ON A THIRD-ORDER MODULUS OF SMOOTHNESS

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Dedicated to Professor Radu Păltănea on the occasion of his 70th anniversary

Abstract

In this paper we define a new third-order modulus of smoothness and we prove the conservation by Stancu operators of the Lipschitz classes defined with this modulus.

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1 Introduction

Let \((L_n)_{n \in \mathbb{N}}\) be a sequence of linear positive operators of approximation on \(C[0,1]\). If global smoothness of a continuous function \(f\) is expressed by a Lipschitz condition with some modulus of continuity, it is of interest if \(L_nf\) verify same condition.

The preservation of global smoothness properties by the Bernstein operators

\[
B_n(f, x) = \sum_{j=0}^{n} p_{n,j}(x)f \left( \frac{j}{n} \right), \quad f \in C[0,1], \quad x \in [0,1],
\]

\[
p_{n,j}(x) = \binom{n}{j} x^j (1-x)^{n-j},
\]

were studied in [6], [7], [4], [2], [5], [3]. In [14], D.-X. Zhou showed that the Lipschitz classes with respect to the second order modulus

\[
\omega_2(f, t) = \sup \{|f(x-h) - 2f(x) + f(x+h)| : x \pm h \in [0,1], 0 < h \leq t\}
\]

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are not preserved by the Bernstein operators. He introduced the following modulus of smoothness of order two

$$\bar{\omega}_2(f, t) = \sup \{|f(x + h_1 + h_2) - f(x + h_1) - f(x + h_2) + f(x)| : x, x + h_1 + h_2 \in [0, 1], h_1, h_2 > 0, h_1 + h_2 \leq 2t\}$$

(1)

and proved the preservation by Bernstein’s operators of Lipschitz classes defined with this modulus.

For the Bernstein-type operators

$$L_n(f, x) = \sum_{j=0}^{n} p_{n,j}(x) F_{n,j}(f), f \in C[0, 1], x \in [0, 1],$$

(2)

where $F_{n,j} : C[0, 1] \rightarrow \mathbb{R}, j = 1, \ldots, n$, are linear positive functionals, in [11] we studied simultaneous global smoothness preservation in terms of modulus of continuity $\omega_2^*$ introduced by R. Păltănea [8], [9] and independently by J. Adell and J. de la Cal [1], defined for $f \in C[0, 1]$ and $t > 0$ by

$$\omega_2^*(f, t) = \sup \{|(1 - \lambda)f(x) + \lambda f(y) - f((1 - \lambda)x + \lambda y)| : x, y \in [0, 1], x < y, y - x \leq 2t, \lambda \in [0, 1]\}.$$ 

(3)

In [13] we proved the preservation of Lipschitz classes by the following generalized Bernstein operators which were introduced by D. D. Stancu (see [10])

$$S_{n,r,s}(f, x) = \sum_{j=0}^{n-rs} p_{n-rs,j}(x) \sum_{i=0}^{s} p_{s,i}(x) f\left(\frac{j + ir}{n}\right),$$

(4)

$f \in C[0, 1], x \in [0, 1]$, where $n \in \mathbb{N}, r, s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$ such that $rs < n$.

Bernstein’s operators are obtained for $s = 0$ or $s = 1, r = 0$ or $s = 1, r = 1$.

In the next section we define a new third-order modulus of smoothness and we prove the conservation by Stancu’s operators of the Lipschitz classes in terms of this modulus.

2 Main result

Definition 1. For $f \in C[0, 1]$ and $t > 0$ we define

$$\omega_3^*(f, t) = \sup \{|(1 - 2\lambda)f(y) - f(\lambda x + (1 - \lambda)y) + f((1 - \lambda)x + \lambda y) - (1 - 2\lambda)f(x)| : x, y \in [0, 1], x < y, y - x \leq 3t, \lambda \in [0, 1]\}.$$ 

(5)

Remark 1. The application

$$\omega_3^* : C[0, 1] \times (0, \infty) \rightarrow [0, \infty)$$

is a modulus of continuity of order 3 normalized on $C[0, 1]$ in the sense of the definition from [9], that is, the following conditions are verified:
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(\textbf{MC1}) \( \omega_3^* (f, t_1) \leq \omega_3^* (f, t_2) \), \( f \in C[0, 1] \), \( 0 < t_1 < t_2 \);

(\textbf{MC2}) \( \omega_3^* (f + p, t) = \omega_3^* (f, t) \), \( f \in C[0, 1] \), \( p \) algebraic polynomial of degree at most 2, \( t > 0 \);

(\textbf{MC3}) \( \omega_3^* (0, t) = 0 \), \( t > 0 \).

(\textbf{N}) \( (\exists) C > 0, (\forall) t > 0 : \omega_3^* (e_3, t) \leq C \cdot t^3 \),

Indeed, properties (MC1) and (MC3) are clear. Property (MC2) results from the fact that for the monomial functions \( e_i(x) = x^i \), \( i = 0, 1, 2 \) we have

\[
(1-2\lambda)e_i(y) - e_i(\lambda x + (1-\lambda)y) + e_i((1-\lambda)x + \lambda y) - (1-2\lambda)e_i(x) = 0, \quad i = 0, 1, 2.
\]

Since \( (1-2\lambda)e_3(y) - e_3(\lambda x + (1-\lambda)y) + e_3((1-\lambda)x + \lambda y) - (1-2\lambda)e_3(x) = \lambda(1-\lambda)(1-2\lambda)(y-x)^3 \), it follows that \( \omega^*_{3} (e_3, t) \leq 9\sqrt{3} \cdot t^3 \), \( (\forall) t > 0 \), therefore the modulus of continuity is normalized.

To prove the global smoothness preservation by Stancu operators in terms of modulus of continuity \( \omega_{3}^* \), we need a representation of Stancu’s operators.

For the Bernstein operator in [11], proceeding similarly as in [14] (see also [4]), we obtained:

\[
B_n (f, (1-\lambda) x + \lambda y) = \sum_{k+l=0}^{n} p_{n,k,l}(x, y - x) \sum_{m=0}^{l} p_{l,m}(\lambda)f \left( \frac{k + m}{n} \right), \quad (6)
\]

where \( p_{n,k,l}(x, y) = \frac{n!}{k!l!(n-k-l)!} x^k y^l (1-x)^{n-k-l} \) is the two-variable Bernstein basis. Repeating the application of an adapted version of relation (6) yields [13] (see also [12])

\[
S_{n,r,s} (f, (1-\lambda) x + \lambda y) = \sum_{k_1+l_1=0}^{n-r} \sum_{k_2+l_2=0}^{s-r} p_{s,k_1,l_1}(x, y - x) p_{n-r,s,k_2,l_2}(x, y - x) \cdot \\
\cdot \sum_{m_1=0}^{l_1} \sum_{m_2=0}^{l_2} p_{n,m_1}(\lambda)p_{l_2,m_2}(\lambda)f \left( \frac{k_2 + m_2 + r(k_1 + m_1)}{n} \right), \quad (7)
\]

\textbf{Theorem 1.} Let \( f \in C[0, 1] \), \( M > 0 \) and \( \alpha \in (0, 1] \). If

\[
\omega_{3}^* (f, t) \leq Mt^\alpha, \quad t \in \left( 0, \frac{1}{3} \right],
\]

then

\[
\omega_{3}^* (S_{n,r,s} f, t) \leq Mt^\alpha, \quad t \in \left( 0, \frac{1}{3} \right].
\]
Proof. Let $t \in (0, \frac{1}{3}]$, $x, y \in [0, 1]$, $x < y$, $y - x \leq 3t$, $\lambda \in [0, 1]$. By using the representation (7), we obtain:

$$\left| (1 - 2\lambda)S_{n,r,s}(f, y) - S_{n,r,s}(f, \lambda x + (1 - \lambda)y) \right.$$  

$$+ S_{n,r,s}(f, (1 - \lambda)x + \lambda y) - (1 - 2\lambda)S_{n,r,s}(f, x) \right|$$  

$$\leq \sum_{k_1+l_1=0}^{s} \sum_{k_2+l_2=0}^{n-rs} p_{s,k_1,l_1}(x, y - x)p_{n-rs,k_2,l_2}(x, y - x) \cdot$$  

$$\cdot \left| (1 - 2\lambda)f \left( \frac{k_2 + l_2 + r(k_1 + l_1)}{n} \right) ight.$$  

$$- \sum_{m_1=0}^{l_1} \sum_{m_2=0}^{l_2} p_{l_1,m_1}(1 - \lambda)p_{l_2,m_2}(1 - \lambda)f \left( \frac{k_2 + m_2 + r(k_1 + m_1)}{n} \right)$$  

$$+ \sum_{m_1=0}^{l_1} \sum_{m_2=0}^{l_2} p_{l_1,m_1}(\lambda)p_{l_2,m_2}(\lambda)f \left( \frac{k_2 + m_2 + r(k_1 + m_1)}{n} \right)$$  

$$- (1 - 2\lambda)f \left( \frac{k_2 + rk_1}{n} \right) \right|$$  

$$= \sum_{k_1+l_1=0}^{s} \sum_{k_2+l_2=0}^{n-rs} p_{s,k_1,l_1}(x, y - x)p_{n-rs,k_2,l_2}(x, y - x) \cdot$$  

$$\cdot \left| \sum_{m_1=0}^{l_1} \sum_{m_2=0}^{l_2} p_{l_1,m_1}(\lambda)p_{l_2,m_2}(\lambda) \left( 1 - 2\frac{m_2 + rm_1}{l_2 + rl_1} \right) f \left( \frac{k_2 + l_2 + r(k_1 + l_1)}{n} \right) ight.$$  

$$- \sum_{m_1=0}^{l_1} \sum_{m_2=0}^{l_2} p_{l_1,m_1}(\lambda)p_{l_2,m_2}(\lambda) f \left( \frac{k_2 + l_2 - m_2 + r(k_1 + l_1 - m_1)}{n} \right)$$  

$$+ \sum_{m_1=0}^{l_1} \sum_{m_2=0}^{l_2} p_{l_1,m_1}(\lambda)p_{l_2,m_2}(\lambda) f \left( \frac{k_2 + m_2 + r(k_1 + m_1)}{n} \right)$$  

$$- \sum_{m_1=0}^{l_1} \sum_{m_2=0}^{l_2} p_{l_1,m_1}(\lambda)p_{l_2,m_2}(\lambda) \left( 1 - 2\frac{m_2 + rm_1}{l_2 + rl_1} \right) f \left( \frac{k_2 + rk_1}{n} \right) \right|$$  

$$\leq \sum_{k_1+l_1=0}^{s} \sum_{k_2+l_2=0}^{n-rs} p_{s,k_1,l_1}(x, y - x)p_{n-rs,k_2,l_2}(x, y - x) \cdot$$  

$$\cdot \sum_{m_1=0}^{l_1} \sum_{m_2=0}^{l_2} p_{l_1,m_1}(\lambda)p_{l_2,m_2}(\lambda) \cdot \left| \left( 1 - 2\frac{m_2 + rm_1}{l_2 + rl_1} \right) f \left( \frac{k_2 + rk_1}{n} + \frac{l_2 + rl_1}{n} \right) ight.$$  

$$- f \left( \frac{k_2 + rk_1}{n} + \frac{l_2 - m_2 + r(l_1 - m_1)}{n} \right) + f \left( \frac{k_2 + rk_1}{n} + \frac{m_2 + rm_1}{n} \right)$$  

$$- \left( 1 - 2\frac{m_2 + rm_1}{l_2 + rl_1} \right) f \left( \frac{k_2 + rk_1}{n} \right) \right|.$$
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\[
\leq \sum_{k_1+l_1=0}^{s} \sum_{k_2+l_2=0}^{n-rs} p_{s,k_1,l_1}(x, y-x)p_{n-rs,k_2,l_2}(x, y-x) \cdot \\
\sum_{m_1=0}^{l_1} \sum_{m_2=0}^{l_2} p_{l_1,m_1}(\lambda)p_{l_2,m_2}(\lambda)\omega_3^*(f, \frac{l_2 + rl_1}{3n})
\leq M \sum_{k_1+l_1=0}^{s} \sum_{k_2+l_2=0}^{n-rs} p_{s,k_1,l_1}(x, y-x)p_{n-rs,k_2,l_2}(x, y-x) \left( \frac{l_2 + rl_1}{3n} \right)^\alpha
\leq M \left( \sum_{k_1+l_1=0}^{s} \sum_{k_2+l_2=0}^{n-rs} p_{s,k_1,l_1}(x, y-x)p_{n-rs,k_2,l_2}(x, y-x) \frac{l_2 + rl_1}{3n} \right)^\alpha
= M \left( \frac{n-rs}{3n} (y-x) + \frac{rs}{3n} (y-x) \right)^\alpha
= M \left( \frac{y-x}{3} \right)^\alpha \leq Mt^\alpha.
\]

Hence $\omega_3^*(S_{n,r,s}f, t) \leq Mt^\alpha$. \qed

References


