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UPDATED OSTROWSKI INEQUALITIES OVER A SPHERICAL SHELL

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Abstract

Here we present general multivariate mixed Ostrowski type inequalities over spherical shells and balls. We cover the radial and not necessarily radial cases. The proofs derive by the use of some estimates coming out of some new trigonometric and hyperbolic Taylor's formulae ([2]) and reducing the multivariate problem to a univariate one via general polar coordinates.

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1 Introduction

We are motivated by the following:

In 1938, A. Ostrowski [4] proved the following famous inequality.

Theorem 1. *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) whose derivative $f':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty=\sup_{t\in(a,b)}|f'(t)|<+\infty$. Then*

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt-f(x)\right|\leq\left[\frac{1}{4}+\frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\left\|f'\right\|_\infty, \tag{1}$$

for any $x\in[a,b]$. The constant $\frac{1}{4}$ is the best.

Ostrowski type inequalities have great applications to numerical analysis and probability and their literature is enormous.

Here $K=\mathbb{R}$ or $\mathbb{C}$.

Recently the author proved:

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Theorem 2. *([2]) Let $f \in C_K^3([c,d])$, $a \in [c,d]$, such that $f'(a) = f''(a) = 0$. Then*

i)

$$\left| \frac{1}{d-c} \int_c^d f(x)\,dx - f(a) \right| \leq \|f''' + f'\|_\infty \frac{\left[(d-a)^3 + (a-c)^3 \right]}{6(d-c)}, \tag{2}$$

ii) when $f'\left(\frac{c+d}{2}\right) = f''\left(\frac{c+d}{2}\right) = 0$, and $a = \frac{c+d}{2}$, we get

$$\left| \frac{1}{d-c} \int_c^d f(x)\,dx - f\left(\frac{c+d}{2}\right) \right| \leq \|f' + f'''\|_\infty \frac{(d-c)^2}{24}. \tag{3}$$

We are also motivated by author's monograph, see chapter 6.

This work is based on author's recent article [2], where we developed some new trigonometric and hyperbolic type Taylor's formulae.

We prove here a collection of multivariate Ostrowski type inequalities related to functions over a spherical shell in $\mathbb{R}^N$, with respect to all norms $\|\cdot\|_p$, $1 \leq p \leq \infty$, and we give also their generalizations.

2 Main results

We need

Remark 3. *Let the spherical shell*

$$A := B(0, R_2) - \overline{B(0, R_1)},$$

$0 < R_1 < R_2$, $A \subseteq \mathbb{R}^N$, $N \geq 2$, $x \in \overline{A}$. *Consider that $f : \overline{A} \to \mathbb{R}$ is radial, thai is, there exists g such that $f(x) = g(r)$, $r = |x|$, $r \in [R_1, R_2]$, $\forall\ x \in \overline{A}$, $|\cdot|$ the Euclidean norm. Here x can be written uniquely as $x = r\omega$, where $r = |x| > 0$ and $\omega = \frac{x}{r} \in S^{N-1}$, $|\omega| = 1$, see ([5], pp. 149-150 and [6], p. 421).*

Furthermore for $F : \overline{A} \to \mathbb{R}$ a Lebesgue integrable function we have that

$$\int_A F(x)\,dx = \int_{S^{N-1}} \left(\int_{R_1}^{R_2} F(r\omega) r^{N-1} dr \right) d\omega, \tag{4}$$

where $S^{N-1} := \left\{ x \in \mathbb{R}^N : |x| = 1 \right\}$.

Let $d\omega$ be the element of surface measure on $\mathbb{S}^{N-1}$ with surface area

$$\omega_N = \int_{S^{N-1}} d\omega = \frac{2\pi^{\frac{N}{2}}}{\Gamma\left(\frac{N}{2}\right)}. \tag{5}$$

Here it is volume of A,

$$vol(A) = \frac{\omega_N \left(R_2^N - R_1^N \right)}{N}. \tag{6}$$

We present the following multivariate radial Ostrowski type inequality on the spherical shell.

Theorem 4. *Let the function $f : \overline{A} \to \mathbb{R}$ be radial, that is, there exists g such that $f(x) = g(r)$, $r = |x|$, $\forall\ x \in \overline{A}$; $\omega \in S^{N-1}$. Assume that $g \in C^3([R_1, R_2])$, and $g^{(k)}(r_0) = 0$, $k = 1, 2$, where $r_0 \in [R_1, R_2]$ is fixed. Then ($\forall\ \omega \in S^{N-1}$)*

$$\left| f(r_0\omega) - \frac{\int_A f(y)\,dy}{Vol(A)} \right| = \left| g(r_0) - \left(\frac{N}{R_2^N - R_1^N} \right) \int_{R_1}^{R_2} g(s)\, s^{N-1} ds \right| \leq \tag{7}$$

$$\left(\frac{N!}{R_2^N - R_1^N} \right) \left\{ \left\| g''' + g' \right\|_{\infty,[R_1,r_0]} \left(\sum_{k=0}^{N-1} \frac{R_1^k (r_0 - R_1)^{3+N-k}}{k!\,(N-k+3)!} \right) + \right.$$

$$\left. \left\| g''' + g' \right\|_{\infty,[r_0,R_2]} \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1}}{k!\,(N-k+3)!} R_2^k (R_2 - r_0)^{3+N-k} \right) \right\}.$$

Proof. As in [2], we get that

$$|g(r) - g(r_0)| \leq \left\| g''' + g' \right\|_{\infty,[r_0,R_2]} \frac{(r - r_0)^3}{3!}, \tag{8}$$

$\forall\ r \in [r_0, R_2]$,

and

$$|g(r) - g(r_0)| \leq \left\| g''' + g' \right\|_{\infty,[R_1,r_0]} \frac{(r_0 - r)^3}{3!}, \tag{9}$$

$\forall\ r \in [R_1, r_0]$.

Next we observe that

$$\left| f(r_0\omega) - \frac{\int_A f(y)\,dy}{Vol(A)} \right| =$$

$$\left| g(r_0) - \left(\frac{N}{R_2^N - R_1^N} \right) \int_{R_1}^{R_2} g(s)\, s^{N-1} ds \right| =$$

$$\left(\frac{N}{R_2^N - R_1^N} \right) \left| \int_{R_1}^{R_2} (g(r_0) - g(s))\, s^{N-1} ds \right| \leq \tag{10}$$

$$\left(\frac{N}{R_2^N - R_1^N} \right) \int_{R_1}^{R_2} |g(r_0) - g(s)|\, s^{N-1} ds =$$

$$\left(\frac{N}{R_2^N - R_1^N} \right) \left\{ \int_{R_1}^{r_0} |g(r_0) - g(s)|\, s^{N-1} ds + \int_{r_0}^{R_2} |g(r_0) - g(s)|\, s^{N-1} ds \right\} \leq$$

$$\left(\frac{N}{6\left(R_2^N - R_1^N\right)} \right) \left\{ \left\| g''' + g' \right\|_{\infty,[R_1,r_0]} \int_{R_1}^{r_0} (r_0 - s)^3\, s^{N-1} ds + \right. \tag{11}$$

$$\left. \left\| g''' + g' \right\|_{\infty,[r_0,R_2]} \int_{r_0}^{R_2} (s - r_0)^3\, s^{N-1} ds \right\} =: (*)\,.$$

Here we calculate

$$I_1 := \int_{R_1}^{r_0} (r_0 - s)^3 s^{N-1} ds =$$

$$\int_{R_1}^{r_0} (r_0 - s)^3 ((s - R_1) + R_1)^{N-1} ds = \tag{12}$$

$$\sum_{k=0}^{N-1} \binom{N-1}{k} R_1^k \int_{R_1}^{r_0} (r_0 - s)^{4-1} (s - R_1)^{N-k-1} ds =$$

$$\sum_{k=0}^{N-1} \frac{(N-1)!}{k!(N-k-1)!} R_1^k \frac{3!(N-k-1)!}{(N-k+3)!} (r_0 - R_1)^{3+N-k} =$$

$$\sum_{k=0}^{N-1} \frac{(N-1)!3!}{k!(N-k+3)!} R_1^k (r_0 - R_1)^{3+N-k}.$$

That is

$$I_1 = 3!(N-1)! \sum_{k=0}^{N-1} \frac{R_1^k (r_0 - R_1)^{3+N-k}}{k!(N-k+3)!}. \tag{13}$$

Also

$$I_2 := \int_{r_0}^{R_2} (s - r_0)^3 s^{N-1} ds =$$

$$(-1)^{N-1} \int_{r_0}^{R_2} ((R_2 - s) - R_2)^{N-1} (s - r_0)^3 ds =$$

$$(-1)^{N-1} \sum_{k=0}^{N-1} \frac{(N-1)!}{k!(N-k-1)!} (-1)^k R_2^k \int_{r_0}^{R_2} (R_2 - s)^{(N-k)-1} (s - r_0)^{4-1} ds =$$

$$\sum_{k=0}^{N-1} \frac{(N-1)!}{k!(N-k-1)!} (-1)^{N+k-1} R_2^k \frac{(N-k-1)!3!}{(N-k+3)!} (R_2 - r_0)^{3+N-k}. \tag{14}$$

That is

$$I_2 = 3!(N-1)! \sum_{k=0}^{N-1} \frac{(-1)^{N+k-1}}{k!(N-k+3)!} R_2^k (R_2 - r_0)^{3+N-k}. \tag{15}$$

Consequently we obtain

$$(*) = \left(\frac{N!}{R_2^N - R_1^N}\right) \left\{ \left\| g''' + g' \right\|_{\infty,[R_1,r_0]} \sum_{k=0}^{N-1} \frac{R_1^k (r_0 - R_1)^{3+N-k}}{k!(N-k+3)!} + \right.$$

$$\left. \left\| g''' + g' \right\|_{\infty,[r_0,R_2]} \sum_{k=0}^{N-1} \frac{(-1)^{N+k-1}}{k!(N-k+3)!} R_2^k (R_2 - r_0)^{3+N-k} \right\}. \tag{16}$$

The proof is completed. □

It follows an L_1 Ostrowski inequality.

Theorem 5. *All as in Theorem 4, except now $g \in C^2([R_1, R_2])$. Then ($\forall\ \omega \in S^{N-1}$)*

$$\left| f(r_0\omega) - \frac{\int_A f(y)\,dy}{Vol(A)} \right| = \left| g(r_0) - \left(\frac{N}{R_2^N - R_1^N}\right) \int_{R_1}^{R_2} g(s)\, s^{N-1} ds \right| \leq$$

$$\left(\frac{N!}{R_2^N - R_1^N}\right) \left\{ \left\| g'' + g - g(r_0) \right\|_{L_1([R_1, r_0])} \left(\sum_{k=0}^{N-1} \frac{R_1^k (r_0 - R_1)^{N-k+1}}{k!\,(N-k+1)!} \right) + \right.$$

$$\left. \left\| g'' + g - g(r_0) \right\|_{L_1([r_0, R_2])} \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1}}{k!\,(N-k+1)!} R_2^k (R_2 - r_0)^{N-k+1} \right) \right\}. \tag{17}$$

Proof. As in [2], we get that

$$|g(r) - g(r_0)| \leq \left\| g'' + g - g(r_0) \right\|_{L_1([r_0, R_2])} (r - r_0), \tag{18}$$

$\forall\ r \in [r_0, R_2]$,

and

$$|g(r) - g(r_0)| \leq \left\| g'' + g - g(r_0) \right\|_{L_1([R_1, r_0])} (r_0 - r), \tag{19}$$

$\forall\ r \in [R_1, r_0]$.

Next we observe

$$\left| f(r_0\omega) - \frac{\int_A f(y)\,dy}{Vol(A)} \right| = \left| g(r_0) - \left(\frac{N}{R_2^N - R_1^N}\right) \int_{R_1}^{R_2} g(s)\, s^{N-1} ds \right|$$

$$\leq \text{(as earlier)} \leq$$

$$\left(\frac{N}{R_2^N - R_1^N}\right) \left\{ \int_{R_1}^{r_0} |g(r_0) - g(s)|\, s^{N-1} ds + \int_{r_0}^{R_2} |g(r_0) - g(s)|\, s^{N-1} ds \right\} \leq$$

$$\left(\frac{N}{R_2^N - R_1^N}\right) \left\{ \left\| g'' + g - g(r_0) \right\|_{L_1([R_1, r_0])} \int_{R_1}^{r_0} (r_0 - s)\, s^{N-1} ds + \right. \tag{20}$$

$$\left. \left\| g'' + g - g(r_0) \right\|_{L_1([r_0, R_2])} \int_{r_0}^{R_2} (s - r_0)\, s^{N-1} ds \right\}.$$

Here we calculate

$$I_1^* := \int_{R_1}^{r_0} (r_0 - s)\, s^{N-1} ds =$$

$$\int_{R_1}^{r_0} (r_0 - s) \left((s - R_1) + R_1\right)^{N-1} ds =$$

$$\sum_{k=0}^{N-1} \binom{N-1}{k} R_1^k \int_{R_1}^{r_0} (r_0 - s)^{2-1} (s - R_1)^{N-k-1} ds = \tag{21}$$

$$\sum_{k=0}^{N-1} \frac{(N-1)!}{k!\,(N-k-1)!} R_1^k \frac{(N-k-1)!}{(N-k+1)!} (r_0 - R_1)^{N-k+1} =$$

$$\sum_{k=0}^{N-1} \frac{(N-1)!}{k!\,(N-k+1)!} R_1^k \left(r_0 - R_1\right)^{N-k+1}.$$

That is

$$I_1^* = (N-1)! \sum_{k=0}^{N-1} \frac{R_1^k \left(r_0 - R_1\right)^{N-k+1}}{k!\,(N-k+1)!}. \tag{22}$$

Also

$$I_2^* := \int_{r_0}^{R_2} \left(s - r_0\right) s^{N-1} ds =$$

$$(-1)^{N-1} \int_{r_0}^{R_2} \left(\left(R_2 - s\right) - R_2\right)^{N-1} \left(s - r_0\right)^{2-1} ds =$$

$$(-1)^{N-1} \sum_{k=0}^{N-1} \frac{(N-1)!}{k!\,(N-k-1)!} (-1)^k R_2^k \int_{r_0}^{R_2} \left(R_2 - s\right)^{(N-k)-1} \left(s - r_0\right)^{2-1} ds =$$

$$\sum_{k=0}^{N-1} \frac{(N-1)!}{k!\,(N-k-1)!} (-1)^{N+k-1} R_2^k \frac{(N-k-1)!}{(N-k+1)!} \left(R_2 - r_0\right)^{N-k+1} = \tag{23}$$

$$\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1} (N-1)!}{k!\,(N-k+1)!} R_2^k \left(R_2 - r_0\right)^{N-k+1}.$$

That is

$$I_2^* = (N-1)! \sum_{k=0}^{N-1} \frac{(-1)^{N+k-1}}{k!\,(N-k+1)!} R_2^k \left(R_2 - r_0\right)^{N-k+1}. \tag{24}$$

The theorem is proved. □

Next comes an Ostrowski multivariate radial inequality for $\|\cdot\|_p$, $p > 1$.

Theorem 6. *Let $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$. The rest as in Theorem 5. Then ($\forall$ $\omega \in S^{N-1}$)*

$$\left| f\left(r_0 \omega\right) - \frac{\int_A f(y)\,dy}{Vol(A)} \right| = \left| g\left(r_0\right) - \left(\frac{N}{R_2^N - R_1^N} \right) \int_{R_1}^{R_2} g(s)\, s^{N-1} ds \right| \leq$$

$$\frac{N!\,\Gamma\left(2 + \frac{1}{q}\right)}{\left(R_2^N - R_1^N\right)(q+1)^{\frac{1}{q}}} \tag{25}$$

$$\left\{ \left\| g'' + g - g\left(r_0\right) \right\|_{L_p([R_1, r_0])} \left(\sum_{k=0}^{N-1} \frac{1}{k!\,\Gamma\left(2 + \frac{1}{q} + N - k\right)} R_1^k \left(r_0 - R_1\right)^{N-k+1+\frac{1}{q}} \right) + \right.$$

$$\left. \left\| g'' + g - g\left(r_0\right) \right\|_{L_p([r_0, R_2])} \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1}}{k!\,\Gamma\left(N - k + 2 + \frac{1}{q}\right)} R_2^k \left(R_2 - r_0\right)^{N-k+1+\frac{1}{q}} \right) \right\}.$$

Proof. As in [2], we get that

$$\left|g\left(r\right)-g\left(r_0\right)\right| \leq \left\|g''+g-g\left(r_0\right)\right\|_{L_p([r_0,R_2])}\frac{\left(r-r_0\right)^{\frac{q+1}{1}}}{\left(q+1\right)^{\frac{1}{q}}}, \tag{26}$$

$\forall\ r \in \left[r_0, R_2\right]$,

and

$$\left|g\left(r\right)-g\left(r_0\right)\right| \leq \left\|g''+g-g\left(r_0\right)\right\|_{L_p([R_1,r_0])}\frac{\left(r_0-r\right)^{\frac{q+1}{q}}}{\left(q+1\right)^{\frac{1}{q}}}, \tag{27}$$

$\forall\ r \in \left[R_1, r_0\right]$.

Next we observe

$$\left|f\left(r_0\omega\right)-\frac{\int_A f\left(y\right)dy}{Vol\left(A\right)}\right| =$$

$$\left|g\left(r_0\right)-\left(\frac{N}{R_2^N-R_1^N}\right)\int_{R_1}^{R_2} g\left(s\right)s^{N-1}ds\right|$$

$$\leq \text{(as earlier)} \leq$$

$$\left(\frac{N}{R_2^N-R_1^N}\right)\left\{\int_{R_1}^{r_0}\left|g\left(r_0\right)-g\left(s\right)\right|s^{N-1}ds+\int_{r_0}^{R_2}\left|g\left(r_0\right)-g\left(s\right)\right|s^{N-1}ds\right\} \leq$$

$$\left(\frac{N}{R_2^N-R_1^N}\right)\frac{1}{\left(q+1\right)^{\frac{1}{q}}}\left\{\left\|g''+g-g\left(r_0\right)\right\|_{L_p([R_1,r_0])}\int_{R_1}^{r_0}\left(r_0-s\right)^{\frac{q+1}{q}}s^{N-1}ds+\right.$$
$$\left.\left\|g''+g-g\left(r_0\right)\right\|_{L_p([r_0,R_2])}\int_{r_0}^{R_2}\left(s-r_0\right)^{\frac{q+1}{q}}s^{N-1}ds\right\}. \tag{28}$$

Here we calculate

$$J_1 := \int_{R_1}^{r_0}\left(r_0-s\right)^{\frac{q+1}{q}}s^{N-1}ds =$$

$$\int_{R_1}^{r_0}\left(r_0-s\right)^{\frac{q+1}{q}}\left(\left(s-R_1\right)+R_1\right)^{N-1}ds =$$

$$\sum_{k=0}^{N-1}\binom{N-1}{k}R_1^k\int_{R_1}^{r_0}\left(r_0-s\right)^{\left(2+\frac{1}{q}\right)-1}\left(s-R_1\right)^{N-k-1}ds =$$

$$\sum_{k=0}^{N-1}\binom{N-1}{k}R_1^k\frac{\Gamma\left(2+\frac{1}{q}\right)\left(N-k-1\right)!}{\Gamma\left(2+\frac{1}{q}+N-k\right)}\left(r_0-R_1\right)^{1+\frac{1}{q}+N-k} =$$

$$\sum_{k=0}^{N-1}\frac{\left(N-1\right)!}{k!}\frac{\Gamma\left(2+\frac{1}{q}\right)}{\Gamma\left(2+\frac{1}{q}+N-k\right)}R_1^k\left(r_0-R_1\right)^{1+\frac{1}{q}+N-k}. \tag{29}$$

We have found that

$$J_1 = (N-1)! \sum_{k=0}^{N-1} \frac{\Gamma\left(2+\frac{1}{q}\right)}{k!\Gamma\left(2+\frac{1}{q}+N-k\right)} R_1^k \left(r_0 - R_1\right)^{1+\frac{1}{q}+N-k}. \qquad (30)$$

Also we have

$$J_2 := \int_{r_0}^{R_2} (s-r_0)^{\left(1+\frac{1}{q}\right)} s^{N-1} ds =$$

$$(-1)^{N-1} \int_{r_0}^{R_2} ((R_2 - s) - R_2)^{N-1} (s-r_0)^{\left(1+\frac{1}{q}\right)} ds =$$

$$(-1)^{N-1} \sum_{k=0}^{N-1} \frac{(N-1)!}{k!(N-k-1)!} (-1)^k R_2^k \int_{r_0}^{R_2} (R_2 - s)^{(N-k)-1} (s-r_0)^{\left(2+\frac{1}{q}\right)-1} ds =$$

$$\sum_{k=0}^{N-1} \frac{(N-1)!}{k!(N-k-1)!} (-1)^{k+N-1} R_2^k \frac{(N-k-1)!\Gamma\left(2+\frac{1}{q}\right)}{\Gamma\left(N-k+2+\frac{1}{q}\right)} (R_2 - r_0)^{N-k+1+\frac{1}{q}} =$$

$$(N-1)! \sum_{k=0}^{N-1} \frac{(-1)^{k+N-1}\Gamma\left(2+\frac{1}{q}\right)}{k!\Gamma\left(N-k+2+\frac{1}{q}\right)} R_2^k (R_2 - r_0)^{N-k+1+\frac{1}{q}}. \qquad (31)$$

That is

$$J_2 = (N-1)! \sum_{k=0}^{N-1} \frac{(-1)^{k+N-1}\Gamma\left(2+\frac{1}{q}\right)}{k!\Gamma\left(N-k+2+\frac{1}{q}\right)} R_2^k (R_2 - r_0)^{N-k+1+\frac{1}{q}}. \qquad (32)$$

The proof of the theorem is finished. $\square$

We make

Remark 7. *We treat here the general, not necessarily radial case of f. We apply Theorem 4 to $f(r\omega)$, ω fixed, $r \in [R_1, R_2]$, under the following assumptions:*

$f(\cdot\omega) \in C^3([R_1, R_2])$, $\forall\ \omega \in S^{N-1}$, where $f : \overline{A} \to \mathbb{R}$ is Lebesgue integrable; $\frac{\partial^k f}{\partial r^k}$, $k = 1, 2$, vanish on $\partial B(0, r_0)$, r_0 is fixed in $[R_1, R_2]$; $\left(\frac{\partial^3 f}{\partial r^3} + \frac{\partial f}{\partial r}\right) \in B\left(\overline{A_1}\right)$ (bounded functions), where $A_1 := B(0, R_2) - \overline{B(0, r_0)}$, and $\left(\frac{\partial^3 f}{\partial r^3} + \frac{\partial f}{\partial r}\right) \in B\left(\overline{A_2}\right)$, where $A_2 := B(0, r_0) - \overline{B(0, R_1)}$.

Then ($\forall\ \omega \in S^{N-1}$)

$$\left| f(r_0\omega) - \left(\frac{N}{R_2^N - R_1^N}\right) \int_{R_1}^{R_2} f(s\omega) s^{N-1} ds \right| \leq$$

$$\left(\frac{N!}{R_2^N - R_1^N}\right) \left\{ \left\| \frac{\partial^3 f}{\partial r^3} + \frac{\partial f}{\partial r} \right\|_{\infty, \overline{A_2}} \left(\sum_{k=0}^{N-1} \frac{R_1^k (r_0 - R_1)^{N-k+3}}{k!(N-k+3)!} \right) + \right.$$

$$\left\|\frac{\partial^3 f}{\partial r^3}+\frac{\partial f}{\partial r}\right\|_{\infty,\overline{A_1}}\left(\sum_{k=0}^{N-1}\frac{(-1)^{N+k-1}}{k!\,(N-k+3)!}R_2^k\,(R_2-r_0)^{N-k+3}\right)\Bigg\} =: \lambda_1. \tag{33}$$

Therefore

$$\left|\frac{\int_{S^{N-1}} f(r_0\omega)\,d\omega}{\omega_N}-\frac{N}{\left(R_2^N-R_1^N\right)\omega_N}\int_{S^{N-1}}\left(\int_{R_1}^{R_2} f(s\omega)\,s^{N-1}ds\right)d\omega\right|\leq \lambda_1. \tag{34}$$

That is

$$\left|\frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}}\int_{S^{N-1}} f(r_0\omega)\,d\omega-\frac{\int_A f(x)\,dx}{Vol(A)}\right|\leq \lambda_1. \tag{35}$$

Therefore it holds for $x\in\overline{A}$ that

$$\left|f(x)-\frac{\int_A f(x)\,dx}{Vol(A)}\right|\leq\left|f(x)-\frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}}\int_{S^{N-1}} f(r_0\omega)\,d\omega\right|+\lambda_1. \tag{36}$$

We have proved the following:

Theorem 8. *Let $f:\overline{A}\to\mathbb{R}$ be Lebesgue integrable with $f(\cdot\omega)\in C^3([R_1,R_2])$, $\forall\ \omega\in S^{N-1}$; $\frac{\partial^k f}{\partial r^k}$, $k=1,2$, vanish on $\partial B(0,r_0)$, r_0 is fixed in $[R_1,R_2]$; and $\left(\frac{\partial^3 f}{\partial r^3}+\frac{\partial f}{\partial r}\right)\in B\left(\overline{A_1}\right)$, $\left(\frac{\partial^3 f}{\partial r^3}+\frac{\partial f}{\partial r}\right)\in B\left(\overline{A_2}\right)$. Then for $x\in\overline{A}$ we have*

$$\left|f(x)-\frac{\int_A f(x)\,dx}{Vol(A)}\right|\leq\left|f(x)-\frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}}\int_{S^{N-1}} f(r_0\omega)\,d\omega\right|+$$

$$\left(\frac{N!}{R_2^N-R_1^N}\right)\Bigg\{\left\|\frac{\partial^3 f}{\partial r^3}+\frac{\partial f}{\partial r}\right\|_{\infty,\overline{A_2}}\left(\sum_{k=0}^{N-1}\frac{R_1^k\,(r_0-R_1)^{N-k+3}}{k!\,(N-k+3)!}\right)+$$

$$\left\|\frac{\partial^3 f}{\partial r^3}+\frac{\partial f}{\partial r}\right\|_{\infty,\overline{A_1}}\left(\sum_{k=0}^{N-1}\frac{(-1)^{N+k-1}}{k!\,(N-k+3)!}R_2^k\,(R_2-r_0)^{N-k+3}\right)\Bigg\}. \tag{37}$$

We need the following multivariate radial Ostrowski type inequality.

Theorem 9. *([3]) Let the function $f:\overline{B(0,R)}\to\mathbb{R}$ be radial, that is, there exists a function g such that $f(x)=g(r)$, where $r=|x|$, $r\in[0,R]$, $\forall\ x\in\overline{B(0,R)}$. We further assume that $g\in C^3([0,R])$, and $g^{(k)}(r_0)=0$, $k=1,2$, where $r_0\in[0,R]$ is fixed. Then ($\forall\ \omega\in S^{N-1}$)*

$$\left|f(r_0\omega)-\frac{\int_{B(0,R)} f(y)\,dy}{Vol(B(0,R))}\right|=\left|g(r_0)-\frac{N}{R^N}\int_0^R g(s)\,s^{N-1}ds\right|\leq$$

$$\frac{N!}{R^N}\left[\left\|g'''+g'\right\|_{\infty,[0,r_0]}\frac{r_0^{3+N}}{(3+N)!}+\right.$$

$$\left.\left\|g'''+g'\right\|_{\infty,[r_0,R]}\sum_{k=0}^{N-1}\frac{(-1)^{N+k-1}}{k!\,(N-k+3)!}R^k\,(R-r_0)^{N-k+3}\right]. \tag{38}$$

We make

Remark 10. *Let $f : \overline{B(0,R)} \to \mathbb{R}$ be a Lebesgue integrable function, that is, not necessarily a radial function. We apply Theorem 9 to $f(r\omega)$, where ω is fixed, $\forall$ $r \in [0,R]$, under the following assumptions:*

$f(\cdot\omega) \in C^3([0,R])$, $\forall$ $\omega \in S^{N-1}$. Furthermore $\frac{\partial^k f(r_0\omega)}{\partial r^k} = 0$, $k = 1,2$, $\forall$ $\omega \in S^{N-1}$, where $r_0 \in [0,R]$ is fixed. Finally, we assume that

$$\left\| \left(\frac{\partial^3 f(t\omega)}{\partial r^3} + \frac{\partial f(t\omega)}{\partial r} \right) \right\|_{\infty,(t\in[0,r_0])}, \left\| \left(\frac{\partial^3 f(t\omega)}{\partial r^3} + \frac{\partial f(t\omega)}{\partial r} \right) \right\|_{\infty,(t\in[r_0,R])} \leq K, \tag{39}$$

$\forall$ $\omega \in S^{N-1}$, where $K > 0$.

By (38) we obtain ($\forall$ $\omega \in S^{N-1}$)

$$\left| f(r_0\omega) - \frac{N}{R^N} \int_0^R f(s\omega) s^{N-1} ds \right| \leq$$

$$\frac{KN!}{R^N} \left[\frac{r_0^{3+N}}{(3+N)!} + \sum_{k=0}^{N-1} \frac{(-1)^{N+k-1}}{k!(N-k+3)!} R^k (R-r_0)^{N-k+3} \right] =: \lambda_2. \tag{40}$$

Therefore

$$\left| \frac{\int_{S^{N-1}} f(r_0\omega) d\omega}{\omega_N} - \frac{N}{R^N \omega_N} \int_{S^{N-1}} \left(\int_0^R f(s\omega) s^{N-1} ds \right) d\omega \right| \leq \lambda_2. \tag{41}$$

That is

$$\left| \frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}} \int_{S^{N-1}} f(r_0\omega) d\omega - \frac{\int_{B(0,R)} f(x) dx}{Vol(B(0,R))} \right| \leq \lambda_2. \tag{42}$$

Therefore, it holds for $x \in \overline{B(0,R)}$ that

$$\left| f(x) - \frac{\int_{B(0,R)} f(x) dx}{Vol(B(0,R))} \right| \leq \left| f(x) - \frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}} \int_{S^{N-1}} f(r_0\omega) d\omega \right| + \lambda_2. \tag{43}$$

We have proved on the ball the following:

Theorem 11. *Let $f : \overline{B(0,R)} \to \mathbb{R}$ be a Lebesgue integrable function, that is, not necessarily a radial function. Here $f(\cdot\omega) \in C^3([0,R])$, $R > 0$, $\forall$ $\omega \in S^{N-1}$. Furthermore $\frac{\partial^k f(r_0\omega)}{\partial r^k} = 0$, $k = 1,2$, $\forall$ $\omega \in S^{N-1}$, where $r_0 \in [0,R]$ is fixed. Finally, we assume that*

$$\left\| \left(\frac{\partial^3 f(t\omega)}{\partial r^3} + \frac{\partial f(t\omega)}{\partial r} \right) \right\|_{\infty,(t\in[0,r_0])}, \left\| \left(\frac{\partial^3 f(t\omega)}{\partial r^3} + \frac{\partial f(t\omega)}{\partial r} \right) \right\|_{\infty,(t\in[r_0,R])} \leq K, \tag{44}$$

$\forall$ $\omega \in S^{N-1}$, where $K > 0$.

Then, for $x \in \overline{B(0,R)}$ we have

$$\left| f(x) - \frac{\int_{B(0,R)} f(x)\,dx}{Vol(B(0,R))} \right| \leq \left| f(x) - \frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}} \int_{S^{N-1}} f(r_0\omega)\,d\omega \right| +$$

$$\frac{KN!}{R^N}\left[\frac{r_0^{3+N}}{(3+N)!} + \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1}}{k!\,(N-k+3)!} R^k (R-r_0)^{N-k+3} \right) \right]. \tag{45}$$

Next we generalize Theorem 4.

Theorem 12. *Let the function $f : \overline{A} \to \mathbb{R}$ be radial, that is, there exists g such that $f(x) = g(r)$, $r = |x|$, $\forall\ x \in \overline{A}$; $\omega \in S^{N-1}$. Assume that $g \in C^5([R_1, R_2])$, and $g^{(k)}(r_0) = 0$, $k = 1,2,3,4$, where $r_0 \in [R_1, R_2]$ is fixed. Here $\alpha, \beta \in \mathbb{R}$: $\alpha\beta\left(\alpha^2 - \beta^2\right) \neq 0$. Then ($\forall\ \omega \in S^{N-1}$)*

$$\left| f(r_0\omega) - \frac{\int_A f(y)\,dy}{Vol(A)} \right| = \left| g(r_0) - \left(\frac{N}{R_2^N - R_1^N} \right) \int_{R_1}^{R_2} g(s)\,s^{N-1} ds \right| \leq$$

$$\left(\frac{N!}{R_2^N - R_1^N} \right) \frac{2}{\left|\beta^2 - \alpha^2\right|}$$

$$\left\{ \left\| g^{(5)} + \left(\alpha^2 + \beta^2\right) g^{(3)} + \alpha^2\beta^2 g' \right\|_{\infty,[R_1,r_0]} \left(\sum_{k=0}^{N-1} \frac{R_1^k (r_0 - R_1)^{N-k+3}}{k!\,(N-k+3)!} \right) + \right.$$

$$\left. \left\| g^{(5)} + \left(\alpha^2 + \beta^2\right) g^{(3)} + \alpha^2\beta^2 g' \right\|_{\infty,[r_0,R_2]} \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1} R_2^k (R_2 - r_0)^{N-k+3}}{k!\,(N-k+3)!} \right) \right\}. \tag{46}$$

Proof. As in [2], we get that

$$|g(r) - g(r_0)| \leq \frac{1}{3\left|\beta^2 - \alpha^2\right|} \left\| g^{(5)} + \left(\alpha^2 + \beta^2\right) g^{(3)} + \alpha^2\beta^2 g' \right\|_{\infty,[r_0,R_2]} (r - r_0)^3$$

$$=: A^* (r - r_0)^3, \quad \forall\ r \in [r_0, R_2], \tag{47}$$

and

$$|g(r) - g(r_0)| \leq \frac{1}{3\left|\beta^2 - \alpha^2\right|} \left\| g^{(5)} + \left(\alpha^2 + \beta^2\right) g^{(3)} + \alpha^2\beta^2 g' \right\|_{\infty,[R_1,r_0]} (r_0 - r)^3$$

$$=: B (r_0 - r)^3, \quad \forall\ r \in [R_1, r_0]. \tag{48}$$

We observe that

$$\left| f(r_0\omega) - \frac{\int_A f(y)\,dy}{Vol(A)} \right| \leq \text{(as earlier)} \leq$$

$$\left(\frac{N}{R_2^N - R_1^N} \right) \left\{ \int_{R_1}^{r_0} |g(r_0) - g(s)|\, s^{N-1} ds + \int_{r_0}^{R_2} |g(r_0) - g(s)|\, s^{N-1} ds \right\} \leq \tag{49}$$

$$\left(\frac{N}{R_2^N - R_1^N}\right)\left\{B\int_{R_1}^{r_0}(r_0-s)^3 s^{N-1}ds + A^*\int_{r_0}^{R_2}(s-r_0)^3 s^{N-1}ds\right\} \overset{\text{(by (13), (15))}}{=}$$

$$\left(\frac{N}{R_2^N - R_1^N}\right)\left\{B3!\,(N-1)!\left(\sum_{k=0}^{N-1}\frac{R_1^k (r_0-R_1)^{3+N-k}}{k!\,(N-k+3)!}\right)+\right.$$

$$\left.A^*3!\,(N-1)!\left(\sum_{k=0}^{N-1}\frac{(-1)^{N+k-1}}{k!\,(N-k+3)!}R_2^k (R_2-r_0)^{3+N-k}\right)\right\} =$$

$$\left(\frac{N!3!}{R_2^N - R_1^N}\right)\left\{B\left(\sum_{k=0}^{N-1}\frac{R_1^k (r_0-R_1)^{N-k+3}}{k!\,(N-k+3)!}\right)+\right. \tag{50}$$

$$\left.A^*\left(\sum_{k=0}^{N-1}\frac{(-1)^{N+k-1}}{k!\,(N-k+3)!}R_2^k (R_2-r_0)^{N-k+3}\right)\right\}.$$

The theorem is proved. □

A generalized radial L_1 result follows.

Theorem 13. *All as in Theorem 12, except now $f \in C^4([R_1, R_2])$. Then ($\forall\ \omega \in S^{N-1}$)*

$$\left|f(r_0\omega) - \frac{\int_A f(y)\,dy}{Vol(A)}\right| = \left|g(r_0) - \left(\frac{N}{R_2^N - R_1^N}\right)\int_{R_1}^{R_2} g(s)\,s^{N-1}ds\right| \le$$

$$\left(\frac{N!}{R_2^N - R_1^N}\right)\frac{2}{|\beta^2-\alpha^2|} \tag{51}$$

$$\left\{\left\|g'''' + \left(\alpha^2+\beta^2\right)g'' + \alpha^2\beta^2 g - \alpha^2\beta^2 g(r_0)\right\|_{L_1([R_1,r_0])}\right.$$

$$\left(\sum_{k=0}^{N-1}\frac{R_1^k (r_0-R_1)^{N-k+1}}{k!\,(N-k+1)!}\right)+$$

$$\left\|g'''' + \left(\alpha^2+\beta^2\right)g'' + \alpha^2\beta^2 g - \alpha^2\beta^2 g(r_0)\right\|_{L_1([r_0,R_2])}$$

$$\left.\left(\sum_{k=0}^{N-1}\frac{(-1)^{N+k-1}R_2^k (R_2-r_0)^{N-k+1}}{k!\,(N-k+1)!}\right)\right\}.$$

Proof. As in [2], we get that

$$|g(r) - g(r_0)| \le$$

$$\frac{2}{|\beta^2-\alpha^2|}\left\|g'''' + \left(\alpha^2+\beta^2\right)g'' + \alpha^2\beta^2 g - \alpha^2\beta^2 g(r_0)\right\|_{L_1([R_1,r_0])}(r_0-r), \tag{52}$$

$\forall\ r \in [R_1, r_0]$,

and

$$|g(r) - g(r_0)| \le$$

$$\frac{2}{|\beta^2-\alpha^2|}\left\|g^{''''}+\left(\alpha^2+\beta^2\right)g^{''}+\alpha^2\beta^2 g-\alpha^2\beta^2 g\left(r_0\right)\right\|_{L_1([r_0,R_2])}\left(r-r_0\right), \quad (53)$$

$\forall\ r\in[r_0,R_2]$.

The rest of the proof goes as in Theorem 5. □

It follows a generalized Ostrowski multivariate radial inequality for $\|\cdot\|_p$, $p>1$.

Theorem 14. *All as in Theorem 13, plus let $p,q>1:\frac{1}{p}+\frac{1}{q}=1$. Then ($\forall\ \omega\in S^{N-1}$)*

$$\left|f\left(r_0\omega\right)-\frac{\int_A f\left(y\right)dy}{Vol\left(A\right)}\right|=\left|g\left(r_0\right)-\left(\frac{N}{R_2^N-R_1^N}\right)\int_{R_1}^{R_2}g\left(s\right)s^{N-1}ds\right|\leq$$

$$\left(\frac{N!}{R_2^N-R_1^N}\right)\frac{\Gamma\left(2+\frac{1}{q}\right)}{\left(q+1\right)^{\frac{1}{q}}}$$

$$\Bigg\{\left\|g^{''''}+\left(\alpha^2+\beta^2\right)g^{''}+\alpha^2\beta^2 g-\alpha^2\beta^2 g\left(r_0\right)\right\|_{L_p([R_1,r_0])}$$

$$\left(\sum_{k=0}^{N-1}\frac{R_1^k\left(r_0-R_1\right)^{N-k+1+\frac{1}{q}}}{k!\Gamma\left(2+\frac{1}{q}+N-k\right)}\right)+ \quad (54)$$

$$\left\|g^{''''}+\left(\alpha^2+\beta^2\right)g^{''}+\alpha^2\beta^2 g-\alpha^2\beta^2 g\left(r_0\right)\right\|_{L_p([r_0,R_2])}$$

$$\left(\sum_{k=0}^{N-1}\frac{\left(-1\right)^{N+k-1}R_2^k\left(R_2-r_0\right)^{N-k+1+\frac{1}{q}}}{k!\Gamma\left(N-k+2+\frac{1}{q}\right)}\right)\Bigg\}.$$

Proof. As in [2], we have that

$$\left|g\left(r\right)-g\left(r_0\right)\right|\leq\frac{2}{|\beta^2-\alpha^2|\left(q+1\right)^{\frac{1}{q}}}$$

$$\left\|g^{''''}+\left(\alpha^2+\beta^2\right)g^{''}+\alpha^2\beta^2 g-\alpha^2\beta^2 g\left(r_0\right)\right\|_{L_p([R_1,r_0])}\left(r_0-r\right)^{\frac{q+1}{q}}, \quad (55)$$

$\forall\ r\in[R_1,r_0]$,

and

$$\left|g\left(r\right)-g\left(r_0\right)\right|\leq\frac{2}{|\beta^2-\alpha^2|\left(q+1\right)^{\frac{1}{q}}}$$

$$\left\|g^{''''}+\left(\alpha^2+\beta^2\right)g^{''}+\alpha^2\beta^2 g-\alpha^2\beta^2 g\left(r_0\right)\right\|_{L_p([r_0,R_2])}\left(r-r_0\right)^{\frac{q+1}{q}}, \quad (56)$$

$\forall\ r\in[r_0,R_2]$.

The rest of the proof goes as in Theorem 6. □

It follows a non-radial generalization of Theorem 8.

Theorem 15. *Let $f : \overline{A} \rightarrow \mathbb{R}$ be Lebesgue measurable with $f(\cdot\omega) \in C^5([R_1, R_2])$, $\forall\ \omega \in S^{N-1}$; $\frac{\partial^k f}{\partial r^k}$, $k = 1, 2, 3, 4$, vanish on $\partial B(0, r_0)$, r_0 is fixed in $[R_1, R_2]$; and $\left(\frac{\partial^5 f}{\partial r^5} + (\alpha^2 + \beta^2)\frac{\partial^3 f}{\partial r^3} + \alpha^2\beta^2\frac{\partial f}{\partial r}\right) \in B\left(\overline{A_1}\right)$, $\left(\frac{\partial^5 f}{\partial r^5} + (\alpha^2 + \beta^2)\frac{\partial^3 f}{\partial r^3} + \alpha^2\beta^2\frac{\partial f}{\partial r}\right) \in B\left(\overline{A_2}\right)$. Here $\alpha, \beta \in \mathbb{R} : \alpha\beta(\alpha^2 - \beta^2) \neq 0$. Then for $x \in \overline{A}$ we have*

$$\left|f(x) - \frac{\int_A f(x)\,dx}{Vol(A)}\right| \leq \left|f(x) - \frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}}\int_{S^{N-1}} f(r_0\omega)\,d\omega\right| +$$

$$\left(\frac{N!}{R_2^N - R_1^N}\right)\frac{2}{|\beta^2 - \alpha^2|} \tag{57}$$

$$\left\{\left\|\frac{\partial^5 f}{\partial r^5} + (\alpha^2 + \beta^2)\frac{\partial^3 f}{\partial r^3} + \alpha^2\beta^2\frac{\partial f}{\partial r}\right\|_{\infty,\overline{A_2}}\left(\sum_{k=0}^{N-1}\frac{R_1^k(r_0 - R_1)^{N-k+3}}{k!(N-k+3)!}\right) +\right.$$

$$\left.\left\|\frac{\partial^5 f}{\partial r^5} + (\alpha^2 + \beta^2)\frac{\partial^3 f}{\partial r^3} + \alpha^2\beta^2\frac{\partial f}{\partial r}\right\|_{\infty,\overline{A_1}}\left(\sum_{k=0}^{N-1}\frac{(-1)^{N+k-1}R_2^k(R_2 - r_0)^{N-k+3}}{k!(N-k+3)!}\right)\right\}.$$

Proof. As very similar to Theorem 8 is omitted. Now it is based on Theorem 12. □

We need the following multivariate radial Ostrowski type general inequality.

Theorem 16. *([3]) Let the function $f : \overline{B(0, R)} \rightarrow \mathbb{R}$ be radial, that is there exists a function g such that $f(x) = g(r)$, where $r = |x|$, $r \in [0, R]$, $\forall\ x \in \overline{B(0, R)}$. We further assume that $g \in C^5([0, R])$, and $g^{(k)}(r_0) = 0$, $k = 1, 2, 3, 4$, where $r_0 \in [0, R]$ is fixed. Here $\alpha, \beta \in \mathbb{R} : \alpha\beta(\alpha^2 - \beta^2) \neq 0$. Then ($\forall\ \omega \in S^{N-1}$)*

$$\left|f(r_0\omega) - \frac{\int_{B(0,R)} f(y)\,dy}{Vol(B(0, R))}\right| = \left|g(r_0) - \frac{N}{R^N}\int_0^R g(s)\,s^{N-1}ds\right| \leq \tag{58}$$

$$\frac{2N!}{R^N|\beta^2 - \alpha^2|}\left\{\left\|g^{(5)} + (\alpha^2 + \beta^2)g^{(3)} + \alpha^2\beta^2 g'\right\|_{\infty,[0,r_0]}\left(\frac{r_0^{3+N}}{(3+N)!}\right)\right.$$

$$\left.\left\|g^{(5)} + (\alpha^2 + \beta^2)g^{(3)} + \alpha^2\beta^2 g'\right\|_{\infty,[r_0,R]}\left(\sum_{k=0}^{N-1}\frac{(-1)^{N+k-1}}{k!(N-k+3)!}R^k(R - r_0)^{N-k+3}\right)\right\}.$$

We finish with a non-radial necessarily Ostrowski type general inequality on the ball.

Theorem 17. *Let $f : \overline{B(0, R)} \rightarrow \mathbb{R}$ be a Lebesgue integrable function, that is not necessarily a radial function. Here $f(\cdot\omega) \in C^5([0, R])$, $R > 0$, $\forall\ \omega \in S^{N-1}$. Furthermore $\frac{\partial^k f(r_0\omega)}{\partial r^k} = 0$, $k = 1, 2, 3, 4$, $\forall\ \omega \in S^{N-1}$, where $r_0 \in [0, R]$ is fixed. Assume that $\left\|\left(\frac{\partial^5 f(t\omega)}{\partial r^5} + (\alpha^2 + \beta^2)\frac{\partial^3 f(t\omega)}{\partial r^3} + \alpha^2\beta^2\frac{\partial f(t\omega)}{\partial r}\right)\right\|_{\infty,(t\in[0,r_0])} \leq K^*$*

and $\left\| \left(\frac{\partial^5 f(t\omega)}{\partial r^5} + \left(\alpha^2+\beta^2\right) \frac{\partial^3 f(t\omega)}{\partial r^3} + \alpha^2\beta^2 \frac{\partial f(t\omega)}{\partial r} \right) \right\|_{\infty,(t\in[r_0,R])} \leq K^*, \forall\ \omega \in S^{N-1}$, *where* $K^* > 0$. *Here* $\alpha, \beta \in \mathbb{R} : \alpha\beta\left(\alpha^2-\beta^2\right) \neq 0$.

Then, for $x \in \overline{B(0,R)}$ *we have*

$$\left| f(x) - \frac{\int_{B(0,R)} f(x)\,dx}{Vol(B(0,R))} \right| \leq \left| f(x) - \frac{\Gamma\left(\frac{N}{2}\right)}{2\pi^{\frac{N}{2}}} \int_{S^{N-1}} f(r_0\omega)\,d\omega \right| +$$

$$\frac{2K^* N!}{R^N \left|\beta^2-\alpha^2\right|} \left\{ \left(\frac{r_0^{3+N}}{(3+N)!} \right) + \left(\sum_{k=0}^{N-1} \frac{(-1)^{N+k-1}}{k!\,(N-k+3)!} R^k (R-r_0)^{N-k+3} \right) \right\}. \tag{59}$$

Proof. Very similar to Theorem 11, now it is based on Theorem 16. □

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