z)}\Big\}\\&+B(z)C(z)\Big\{e^{P(z+1)}-e^{P(z)}\Big\}\Big]g(z)+B(z)D(z)\Big[1-e^{P(z+1)}\Big]=0.\end{aligned} \tag{4.10}$$

Now in equation (1.3) as $A(z)$, $B(z)$, $C(z)$, $D(z)$ are non-zero rational functions and $f(z)$, $g(z)$ are transcendental meromorphic functions so we consider the following cases.

Case 1. Let $A(z)D(z)+C(z)B(z)=0$, i.e., $\frac{A(z)}{B(z)}=-\frac{C(z)}{D(z)}=\alpha(z)$, where $\alpha(z)$ is a rational function. Then (4.10) becomes

$$(\alpha(z)g(z)+1)(\alpha(z)e^{P(z)}g(z)+1)(1-e^{P(z+1)})=0.$$

Since $g(z)$ is a transcendental function so $g(z)\neq-\frac{1}{\alpha(z)}$. Thus

$$g(z)=-\frac{e^{-P(z)}}{\alpha(z)}. \tag{4.11}$$

Let us consider

$$P(z)=a_nz^n+a_{n-1}z^{n-1}+\ldots+a_1z+a_0,$$

where $a_n(\neq 0),\ldots,a_1,a_0$ are constants and n is an integer.

Then clearly

$$\rho(e^{P(z+1)-P(z)})=n-1. \tag{4.12}$$

Now combining (4.9) and (4.11), we get

$$e^{-P(z+1)}=\frac{D(z)}{B(z)}\alpha(z+1).$$

Clearly, we see that

$$n=\rho(e^{-P(z+1)})=\rho\bigg(\frac{D(z)}{B(z)}\alpha(z+1)\bigg)=0,$$

which is a contradiction.

Case 2. Let $A(z)D(z)+C(z)B(z)\neq 0$, i.e., $\frac{A(z)}{B(z)}=\alpha(z)$ and $\frac{C(z)}{D(z)}=\beta(z)$, where $\alpha(z)$, $\beta(z)$ are rational functions. Then (4.10) becomes

$$g^2(z)=E(z)g(z)+F(z), \tag{4.13}$$

where $E(z)=\frac{\beta(z)e^{P(z+1)}-\alpha(z)e^{P(z)+P(z+1)}+\alpha(z)-\beta(z)e^{P(z)}}{\alpha(z)\beta(z)e^{P(z)}(1-e^{P(z+1)})}$ and $F(z)=-\frac{1}{\alpha(z)\beta(z)e^{P(z)}}$.

Subcase 2.1. Let $g(z)$ has infinitely many poles of multiplicity ≥ 2. Now we consider the following subcases.

Subcase 2.1.1. Let $P(z)$ is a constant. Then, $E(z)$ and $F(z)$ are rational functions and hence have at most finitely many poles. Now if z_1 is a pole of $g(z)$ with multiplicity $k_1 \geq 2$ such that $E(z_1) \neq \infty$ and $F(z_1) \neq \infty$, then z_1 is a pole of $g^2(z)$ with multiplicity $2k_1$ and a pole of $E(z)g(z)+F(z)$ with multiplicity k_1, which is not possible by (4.13).

Subcase 2.1.2. Let $P(z)$ is a non-constant polynomial such that $\deg P(z) = n \geq 1$.

Now as $g(z)$ has infinitely many poles of multiplicity ≥ 2, so from (4.13) we see that the multiplicity of the zeros of $1-e^{P(z+1)}$ cannot be less than 2. Thus it is clear from

$$(1-e^{P(z+1)})' = P'(z+1)e^{P(z+1)},$$

that $1-e^{P(z+1)}$ has at most n zeros of multiplicity ≥ 2. Thus $E(z)$ is a meromorphic functions which have atmost finitely many poles of multiplicity ≥ 2. Now if z_2 is pole of $g(z)$ with multiplicity $k_2 \geq 2$ such that z_2 is not a pole of $E(z)$ of multiplicity ≥ 2, then z_2 is a pole of $E(z)g(z)+F(z)$ with multiplicity at most k_2. But from (4.13) and $k_2 < 2k_2$, it is not possible.

Subcase 2.2. Let $\rho(g)$ is not an integer and $g(z)$ has atmost finitely many poles. Since $\deg P(z) \leq \rho(g)$, hence we have $\deg P(z) < \rho(g)$.

Now if $g(z)$ has at most finitely many poles of multiplicity ≥ 2, then from *Subcase 2.1.2.* the result is obvious.

Now let $g(z)$ has at most finitely many simple poles. We have,

$$m(r,g) = T(r,g) - N(r,g) = T(r,g) + S(r,g). \tag{4.14}$$

Again, since $\deg P(z) < \rho(g)$, we observe that

$$m(r,E) \leq T(r,E) = S(r,g),\ m(r,F) \leq T(r,F) = S(r,g). \tag{4.15}$$

Applying *Lemma 2* to (4.13), we get

$$m(r,g) = S(r,g),$$

which contradicts (4.14).

This proves the theorem.

□

Proof of Theorem 3. Since $f(z)$ and $g(z)$ are finite order transcendental meromorphic solutions of the equation (1.4) for all z in $\mathbb{C}$, the equation (4.7) still holds.

If $e^{P(z)}=1$, then clearly from (4.7), we obtain $f(z)=g(z)$.

Let $e^{P(z)} \neq 1$, then $e^{P(z+1)} \neq 1$.

Now in the similar way as in the proof of *Theorem 2*, considering equation (1.4) we obtain equation (4.10) with $D(z) = 0$, i.e.,

$$A(z)C(z)e^{P(z)}\left[e^{P(z+1)} - 1\right]g^2(z) + B(z)C(z)\left\{e^{P(z+1)} - e^{P(z)}\right\}g(z) = 0,$$

or,

$$g(z) = \frac{B(z)[e^{P(z+1)-P(z)} - 1]}{A(z)[1 - e^{P(z+1)}]}. \tag{4.16}$$

Also from equation (1.4), we have

$$A(z)g(z)g(z+1) + B(z)g(z+1) + C(z)g(z) = 0. \tag{4.17}$$

Combining (4.16) and (4.17), we get

$$\begin{aligned} &A(z)B(z+1)e^{P(z+1)}h(z+1)[e^{-P(z)} - 1] \\ &+A(z+1)C(z)h(z)[1 - e^{P(z+2)}] = 0, \end{aligned} \tag{4.18}$$

where

$$h(z) = e^{P(z+1)-P(z)} - 1.$$

Let us consider

$$P(z) = a_n z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0, \tag{4.19}$$

where $a_n(\neq 0), \ldots, a_1, a_0$ are constants and n is an integer. Since $g(z)$ is transcendental, so from (4.18), we have $\deg P(z) \geq 1$.

We claim that $\deg P(z) = 1$. So if possible, $\deg P(z) \geq 2$. Then clearly

$$\deg[P(z+2) - P(z+1)] = \deg[P(z+1) - P(z)] = n - 1. \tag{4.20}$$

Thus $\rho(e^{P(z+1)-P(z)}) = n - 1$. Now

$$T(r, h) = T(e^{P(z+1)-P(z)} - 1) = T(e^{P(z+1)-P(z)}) + O(\log r),$$

which yields

$$\rho(h) = n - 1. \tag{4.21}$$

Now from (4.18), we obtain

$$\begin{aligned} &\left\{A(z)B(z+1)h(z+1) + A(z+1)C(z)h(z)e^{P(z+2)-P(z+1)}\right\}e^{P(z+1)} \\ &= A(z)B(z+1)e^{P(z+1)-P(z)}h(z+1) + A(z+1)C(z)h(z). \end{aligned}$$

If possible, suppose $A(z)B(z+1)e^{P(z+1)-P(z)}h(z+1) + A(z+1)C(z)h(z) \neq 0$.

Then from (4.20) and (4.21), we deduce

$$n = \rho\bigg(\bigg\{A(z)B(z+1)h(z+1) + A(z+1)C(z)h(z)e^{P(z+2)-P(z+1)}\bigg\}e^{P(z+1)}\bigg)$$
$$= \rho\bigg(A(z)B(z+1)e^{P(z+1)-P(z)}h(z+1) + A(z+1)C(z)h(z)\bigg) \leq n-1,$$

which is a contradiction.

Thus $A(z)B(z+1)e^{P(z+1)-P(z)}h(z+1) + A(z+1)C(z)h(z) = 0$ and thus we obtain

$$T(r, e^{P(z+1)-P(z)}) = m(r, e^{P(z+1)-P(z)}) = m\bigg(r, -\frac{A(z+1)C(z)h(z)}{A(z)B(z+1)h(z+1)}\bigg). \quad (4.22)$$

Now from *Lemma 1* we see that for $\epsilon > 0$,

$$m\bigg(r, \frac{h(z)}{h(z+1)}\bigg) = O(r^{\rho(h)-1+\epsilon}) = O(r^{n-2+\epsilon}) = o(r^{n-1}). \quad (4.23)$$

Hence from (4.22) and (4.23), we deduce

$$T(r, e^{P(z+1)-P(z)}) \leq o(r^{n-1}) + O(\log r),$$

which again contradicts $\rho(e^{P(z+1)-P(z)}) = n-1 \geq 1$.

So clearly $\deg P(z) = 1$ and hence from (4.19), we obtain $P(z) = a_1 z + a_0$, where $a_1 \neq 0$. Now from (4.16), we obtain

$$g(z) = \frac{B(z)[e^{a_1} - 1]}{A(z)[1 - e^{a_1(z+1)+a_0}]}. \quad (4.24)$$

Hence from (4.17) and (4.24), we have

$$(e^{a_1} - 1)\bigg[e^{a_1 z}\bigg\{A(z)B(z+1)e^{a_1+a_0} + A(z+1)C(z)e^{2a_1+a_0}\bigg\}$$
$$-\bigg\{A(z)B(z+1)e^{a_1} + A(z+1)C(z)\bigg\}\bigg] = 0,$$

which implies either

$$e^{a_1} = 1, \quad (4.25)$$

or

$$e^{a_1 z}\bigg\{A(z)B(z+1)e^{a_1+a_0} + A(z+1)C(z)e^{2a_1+a_0}\bigg\} \quad (4.26)$$
$$-\bigg\{A(z)B(z+1)e^{a_1} + A(z+1)C(z)\bigg\} = 0.$$

Comparing the orders of both sides of the equation (4.26), we observe that

$$A(z)B(z+1)e^{a_1+a_0} + A(z+1)C(z)e^{2a_1+a_0} = 0 \quad (4.27)$$

and hence using (4.27) in (4.26), we get

$$A(z)B(z+1)e^{a_1}+A(z+1)C(z)=0. \tag{4.28}$$

So from (4.27) and (4.28), we deduce that

$$e^{2a_1}=1.$$

Therefore, from (4.7) and (4.24), we obtain

$$f(z)=\frac{B(z)e^{a_1z+a_0}[e^{a_1}-1]}{A(z)[1-e^{a_1(z+1)+a_0}]},$$

where $e^{2a_1}=1$.

Also from (4.27) and (4.28), we obtain the relation $A^2(z)B^2(z+1)=A^2(z+1)C^2(z)$ holds. This proves the theorem. □

Proof of Theorem 4. Since $f(z)$ and $g(z)$ are finite order transcendental meromorphic solutions of the equations (1.4) and (1.5) respectively for all z in $\mathbb{C}$ and share 0, ∞ CM, the equation (4.7) in this case still holds. Here $B(z)C(z)\not\equiv 0$ implies (1.5) admits transcendental meromorphic solution.

Case 1. Let $e^{P(z)}=1$. Then clearly from (4.7), we have $f(z)\equiv g(z)$.

Case 2. Let $e^{P(z)}\neq 1$.

Subcase 2.1. Let $D(z)\equiv 0$. Then from (1.4) and (4.7), we get

$$A(z)e^{P(z)}g(z).e^{P(z+1)}g(z+1)+B(z)e^{P(z+1)}g(z+1)+C(z)e^{P(z)}g(z)=0. \tag{4.29}$$

Again from (1.5), we have

$$B(z)g(z+1)+C(z)g(z)=0. \tag{4.30}$$

By (4.29) and (4.30) we obtain

$$g(z)=\frac{B(z)}{A(z)}\{e^{-P(z+1)}-e^{-P(z)}\}. \tag{4.31}$$

Combining (4.30) and (4.31), we get

$$\frac{A(z)B(z+1)}{A(z+1)C(z)}=\frac{e^{-P(z)}-e^{-P(z+1)}}{e^{-P(z+2)}-e^{-P(z+1)}}=\frac{e^{P(z+1)-P(z)}-1}{e^{P(z+1)-P(z+2)}-1}. \tag{4.32}$$

Let us consider

$$P(z)=a_nz^n+a_{n-1}z^{n-1}+\ldots+a_1z+a_0,$$

where $a_n\neq 0$. Denote $a_{n-1}=r_1e^{i\theta_1}$, where $\theta_1\in[-\pi,\pi)$.

So there exist some $\theta=\theta_2$ such that for $z=re^{i\theta_2}$, we get $\theta_1+(n-1)\theta_2=0$. Thus we have

$$a_{n-1}z^{n-1}=r_1r^{n-1}e^{i(\theta_1+(n-1)\theta_2)}=r_1r^{n-1}. \tag{4.33}$$

Hence from (4.32) and (4.33), we obtain

$$\lim_{r\to\infty}\frac{A(re^{i\theta_2})B(re^{i\theta_2}+1)}{A(re^{i\theta_2}+1)C(re^{i\theta_2})}=\lim_{r\to\infty}\frac{e^{r_1r^{n-1}(1+o(1))}-1}{e^{r_1r^{n-1}(1+o(1))}-1}=1. \tag{4.34}$$

As $\frac{A(z)B(z+1)}{A(z+1)C(z)}$ is a rational function so for all $\theta\in[-\pi,\pi)$, from (4.34), we obtain

$$\lim_{r\to\infty}\frac{A(re^{i\theta})B(re^{i\theta}+1)}{A(re^{i\theta}+1)C(re^{i\theta})}=1,$$

which contradicts (2.3).

Subcase 2.2. Let $D(z)\not\equiv 0$. Then from (1.5), we have

$$B(z)g(z+1)+C(z)g(z)=D(z). \tag{4.35}$$

Also as f is of non-integer finite order, so $P(z)$ is a constant and let us consider $e^{P(z)}=k$. Thus the equation (4.29) becomes

$$kA(z)g(z)g(z+1)+B(z)g(z+1)+C(z)g(z)=0. \tag{4.36}$$

Substituting the value of $g(z+1)$ from (4.35) in (4.36), we get

$$kA(z)C(z)g^2(z)-kA(z)D(z)g(z)-B(z)D(z)=0. \tag{4.37}$$

Thus from (4.37) we deduce that

$$g(z)=\frac{D(z)}{2C(z)}\pm\frac{\sqrt{k^2A^2(z)D^2(z)-4kA(z)B(z)C(z)D(z)}}{2kA(z)C(z)},$$

which is a contradiction as $g(z)$ is transcendental meromorphic function.

This proves the theorem. □

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