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NON-LINEAR MODEL OF THE MACROECONOMIC SYSTEM DYNAMICS: MULTIPLIER-ACCELERATOR

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Dedicated to Professor Radu Păltănea on the occasion of his 70th anniversary

Abstract

The article discusses one of the classic problems of macroeconomic dynamics, on which the neo-Keynesian theory of growth is based, so that the interaction between the multiplier and the accelerator. A new approach was proposed to develop a mathematical model of the dynamics of the mutual multiplier, which indicates the marginal propensity to save as a result of the GDP growth, and the accelerator, which reflects the growth of capital of the national income in the multiplier. The model is based on the hypothesis of non-linear dependence of the consumption on the amount of profit. This assumes that consumption growth is limited, i.e., a saturation effect occurs. In addition, the model took into account the delayed reaction of the accelerator to the influence of the multiplier. Under building the model, the considered processes were considered continuous in time. This made it possible to provide the mathematical model "accelerator - multiplier" as a system of two differential equations of the first order. The application of the developed model to the analysis of the dynamic properties of the macroeconomic system makes it possible to evaluate the parameters at which the "multiplier-accelerator" system enters a critical state. It has been proven that the conditions for the occurrence of self-oscillations depend on the critical value of the accelerator power. The presence of a two-fold limit cycle with corresponding "soft" and "hard" modes of birth (death) of the limit cycle was also founded. The applied usefulness of the model is that the choice of

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management decisions taking into account these results allow us preventing the occurrence of bifurcations and disasters in the process of evolution of the macroeconomic systems.

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1 Introduction

The study of the "multiplier-accelerator" system, on which Neo-Keynesian theories of growth are based, and the construction of economic dynamics models which takes into account the characteristics of the interaction between accumulation and consumption are leading tasks of macroeconomics theory. One of the most difficult problems that arise when constructing mathematical models of the interaction of the investment multiplier, which shows how much the marginal propensity to save limits GDP growth, and the accelerator, which reflects the growth of the capital intensity of national income as a result of the initial investments multiplier effect, is the justification of the choice of the structure of this delay impact [23], [35], [38]. Thanks to the action of the multiplier, a change in investment causes a change in national income, which in turn causes a further change in investment, and then - a further change in income, etc. Therefore, such an interaction can lead to cyclical changes in national income over time. Besides, minor changes in the parameters characterizing the state of the macroeconomic system can lead to qualitative changes in the behavior of this system. It is the consideration of the delayed impact of additional investments on GDP growth that determines the fundamental difference between mathematical models of system dynamics and static models. In this case, the mathematical model of any economic object has the form of a system of functional (differential, integral or difference) equations. Recently, system dynamics has attracted more and more attention of researchers, since, relying on such mathematical models, it is possible to form the state policy of managing macroeconomic processes, taking into account not only the consequences for which this management is carried out, but also to determine the possibility of undesirable consequences, which will allow developing measures in advance to prevent them.

The presented work provides a Literature Review of sources that are devoted to issues of bifurcation analysis in the construction of models of nonlinear dynamics. This allows us to formulate the main problems that arise in the study of the interaction of the structural elements of the "multiplier-accelerator" macroeconomic system in conditions where such interaction is determined by the presence of non-linear dependencies. In the section Data and Methodology, a new version of the multiplier-accelerator model is proposed, in which, unlike the previous ones, the dependence of consumption on the amount of profit is described by a nonlinear function, while the behavior of the accelerator is described by a linear function. Within this model, the parameters that control the boundaries between the regions of the bifurcation diagram are defined. The Conclusions section contains generalizations about the nature of the evolution of the system, which relates the saving function, the amount of government spending, and directly the value of profit. and the direction of further improvement of the model is also determined.

2 Literature review

Within the framework of the theory of dynamic systems, the limit cycle is considered as one of the possible variants of the steady state of the system. This is a certain closed (periodic) trajectory of the vector field, in the vicinity of which there are no other periodic trajectories, and any trajectory sufficiently close to the limit cycle tends to it either in forward or reverse time.The main feature of complex systems is the non-linearity of their evolution, that is, the development of the system is not linear, but can lead to both the formation of a new ordered structure or chaos. And such structural transitions can occur with a slight change in system parameters, that is, they are the result of the implementation of bifurcations of various natures. It should be emphasized that bifurcation means a change in qualitative, not quantitative, characteristics of a dynamic system. If such properties as the period or amplitude of cycles can be attributed to the quantitative characteristics of a dynamic system, then the change of qualitative characteristics means the transition from one type of stability (or instability) of the system to another type of its stability (or instability). Accordingly, the set of parameters determining the state of the dynamic system (the position of a point in the phase space) can be divided into several subsets with different types of dynamics within each subset. The boundaries between these subsets are bifurcation boundaries. Depending on the type of changes that occur at the bifurcation boundary, bifurcations that may appear in the system are classified. Thus, the cyclicality of the development of the economic system in terms of the theory of dynamic systems can be represented as a sequence of limit cycles, and the transition from one limit cycle to another is a consequence of bifurcation changes.

The concept of "bifurcation" was introduced by Henri Poincaré in his paper in mathematical physics and developed in the classic works of O.M. Lyapunov on the theory of stability of differential equations solutions and trajectories of dynamic systems. Further theoretical studies have shown that dynamic systems of different nature have common features, for the definition of which it is appropriate to use the theory of bifurcations. Since then, system dynamics has become the subject of research by specialists in various fields of science, namely, in the field of mechanics and electromechanics [37], [41], in chemistry [40], in the field of economics [4], [10], [31], sociology [14], [19], biology and ecology [22], health care [39] and many others. The presence of bifurcations is inherent in most dynamic systems, and this should be considered the rule rather than the exception, so bifurcation analysis is very important for understanding the dynamic properties of complex systems. In this regard, the theory of nonlinear dynamic systems, and in particular the theory of bifurcations, have gone beyond the interests of a single branch of science and acquired an interdisciplinary orientation, therefore, the construction of mathematical models of complex systems of nonlinear dynamics requires the development of a universal mathematical apparatus within the limits of differential calculus.

The fact that bifurcation analysis acquires special importance in the construction of macroeconomic models of dynamics is emphasized by many theoretical economists. Nonlinear dynamic modeling is becoming a popular methodology among economists investigating various models of social networks, regional economics, and environmental economics. The reasons for such interest are the limitations of standard linear-stochastic models, that are unable to describe certain features of economic reality. In addition, there was a growing awareness that the dynamics of a complex economic system can be determined by endogenous factors [36]. The application of dynamic analysis in the economy is primarily connected with the development and flourishing of macroeconomics and, in particular, with the study of business cycles and fluctuations.

Since the presence of bifurcations disrupts the stable functioning of economic systems, which can lead to unpredictable consequences, bifurcation analysis is important not only for understanding the dynamic properties of this system, but also for choosing ways to stabilize it. The causes of structural instability can be both external [1] and internal [6]. If we consider that bifurcations are generated by endogenous mechanisms inherent in the very nature of economic systems, then by changing the parameters of the system in a certain range, it is possible to avoid undesirable irregular or even cyclical behavior. At the same time, the purpose of such research is to identify a deterministic endogenous mechanism of irregular fluctuations in the economy. As demonstrated in the paper [24], the application of the nonlinear dynamics toolkit makes it possible to develop a strategy for stabilizing an unstable Walrasian equilibrium due to a change in the income tax rate or government spending. Another example of the usefulness of this approach is the model of a medium-sized enterprise [2], [3], within which the quantitative characteristics of irregular boundary dynamics are calculated based on such Lyapunov indicators as dimensionality and entropy. This allows us to control the development of the system, to adjust the factors that serve as bifurcation parameters, and with the help of a controlling influence with delayed feedback to create the conditions for the transition from irregular limit dynamics to regular periodic ones.

Barnett and Chen [5], having reviewed a large number of macroeconomic models, came to the conclusion that no case has yet been found in which the parameter space of a macroeconomic model did not exhibit bifurcation stratification. Most often, in macroeconomic models of dynamics, there are Hopf bifurcations, when the loss of stability of the system is local, and saddle-node bifurcation, or a fold, when a pair of singular points (stable and unstable) merge into a semi-stable singular point (saddle-node) and then disappear. The following works can be cited as several examples. Thus, the work [7] compares various implementations of monetary policy in Neo-Keynesian conditions. It is determined that the transition from the optimal short-term commitment policy based on the negative feedback mechanism to the policy based on the positive feedback mechanism corresponds to a Hopf bifurcation with opposite recommendations regulation policy of such a system. In the work [33] it is shown that when justifying state policy on any issues, which is carried out on the basis of mathematical models, it is necessary to take into account the possibility of bifurcations of various types. The implementation of bifurcations can lead the system to stagnation or a rigid state, accordingly, it is necessary to adjust each parameter of the system in order to control the development of the system and return it to a normal state. As another example, we can cite the work [25], which considers a macroeconomic model of the interaction of a non-linear investment function, which is considered continuous, and a logistic production function, i.e., the IS – LM model, which is a development of the mathematical representation of Keynesian macroeconomic theory. The intersection of the investment-savings (IS) and the liquidity preference-money supply (LM) curves defines an equilibrium that occurs simultaneously in both goods and asset markets. When studying the dynamics of this system, the main attention was paid to determining the impact of the export multiplier. When conducting numerical experiments, it was found that at a relatively low value of the export multiplier, a fairly complex dependence of production on net exports was observed.

As already mentioned above, one of the urgent problems faced by researchers when building dynamics models is the determination of the structure of the time lag. For example, a delay political reaction to a change in the country's macroeconomic stability can lead to Hopf bifurcation [30]. In his work [9] carried out a macroeconomic analysis of the IS-LM model as an interaction of current and previous incomes, the relationship between which is characterized by a fixed time delay. He showed that in the case of a low tax rate and a small share of deferred tax revenues, the loci of the Hopf bifurcation points form closed curves, some of which overlap, and the equilibrium within these loci is unstable. Another example is the work [13], in which a mathematical model of interactions between an economically active population and economic growth is built. It is shown that such a dynamic system can have supercritical or subcritical Hopf bifurcations, and the stability of the bifurcation periodic solution depends on the choice of parameters characterizing the delay. Economic processes in most cases depend on past events, so one of the urgent problems faced by researchers when building dynamic models is determining the structure of the time lag. Therefore, when modeling economic phenomena, which can be discrete or continuous in time, it is natural to use differential equations with various types of delays [16], [18], [34] , etc.

In the previous works of one of the authors of this study, two mathematical models of the interaction mechanism of the multiplier and the accelerator were considered. And on the example of these models, the presence of various components of economic cycles was demonstrated both at the substantive and methodological levels. What these two models have in common is that they have a number of built-in nonlinear elements and are formed in the tradition of modern Keynesianism. The first of these models [42] is characterized by the presence of two types of delay: from the point of view of investment demand, it is a lag with the finite time of the accelerator, and from the supply side, it is a continuous distribution of delay, which is described by a linear differential equation of the first order. In other words, the main postulate in the construction of this mathematical model was the essentially non-linear dependence of the accelerator on the first derivative of profit and the linear function of consumption on the size of profit. The main result of the analysis of the "multiplier-accelerator" system according to the considered model is the detection of a stable limit cycle characterized by a "soft" mode of occurrence of self-oscillations around the equilibrium value of profit. And the stability of the limit cycle is determined by the structure of the nonlinear function of the accelerator, which tends to saturation. The second model of the macroeconomic system [43], system [44] has another basis. A significant difference from the first model is the hypothesis of a continuously distributed lag of the accelerator. As for the action of the multiplier, a continuously distributed delay in the form of a first-order differential equation describing the evolution of profit was also used in relation to it. However, unlike the first model, the consumption function was considered as a non-linear profit function. That is, there are two nonlinear dependencies both on the side of the multiplier and on the side of the accelerator. The presence of these two fundamentally different nonlinearities, as well as their dynamic interaction, suggest the appearance of stable and unstable limit cycles, which, when combined, generate a double limit cycle. Thus, the second model can generate a "hard" self-oscillation excitation with a catastrophic loss of stability. In general, this model of the macroeconomic system has a more complex dynamic behavior compared to the previous model.

The purpose of this work is to analyze the interaction of the structural elements of the "multiplier-accelerator" macroeconomic system in the presence of nonlinear relationships. Such an interaction can lead to the formation of a complex architecture of instability with the appearance of bifurcations of various types and catastrophes, limit cycles, both stable and unstable, or homoclinic structures. Such models are focused on the implementation of qualitative forecasting of the nonlinear evolution of the macroeconomic system in order to identify equilibrium states, predict the appearance of dynamic chaos, and so on.

3 Data and methodology

Let us consider one more model, i.e. the third, version of dynamics of the multiplier-accelerator interaction. Unlike the previous two, it will use the nonlinear dependence of the consumption function on the amount of profit and the simple linear dependence of the accelerator on the derivative of the profit within one dynamic system.

First, let's form a functional equation to describe the dynamics of the multiplier. Aggregate demand is given in the form of the well-known Keynesian ratio:

$$D(Y) = C(Y) + I + G \tag{1}$$

where Y is the amount of profit; I is the volume of investment contributions;

G is independent government expenditures; $C(Y)$ is consumption function; $D(Y)$ is aggregate macroeconomic demand. Suppose that $C(Y)$ is a non-linear function that grows more slowly than a linear one, which corresponds to the so-called "saturation effect", which is associated with the fact that consumption growth is limited. Let us represent the function $C(Y)$ in the form of a polynomial whose degree is not higher than the third:

$$C(Y) = c_1 Y + c_2 \frac{Y^2}{2} - c_3 \frac{Y^3}{6} \tag{2}$$

where all coefficients c_i $(i = \overline{1,3}.)$ are positive, and their values satisfy the condition: $0 < c_i < 1$. It should be emphasized that we consider a real situation when the growth of the consumption function is limited, that is, consumption does not exceed a certain level. Therefore, the second derivative of the consumption function must be negative for the function itself to be convex upward. In order for the growth to slow down, the value of the profit of the third degree with a "minus" sign is entered in the polynomial (2), and the coefficient itself is positive.

The amount of government expenditures, without violating commonality, will be considered constant. The dynamics of a multiplier with a distributed delay can be represented in the form of an integral relation:

$$Y(t) = \int_0^t K_1(t,\ \tau) \cdot D\left(Y(\tau)\right) \cdot d\tau. \tag{3}$$

This means that the profit that exists at a fixed point in time τ depends on all past values of aggregate demand. The function $K_1(t,\ \tau)$, which is the kernel of this integral transformation, is called weight, or "dynamic memory function". As a rule, such functions are decreasing with respect to past moments of time.

Regarding the investment function $I = I(t)$, we note that there is also a distributed delay here, which we give in integral form:

$$I(t) = \int_0^t K_2(t,\ \tau) \cdot v_0 \cdot Y'(\tau) \cdot d\tau. \tag{4}$$

where $Y'(\tau)$ is derivative of the product; v_0 is the power of the accelerator, which has the dimension of time; $K_2(t,\ \tau)$ is the corresponding kernel, to which the same assumptions apply as to the kernel $K_1(t,\ \tau)$.

Thus, if we combine the algebraic equation (1) and the integral equations (3) and (4) into one system, we will have a system of integral-differential equations from which the amount of profit $Y(t)$ and the volume of investments $I(t)$ can be determined. This approach allows you to build a set of different mathematical models, using various structures of kernels of integral transformations $K_1(t,\ \tau)$ and $K_2(t,\ \tau)$. As the simplest types of kernels $K_1(t,\ \tau)$ and $K_2(t,\ \tau)$ we will take functions with an exponential delay relative to previous moments of time:

$$K_1(t,\ \tau) = \frac{1}{T_1} \cdot \exp\left(-\frac{t-\tau}{T_1}\right); \tag{5}$$

$$K_2(t,\ \tau) = \frac{1}{T_2} \cdot \exp\left(-\frac{t-\tau}{T_2}\right) \tag{6}$$

where T_1 and T_2 are fixed values of certain time constants that characterize transient processes, respectively, in the multiplier and accelerator

In this case, the integral relations (3) and (4) taking into account equation (1) can be transformed into the form:

$$Y(t) = \int_0^t \frac{1}{T_1} \cdot \exp\left(-\frac{t-\tau}{T_1}\right) \cdot \left(C\left(Y(t)\right) + I(\tau) + G\right) \cdot d\tau \tag{7}$$

$$I(t) = \int_0^t \frac{1}{T_2} \cdot \exp\left(-\frac{t-\tau}{T_2}\right) \cdot v_0 \cdot Y'(t) \cdot d\tau \tag{8}$$

If we differentiate in time relations (7) and (8) and perform the necessary transformations, we will obtain a system of two ordinary differential equations with respect to the unknowns $Y(t)$ and $I(t)$:

$$\begin{cases} T_1 \cdot Y' = C(Y) - Y + I + G; \\ T_2 \cdot I' = v_0 \cdot Y' - I. \end{cases} \tag{9}$$

Thus, due to differentiation in time of the integral relations, the transition from the original integral relations to the system of differential equations is carried out.

In order to reduce the number of parameters in the system of differential equations (9), we will introduce relative time $\overline{t} = \frac{t}{T_2}$, as well as non-dimensional parameters $v = \frac{v_0}{T_2}$ and $\gamma = \frac{T_2}{T_1}$. In this case, the system of equations takes the following form:

$$\begin{cases} Y' = \gamma \cdot \left(C(Y) - Y + I + G\right); \\ I' = v \cdot Y' - I. \end{cases} \tag{10}$$

Let's reduce the system of two differential equations (10) to one differential equation of the second order with respect to the profit function $Y(t)$. For this purpose, we will introduce the savings function $S(Y) = Y - C(Y)$, which we will use later. Thanks to this, first equation of the system (10) takes the following form:

$$Y' = \gamma \cdot \left(I - S(Y) + G\right). \tag{11}$$

We differentiate equation (11) by time (in the future, we will keep in mind that the differentiation of all functions is carried out by time; in cases where the differentiation is carried out by another variable, this will be indicated separately).

After performing the transformations taking into account equation (10), we will obtain a nonlinear differential equation of the second order:

$$Y'' + \big(1 + \gamma \cdot \big(S'(Y) - v\big)\big) \cdot Y' + \gamma \cdot (S(Y) - G) = 0. \tag{12}$$

where $S'(Y)$ is the derivative of the savings function of income.

We present the savings function and its profit derivative as it is used in the model proposed in this paper:

$$S(Y) = (1 - c_1)Y - c_2 \frac{Y^2}{2} + c_3 \frac{Y^3}{6} \tag{13}$$

$$S'(Y) = 1 - c_1 - c_2 Y + c_3 \frac{Y}{2} \tag{14}$$

The differential equation (12) can have special solutions that are determined based on the condition $S(Y) = G$:

$$c_3 \frac{Y^3}{6} - c_2 \frac{Y^2}{2} + (1 - c_1)Y - G = 0. \tag{15}$$

Analysis of the structure of the cubic equation (15) shows that it has at least one positive root Y^*, which corresponds to the state of equilibrium of the macroeconomic system. Let's introduce a new variable $\tilde{Y} = Y - Y^*$ that determines the deviation of profit from its value corresponding to the equilibrium state. For the following transformations, we will need the values of the three derivatives of the savings function, which correspond to the equilibrium state Y^*. Here are these derivatives:

$$S_1 = S'(Y) = 1 - c_1 - c_2 Y^* + c_3 \frac{(Y^*)^2}{2}; \tag{16}$$

$$S_2 = S''(Y^*) = c_3 Y^* - c_2 \tag{17}$$

$$S_3 = S'''(Y^*) = c_3 \tag{18}$$

After performing the necessary transformations, we obtain the differential equation with respect to the deviation from the product in an explicit form:

$$\tilde{Y}'' + (1 - \gamma v + \gamma S_1)\,\tilde{Y}' + \gamma S_1 \tilde{Y} + \gamma S_2 \tilde{Y} \cdot \tilde{Y}' + \gamma S_2 \frac{\tilde{Y}^2}{2} + \\ + \gamma S_3 \frac{\tilde{Y}^2 \cdot \tilde{Y}'}{2} + \gamma S_3 \frac{\tilde{Y}^3}{6} = 0 \tag{19}$$

The nonlinear differential equation of the second order (19) will be presented in the form of a system of two ordinary differential equations of the first order with respect to the variables $y_1 = \tilde{Y}$ and $y_2 = \tilde{Y}'$:

$$\begin{cases} y'_1 = y_2; \\ y'_2 = -\gamma S_1 y_1 - (1 + \gamma (S_1 - v)) y_2 - \gamma S_2 y_1 y_2 - \gamma S_2 \frac{y_1^2}{2} - \\ \qquad -\gamma S_3 \frac{y_1^2 \cdot y_2}{2} - \gamma S_3 \frac{y_1^3}{6}. \end{cases} \tag{20}$$

The linear part of system (20) has a characteristic polynomial:

$$\lambda^2 + (1 + \gamma (S_1 - v)) \cdot \lambda + \gamma S_1 = 0 \tag{21}$$

Due to the fact that the value S_1, which can be considered as the marginal propensity to save, is a positive value, the real parts of the roots λ_1 and λ_2 equation (20) will have the same signs. This, in turn, means that the type of equilibrium can be a stable (unstable) node or a focus. Since we are interested in periodic processes described by system (20), we will assume the following:

$$1 + \gamma (S_1 - v) = 2\mu. \tag{22}$$

where a parameter μ is a small variable that can be both positive and negative. If we denote by $\omega^2 = \gamma S_1$, then equation (21) with respect to eigenvalues λ_1 and λ_2 will have the form:

$$\lambda^2 - 2\mu \cdot \lambda + \omega^2 = 0 \tag{23}$$

The solution of this quadratic equation can be given in the complex form:

$$\lambda_{1,2} = \mu \pm i \cdot \omega \tag{24}$$

That is, the roots of the characteristic equation (23) contain a small parameter μ only in the first degree, and the values μ in degrees greater than the first can be neglected.

It should be noted that since $\omega^2 = \gamma S_1 > 0$, only nodal or focal types of equilibrium states can occur in the dynamic economic system considered in our study. That is, neither homoclinic nor heteroclinic trajectories can be observed in the studied system for the reason that there are no saddle points among the equilibrium positions.

The study of local bifurcations of the vector field was carried out for the purpose of a detailed analysis of the stability of equilibrium positions and construction of the boundaries of the stability region. In the system under consideration, the existence of both stable and unstable trajectories and orbits is possible.

It is clear that the type of equilibrium for $\mu < 0$ is a stable focus, and for $\mu > 0$, accordingly, is unstable. Thus, the linear operator of this nonlinear system of differential equations has two eigenvalues that correspond to both a stable focus and an unstable one. The derivative of eigenvalues λ_1 and λ_2 by parameter μ is equal to one: $\lambda'_{1,2} = 1$. Therefore, it is possible to assume that all the conditions of Hopf bifurcation theory [20], [21] about the existence of limit cycles in the dynamic system (20) are fulfilled. From relation (22) we get:

$$v = S_1 + \frac{1+2\mu}{\gamma} \tag{25}$$

Assuming $\mu = 0$ and returning to the original notation, we get:

$$v_0 = S_1 \cdot T_2 + T_1 \tag{26}$$

Therefore, formula (26), which has the dimension of time, describes the dependence of the accelerator power v_0 on the marginal propensity to save and the corresponding time constants T_1 and T_2. According to formula (26), the critical value of the accelerator power is a parameter at which the equilibrium value of the profit Y^* changes its stability: an unstable focus becomes stable or vice versa depending on the sign of the small parameter μ. At the same time, if $\mu = 0$, undamped oscillations with frequency ω are observed.

We remind you that in this system there can be neither homoclinic nor heteroclinic bifurcations, which is due to the structure of the model and limitations on the parameters.

To further study the mechanism of the birth (death) of the limit cycle around the equilibrium value, it is necessary to bring the system (20) to the appearance of the Poincaré normal form. In the study of nonlinear vector field Normal form theory is widely used in order to simplify the analysis of the original system of differential equations [8], [32]. Let's do this conversion by replacing the variables $y_1 = x_1$ and $y_2 = \mu x_1 - \omega x_2$. After algebraic transformations, we obtain the following system of differential equations:

$$\begin{cases} x'_1 = \mu x_1 - \omega x_2; \\ x'_2 = \omega x_1 - \mu x_2 - \gamma S_2 x_1 x_2 + \frac{\gamma S_2}{\omega}\left(1+2\mu\right)\frac{x_1^2}{2} \\ \qquad -\gamma S_3 \cdot \frac{x_1^2 x_2}{2} + \frac{\gamma S_3}{\omega}\left(1+3\mu\right)\frac{x_1^3}{6}. \end{cases} \tag{27}$$

Let's once again introduce a new time scale $\bar{t} = \omega \cdot \bar{\bar{t}}$. In this case, system (27) takes the form:

$$\begin{cases} x'_1 = \frac{\mu}{\omega} x_1 - x_2; \\ x'_2 = x_1 - \frac{\mu}{\omega} x_2 - \frac{\gamma S_2}{\omega} x_1 x_2 + \frac{\gamma S_2}{\omega^2}\left(1+2\mu\right)\frac{x_1^2}{2} - \\ \qquad -\frac{\gamma S_3}{\omega} \cdot \frac{x_1^2 x_2}{2} + \frac{\gamma S_3}{\omega^2}\left(1+3\mu\right)\frac{x_1^3}{6}. \end{cases} \tag{28}$$

For the convenience of further transformations, we introduce the following notations:

$$b_{11} = \frac{\gamma S_2}{\omega}; \quad b_{20}(\mu) = \frac{\gamma S_2}{\omega^2}\left(1+2\mu\right); \quad b_{21} = \frac{\gamma S_3}{\omega}; \quad b_{30}(\mu) = \frac{\gamma S_3}{\omega^2}\left(1+3\mu\right).$$

Taking into account the new notations, system (28) will have the form:

$$\begin{cases} x'_1 = \frac{\mu}{\omega}x_1 - x_2; \\ x'_2 = x_1 - \frac{\mu}{\omega}x_2 - b_{11}x_1x_2 + b_{20}(\mu)\cdot\frac{x_1^2}{2} - b_{21}\frac{x_1^2x_2}{2} + b_{30}(\mu)\cdot\frac{x_1^3}{6}. \end{cases} \tag{29}$$

The system of two differential equations (29) can be reduced to one complex-valued differential equation with respect to the variable $z = x_1 + i\cdot x_2$:

$$z' = \left(\frac{\mu}{\omega} + i\right)z + g_{20}\frac{z^2}{2} + g_{11}z\cdot\bar{z} + g_{02}\frac{\bar{z}^2}{2} + g_{30}\frac{z^3}{6} + g_{21}\frac{z^2\cdot\bar{z}}{2} + g_{12}\frac{z\cdot\bar{z}^2}{2} + g_{03}\frac{\bar{z}^3}{6}. \tag{30}$$

where

$$\bar{z} = x_1 - i\cdot x_2; \quad g_{11} = i\frac{b_{20}}{4}; \quad g_{20} = -b_{11} + i\frac{b_{20}}{4} \quad g_{02} = \frac{b_{11}}{2} + i\frac{b_{20}}{4};$$

$$g_{30} = \frac{i\cdot b_{30} - 3b_{21}}{8}; \; g_{21} = \frac{i\cdot b_{30} - b_{21}}{8}; \; g_{12} = \frac{i\cdot b_{30} + b_{21}}{8}; \; g_{03} = \frac{i\cdot b_{30} + 3b_{21}}{8}$$

The nature of the stability of the limit cycle is determined by the sign of the first Lapunov quantity. The limit cycle is stable if the first Lyapunov value is negative, and unstable if the last value is positive. So it is necessary to determine the type of stability of the limit cycle using the first Lyapunov quantity [11] and [27], which has the form:

$$L_1(\mu) = \frac{g_{21}}{2} + \frac{g_{20}g_{11}\left(2\lambda + \bar{\lambda}\right)}{2\cdot|\lambda|^2} + \frac{|g_{11}|^2}{\lambda} + \frac{|g_{02}|^2}{2\left(2\lambda - \bar{\lambda}\right)}, \tag{31}$$

where $\lambda = \frac{\mu}{\omega} + i$ and $\bar{\lambda} = \frac{\mu}{\omega} - i$.

Let's calculate the real part of the first Lyapunov quantity. Applying complex arithmetic, after transformations we get:

$$l_1 = \operatorname{Re} L_1(\mu) = \frac{\gamma^2}{\omega^3}\left(\frac{S_2^2 - S_1S_3}{16} + \frac{5S_2^2}{36}\left(1 - \frac{1}{\omega^2}\right)\mu\right). \tag{32}$$

If we assume that the value $\upsilon = S_2^2 - S_1S_3$ is small and changes in sign, then the value $l_1(\mu,\ \upsilon)$ is also small around zero. Accordingly, we have:

$$l_1(\mu,\ \upsilon) = \frac{\gamma^2}{16\omega^3}\left(\upsilon + \frac{20S_2^2}{9}\left(1 - \frac{1}{\omega^2}\right)\mu\right). \tag{33}$$

It follows from formulas (32) and (33) that the limit cycle is stable at $l_1 < 0$ and unstable at $l_1 > 0$. This conclusion suggests the existence of two cycles with different types of stability at the same time. For this, to check this assumption, it is necessary to find the second Lyapunov quantity l_2 at $\mu = 0$ and $\upsilon = 0$. According to works [28], [29], we will have:

$$l_2(0) = \frac{-5(\gamma S_3)^2}{576\omega^3} < 0. \tag{34}$$

The fact that we found a fixed value $l_2(0)$ proves that there are two limit cycles in this dynamic system.

After applying the polar coordinates $z = \rho \cdot e^{i\varphi}$ to the differential equation (30), we will make the transition from the normal Poincaré form to the canonical form [17]:

$$\begin{cases} \rho' = \rho\left(\frac{\mu}{\omega} + l_1(\mu,\ \upsilon) \cdot \rho^2 + l_2(0) \cdot \rho^4\right); \\ \varphi' = 1. \end{cases} \tag{35}$$

Equations in system (35) are independent. The second equation of the system describes the rotational motion with unit speed (frequency). The trivial solution of the first equation corresponds to the trivial equilibrium $x_1 = 0$ and $x_2 = 0$ ($\rho = 0$). The positive roots of the first equation must satisfy the equilibrium condition:

$$\frac{\mu}{\omega} + l_1(\mu,\ \upsilon) \cdot \rho^2 + l_2(0) \cdot \rho^4 = 0 \tag{36}$$

In this case, they determine the amplitude of limit cycles. Let's rewrite (36) in the form:

$$\theta_1 + \theta_2\rho^2 - \rho^4 = 0, \tag{37}$$

where $\theta_1 = \frac{576\mu\omega^2}{5(\gamma S_3)^2}$ and $\theta_2 = \frac{\left(36\upsilon + 80S_2^2\left(1 - \frac{1}{\omega^2}\right)\mu\right)}{5S_3^2}$ are small parameters.

Equation (37) may have one or two positive roots or none, corresponding to the number of limit cycles. These solutions move away from the trivial solution along the line $H : \theta_1 = 0$ and merge and vanish on the parabola $G : \theta_2^2 + 4\theta_1 = 0$, where $\theta_2 > 0$. Figure 1 shows a bifurcation diagram on the plane of parameters θ_1 and θ_2.

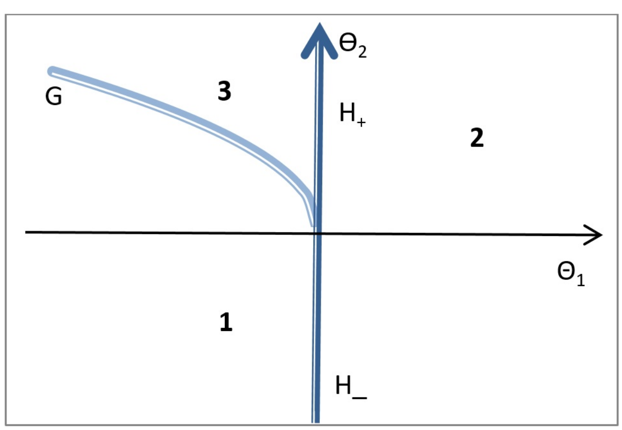

Figure 1: Bifurcation diagram of a double cycle on the plane of parameters θ_1 and θ_2.

The line $H : \theta_1 = 0$ corresponds to the Hopf bifurcation. Along this line, the equilibrium position is determined by the eigenvalues $\lambda_{1,2} = \pm i$ ($\mu = 0$; $\theta_1 = 0$). The equilibrium is stable if $\theta_1 < 0$ and unstable if $\theta_1 > 0$. The first Lyapunov quantity has the value $l_1 = \theta_2$. If there is $\theta_2 < 0$, a stable limit cycle with a "soft" mode of self-oscillation is observed on the line H_-. If there is $\theta_2 > 0$, an unstable limit cycle with "hard" excitation of the periodic regime is observed on the line H_+. A stable limit cycle is born from equilibrium if the line H_- crosses from left to right. An unstable limit cycle occurs when the line H_+ crosses in the opposite direction. Both cycles merge and disappear on the line G. Lines H and G divide the parameter plane into three regions. If you start from region 1, then the system has a resistant state of equilibrium and no cycles. Crossing the line H_- from region 1 to region 2 leads to the birth of a stable limit cycle. When moving from region 2 to region 3, an unstable limit cycle appears inside a stable one. Two cycles of opposite direction coexist in region 3, and merge and disappear on the line G. These results of the bifurcation analysis are similar to those, which were published for the double limit cycle in the work [26].

Of particular interest is the interpretation of a small parameter $\upsilon = S_2^2 - S_1 S_3$ that is determined exclusively by the properties of the savings function $S(Y)$. The construction of the parameter υ with accuracy up to a constant factor corresponds to the so-called Schwartz derivative [15] of the savings function. When $\mu = 0$ we have the value of the Schwartz derivative $\upsilon < 0$. This means stability of the limit cycle. And when $\upsilon > 0$, accordingly, the limit cycle is unstable. It should be noted that υ is an exclusively static parameter [12] that depends exclusively on the structure of the savings function $S(Y)$. When $\upsilon = 0$ we have that $S_2^2 = S_1 S_3$.

Let's return to the original ratios (16-18) that determine the derivatives of the savings function. Accordingly, we get the ratio:

$$(c_3 Y^* - c_2)^2 = c_3 \left(1 - c_1 - c_2 Y^* + \frac{c_3 (Y^*)^2}{2} \right). \tag{38}$$

After transformations, the function looks like this:

$$(c_3 Y^* - c_2)^2 = 2 c_3 (1 - c_1) - c_2^2. \tag{39}$$

Using relation (39), it is easy to determine the equilibrium value Y^*:

$$Y^* = \frac{c_2 + \sqrt{2 c_3 (1 - c_1) - c_2^2}}{c_3}. \tag{40}$$

Let's consider the influence of the parameter c_2 on the properties of the equilibrium value Y^* and peculiarities of macroeconomic dynamics. The coefficient c_2 performs the function of the shift parameter of the equilibrium value Y^* and no more. Therefore, without violating generality, we will assume that $c_2 = 0$. Then from relation (40) we get:

$$Y^* = \sqrt{\frac{2(1 - c_1)}{c_3}}. \tag{41}$$

Using relation (39), it is possible to find the dependence of the constant level of government spending G on the parameters c_1 and c_3 of the consumption function $C(Y)$. From the equation

$$(1 - c_1)\, Y^* + \frac{c_3 (Y^*)^3}{6} = G, \tag{42}$$

we find that

$$G = \sqrt{\frac{8(1 - c_1)^3}{9c_3}}. \tag{43}$$

So, we see that the equilibrium value of the profit Y^* (41) and the value of the function G (43), which defines the border between the regions of the bifurcation diagram of a double cycle, on which the stable and unstable limit cycles merge and disappear, depend only on the coefficients c_1 and c_3 of the third-degree polynomial, with the help of which the consumption function $C(Y)$ was presented (2).

4 Further considerations

In the future, it is advisable to conduct a study of integral ratios between profit and investment using the structure of kernels that differ from the exponent. In this regard, it can be expected that the order of differential equations describing the state of the macroeconomic system will increase to the third or more, which, in turn, will create prerequisites for the appearance of dynamic chaos in the system and will significantly affect the procedure for determining the forecasting horizon.

5 Conclusions

In this work, a study of the mathematical model of the dynamics of the macroeconomic system “multiplier-accelerator” was carried out. Unlike the models considered earlier, this model contains in the consumption function a single nonlinearity in the equation describing the mechanism of action of the multiplier, while the structure of the accelerator is linear. The distributed delay was introduced through the corresponding integral relations, with the help of which a mathematical model of the dynamics of the macroeconomic system was obtained and substantiated in the form of a system of two differential equations of the first order. The analysis of the behavioral properties of the nonlinear dynamic system under study made it possible to identify the conditions for the occurrence of the self-oscillation mode. It is proved that these conditions depend on the critical value of the accelerator power. The conducted studies showed the presence of a two-fold limit cycle with corresponding ”soft” and ”hard” modes of birth (death) of the limit cycle. A condition for separation of the oscillatory behavior of a system with opposite types of stability is a change in the sign of the Schwartz derivative of the savings function. For a two-fold limit cycle in the neighborhood of the equilibrium value of profit, parametric conditions are obtained that connect the saving function, the amount of government expenditures, and directly the value of profit at the equilibrium point.

Analyzing macroeconomic dynamics using the proposed mathematical model makes it possible to identify the values of the parameters at which the system enters a critical state. Taking into account the possible existence of self-oscillations of different stability and determining the parameters of their appearance, when making management decisions, it is possible to prevent the occurrence of bifurcations and disasters in the process of the evolution of the macroeconomic system. As a further development of the proposed model of nonlinear economic dynamics, it is expedient to introduce another periodic component into consideration, namely, to include it in the structure of public expenditures. In this case, we can expect nonlinear resonance and chaotic profit behavior.

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