Bulletin of the *Transilvania* University of Braşov
Series III: Mathematics and Computer Science, Vol. 3(65), No. 2 - 2023, 153-158
https://doi.org/10.31926/but.mif.2023.3.65.2.13

ON A THIRD-ORDER MODULUS OF SMOOTHNESS

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Dedicated to Professor Radu Păltănea on the occasion of his 70th anniversary

Abstract

In this paper we define a new third-order modulus of smoothness and we prove the conservation by Stancu operators of the Lipschitz classes defined with this modulus.

2000 *Mathematics Subject Classification:* 41A36, 41A17.
Key words: Bernstein-Stancu operators, global smoothness preservation, modulus of continuity.

1 Introduction

Let $(L_n)_{n\in\mathbb{N}}$ be a sequence of linear positive operators of approximation on $C[0,1]$. If global smoothness of a continuous function f is expressed by a Lipschitz condition with some modulus of continuity, it is of interest if L_nf verify same condition.

The preservation of global smoothness properties by the Bernstein operators

$$B_n(f,x)=\sum_{j=0}^{n}p_{n,j}(x)f\left(\frac{j}{n}\right),\ f\in C[0,1],\ x\in[0,1],$$

$$p_{n,j}(x)=\binom{n}{j}x^j(1-x)^{n-j},$$

were studied in [6], [7], [4], [2], [5], [3]. In [14], D.-X. Zhou showed that the Lipschitz classes with respect to the second order modulus

$$\omega_2(f,t)=\sup\left\{|f(x-h)-2f(x)+f(x+h)|:\ x\pm h\in[0,1],\ 0<h\leq t\right\}$$

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are not preserved by the Bernstein operators. He introduced the following modulus of smoothness of order two

$$\begin{aligned}\widetilde{\omega}_2(f,t) &= \sup\{|f(x+h_1+h_2)-f(x+h_1)-f(x+h_2)+f(x)| : \\ &\qquad x, x+h_1+h_2 \in [0,1]\,,\, h_1,h_2>0,\, h_1+h_2 \le 2t\} \qquad (1)\end{aligned}$$

and proved the preservation by Bernstein's operators of Lipschitz classes defined with this modulus.

For the Bernstein-type operators

$$L_n(f,x)=\sum_{j=0}^{n} p_{n,j}(x)F_{n,j}(f),\ f\in C[0,1],\ x\in[0,1], \qquad (2)$$

where $F_{n,j} : C[0,1] \longrightarrow \mathbb{R}$, $j = \overline{1,n}$, are linear positive functionals, in [11] we studied simultaneous global smoothness preservation in terms of modulus of continuity ω_2^* introduced by R. Păltănea [8], [9] and independently by J. Adell and J. de la Cal [1], defined for $f \in \mathbf{C}[0,1]$ and $t > 0$ by

$$\begin{aligned}\omega_2^*(f,t) &= \sup\{|(1-\lambda)f(x)+\lambda f(y)-f\left((1-\lambda)x+\lambda y\right)| : \\ &\qquad x,\, y\in[0,1]\,,\, x<y,\, y-x\le 2t,\, \lambda\in[0,1]\}. \qquad (3)\end{aligned}$$

In [13] we proved the preservation of Liplschitz classes by the following generalized Bernstein operators which were introduced by D. D. Stancu (see [10])

$$S_{n,r,s}(f,x)=\sum_{j=0}^{n-rs} p_{n-rs,j}(x)\sum_{i=0}^{s} p_{s,i}(x) f\left(\frac{j+ir}{n}\right), \qquad (4)$$

$f \in C[0,1]$, $x \in [0,1]$, where $n \in \mathbb{N}$, $r,s \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}$ such that $rs < n$. Bernstein's operators are obtained for $s=0$ or $s=1$, $r=0$ or $s=1$, $r=1$.

In the next section we define a new third-order modulus of smoothness and we prove the conservation by Stancu's operators of the Lipschitz classes in terms of this modulus.

2 Main result

Definition 1. *For $f \in C[0,1]$ and $t>0$ we define*

$$\begin{aligned}\omega_3^*(f,t) &= \sup\{|(1-2\lambda)f(y)-f\left(\lambda x+(1-\lambda)y\right)+f\left((1-\lambda)x+\lambda y\right) \\ &\qquad -(1-2\lambda)f(x)| :\ x,\, y\in[0,1]\,,\, x<y,\, y-x\le 3t,\, \lambda\in[0,1]\}. \ (5)\end{aligned}$$

Remark 1. *The application*

$$\omega_3^* : C[0,1]\times(0,\infty)\longrightarrow[0,\infty)$$

is a modulus of continuity of order 3 normalized on $C[0,1]$ in the sense of the definition from [9], that is, the following conditions are verified:

(MC1) $\omega_3^*(f,t_1) \le \omega_3^*(f,t_2)$, $f \in C[0,1]$, $0 < t_1 < t_2$;

(MC2) $\omega_3^*(f+p,t) = \omega_3^*(f,t)$, $f \in C[0,1]$, *p algebraic polynomial of degree at most 2*, $t > 0$;

(MC3) $\omega_3^*(0,t) = 0$, $t > 0$.

(N) $(\exists)C > 0$, $(\forall)t > 0 : \omega_3^*(e_3,t) \le C \cdot t^3$,

Indeed, properties $(MC1)$ *and* $(MC3)$ *are clear. Property* $(MC2)$ *results from the fact that for the monomial functions* $e_i(x) = x^i$, $i = 0,1,2$ *we have*

$$(1-2\lambda)e_i(y) - e_i(\lambda x + (1-\lambda)y) + e_i((1-\lambda)x + \lambda y) - (1-2\lambda)e_i(x) = 0, \; i = 0,1,2.$$

Since $(1-2\lambda)e_3(y) - e_3(\lambda x + (1-\lambda)y) + e_3((1-\lambda)x + \lambda y) - (1-2\lambda)e_3(x) = \lambda(1-\lambda)(1-2\lambda)(y-x)^3$, *it follows that* $\omega_3^*(e_3,t) \le 9\sqrt{3} \cdot t^3$, $(\forall)t > 0$, *therefore the modulus of continuity is normalized.*

To prove the global smoothness preservation by Stancu operators in terms of modulus of continuity ω_3^* we need a representation of Stancu's operators.
For the Bernstein operator in [11], proceeding similarly as in [14] (see also [4]), we obtained:

$$B_n(f,(1-\lambda)x + \lambda y) = \sum_{k+l=0}^{n} p_{n,k,l}(x, y-x) \sum_{m=0}^{l} p_{l,m}(\lambda) f\left(\frac{k+m}{n}\right), \tag{6}$$

where $p_{n,k,l}(x,y) = \dfrac{n!}{k!l!\,(n-k-l)!} x^k y^l (1-x-y)^{n-k-l}$ is the two-variable Bernstein basis. Repeating the application of an adapted version of relation (6) yields [13] (see also [12])

$$\begin{aligned} &S_{n,r,s}(f,(1-\lambda)x + \lambda y) \\ &= \sum_{k_1+l_1=0}^{s} \sum_{k_2+l_2=0}^{n-rs} p_{s,k_1,l_1}(x,y-x) p_{n-rs,k_2,l_2}(x,y-x) \cdot \\ &\cdot \sum_{m_1=0}^{l_1} \sum_{m_2=0}^{l_2} p_{l_1,m_1}(\lambda) p_{l_2,m_2}(\lambda) f\left(\frac{k_2+m_2+r(k_1+m_1)}{n}\right). \end{aligned} \tag{7}$$

Theorem 1. *Let* $f \in C[0,1]$, $M > 0$ *and* $\alpha \in (0,1]$. *If*

$$\omega_3^*(f,t) \le Mt^\alpha, \; t \in \left(0, \frac{1}{3}\right],$$

then

$$\omega_3^*(S_{n,r,s}f,t) \le Mt^\alpha, \; t \in \left(0, \frac{1}{3}\right].$$

Proof. Let $t \in \left(0, \frac{1}{3}\right]$, $x, y \in [0,1]$, $x < y$, $y - x \leq 3t$, $\lambda \in [0,1]$. By using the representation (7), we obtain:

$$\Big|(1-2\lambda)S_{n,r,s}(f,y) - S_{n,r,s}\left(f, \lambda x + (1-\lambda)y\right)$$

$$+S_{n,r,s}\left(f,(1-\lambda)x+\lambda y\right) - (1-2\lambda)S_{n,r,s}(f,x)\Big|$$

$$\leq \sum_{k_1+l_1=0}^{s} \sum_{k_2+l_2=0}^{n-rs} p_{s,k_1,l_1}(x,y-x)p_{n-rs,k_2,l_2}(x,y-x)\cdot$$

$$\cdot\Big|(1-2\lambda)f\left(\frac{k_2+l_2+r(k_1+l_1)}{n}\right)$$

$$-\sum_{m_1=0}^{l_1}\sum_{m_2=0}^{l_2} p_{l_1,m_1}(1-\lambda)p_{l_2,m_2}(1-\lambda)f\left(\frac{k_2+m_2+r(k_1+m_1)}{n}\right)$$

$$+\sum_{m_1=0}^{l_1}\sum_{m_2=0}^{l_2} p_{l_1,m_1}(\lambda)p_{l_2,m_2}(\lambda)f\left(\frac{k_2+m_2+r(k_1+m_1)}{n}\right)$$

$$-(1-2\lambda)f\left(\frac{k_2+rk_1}{n}\right)\Big|$$

$$= \sum_{\substack{k_1+l_1=0 \\ l_2+rl_1\neq 0}}^{s} \sum_{k_2+l_2=0}^{n-rs} p_{s,k_1,l_1}(x,y-x)p_{n-rs,k_2,l_2}(x,y-x)\cdot$$

$$\cdot\Big|\sum_{m_1=0}^{l_1}\sum_{m_2=0}^{l_2} p_{l_1,m_1}(\lambda)p_{l_2,m_2}(\lambda)\left(1-2\frac{m_2+rm_1}{l_2+rl_1}\right)f\left(\frac{k_2+l_2+r(k_1+l_1)}{n}\right)$$

$$-\sum_{m_1=0}^{l_1}\sum_{m_2=0}^{l_2} p_{l_1,m_1}(\lambda)p_{l_2,m_2}(\lambda)f\left(\frac{k_2+l_2-m_2+r(k_1+l_1-m_1)}{n}\right)$$

$$+\sum_{m_1=0}^{l_1}\sum_{m_2=0}^{l_2} p_{l_1,m_1}(\lambda)p_{l_2,m_2}(\lambda)f\left(\frac{k_2+m_2+r(k_1+m_1)}{n}\right)$$

$$-\sum_{m_1=0}^{l_1}\sum_{m_2=0}^{l_2} p_{l_1,m_1}(\lambda)p_{l_2,m_2}(\lambda)\left(1-2\frac{m_2+rm_1}{l_2+rl_1}\right)f\left(\frac{k_2+rk_1}{n}\right)\Big|$$

$$\leq \sum_{\substack{k_1+l_1=0 \\ l_2+rl_1\neq 0}}^{s} \sum_{k_2+l_2=0}^{n-rs} p_{s,k_1,l_1}(x,y-x)p_{n-rs,k_2,l_2}(x,y-x)\cdot$$

$$\cdot\sum_{m_1=0}^{l_1}\sum_{m_2=0}^{l_2} p_{l_1,m_1}(\lambda)p_{l_2,m_2}(\lambda)\cdot\Big|\left(1-2\frac{m_2+rm_1}{l_2+rl_1}\right)f\left(\frac{k_2+rk_1}{n}+\frac{l_2+rl_1}{n}\right)$$

$$-f\left(\frac{k_2+rk_1}{n}+\frac{l_2-m_2+r(l_1-m_1)}{n}\right)+f\left(\frac{k_2+rk_1}{n}+\frac{m_2+rm_1}{n}\right)$$

$$-\left(1-2\frac{m_2+rm_1}{l_2+rl_1}\right)f\left(\frac{k_2+rk_1}{n}\right)\Big|$$

$$
\begin{aligned}
&\leq \sum_{\substack{k_1+l_1=0 \\ l_2+rl_1\neq 0}}^{s} \sum_{k_2+l_2=0}^{n-rs} p_{s,k_1,l_1}(x,y-x)p_{n-rs,k_2,l_2}(x,y-x)\cdot \\
&\cdot \sum_{m_1=0}^{l_1}\sum_{m_2=0}^{l_2} p_{l_1,m_1}(\lambda)p_{l_2,m_2}(\lambda)\omega_3^*\left(f,\frac{l_2+rl_1}{3n}\right) \\
&\leq M \sum_{k_1+l_1=0}^{s} \sum_{k_2+l_2=0}^{n-rs} p_{s,k_1,l_1}(x,y-x)p_{n-rs,k_2,l_2}(x,y-x)\left(\frac{l_2+rl_1}{3n}\right)^{\alpha} \\
&\leq M\left(\sum_{k_1+l_1=0}^{s} \sum_{k_2+l_2=0}^{n-rs} p_{s,k_1,l_1}(x,y-x)p_{n-rs,k_2,l_2}(x,y-x)\frac{l_2+rl_1}{3n}\right)^{\alpha} \\
&= M\left(\frac{n-rs}{3n}(y-x)+\frac{rs}{3n}(y-x)\right)^{\alpha} \\
&= M\left(\frac{y-x}{3}\right)^{\alpha} \leq Mt^{\alpha}.
\end{aligned}
$$

Hence $\omega_3^*\left(S_{n,r,s}f,t\right)\leq Mt^{\alpha}$. □

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