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ON THE ITERATES OF UNI- AND MULTIDIMENSIONAL OPERATORS

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Dedicated to Professor Radu Păltănea on the occasion of his 70th anniversary

Abstract

The problem of the convergence of the iterates of operators or of a sequence of operators is discussed in a general framework related to the fixed point theory, but with a permanent look towards the theory of linear approximation operators. The results are obtained for operators not necessarily linear. Some examples including the class of approximation operators defined by H. Brass are given.

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1 Iterates of unidimensional operators

1.1 Iterates of a single operator

Any discussion about the iterates of an operator should start with the notion of a contraction operator and Banach's contraction principle.

Let (X,d) be a metric space and let $L: X \to X$ an operator. One says that L is Lipschitz continuous with Lipschitz constant λ if

$$d\left(L\left(x\right),L\left(y\right)\right) \leq \lambda d\left(x,y\right) \quad \text{for all } x,y \in X.$$

If $\lambda < 1$, then L is a contraction operator. Observe that if L is Lipschitz continuous with Lipschitz constant λ, then any iterate L^k of L, where $L^k = L\left(L^{k-1}\right)$ $\left(k \geq 1,\ L^0 \text{ being the identity operator } I\right)$, is also Lipschitz continuous with Lipschitz constant λ^k. With this remark, we have the following proposition.

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Theorem 1. *Let $L : X \to X$ be such that:*

(i) *L is a contraction operator with Lipschitz constant $\lambda < 1$;*

(ii) *L has the (unique) fixed point x^*.*

Then for every $x \in X$, the sequence $\left(L^k(x)\right)$ of the iterates of L calculated at x is convergent to x^*.

Proof. Clearly $L^k(x^*) = x^*$ for all k. Then

$$d\left(L^k(x), x^*\right) = d\left(L^k(x), L^k(x^*)\right) \le \lambda^k d(x, x^*).$$

Since $\lambda < 1$, one has $\lambda^k \to 0$ and then $d\left(L^k(x), x^*\right) \to 0$ as $k \to \infty$. Thus $L^k(x)$ converges to x^* as claimed. □

This proposition is useful in case that the fixed point x^* is known for a contraction operator L. This happens for all many linear approximation operators $L : C[0,1] \to C[0,1]$ leaving invariant the functions 1 and x, that is with $L(1) = 1$ and $L(x) = x$. Then, restricted to a set

$$X_{\alpha,\beta} := \{f \in C[0,1] : \ f(0) = \alpha, \ f(1) = \beta\},$$

the operator L maps $X_{\alpha,\beta}$ into itself and the function

$$f^*(x) = \alpha + (\beta - \alpha)x$$

belongs to $X_{\alpha,\beta}$ and is a fixed point of L as can easily be seen. It remains to be clarified the contraction property of L. This aspect in the context of the theory of linear approximation operators was first highlighted by Rus [15] for Bernstein's operators. The contraction property was later highlighted for other well-known classes of operators (see, e.g., [1], [2], [3], [6], [7]).

The proposition above is given in the idea of a priori knowledge of the fixed point. For a contraction operator on a complete metric space, the existence and uniqueness of the fixed point is guaranteed by Banach's contraction principle which in addition gives an estimate of the approximation error, namely

$$d\left(L^k(x), x^*\right) \le \frac{\lambda^k}{1-\lambda} d(L(x), x). \tag{1}$$

In case of linear approximation operators, when metric d is given by the uniform norm $\|.\|$ of the space $C[0,1]$, this estimate reads as

$$\left\|L^k(f) - f^*\right\| \le \frac{\lambda^k}{1-\lambda} \|L(f) - f\| \tag{2}$$

for all $k \ge 1$ and $f \in X_{\alpha,\beta}$. Thus the error $\left\|L^k(f) - f^*\right\|$ of the approximation of f^* by the k-th iterate is expressed by the error $\|L(f) - f\|$ with which the arbitrary start function f is approximated by the operator L. This error is known for many operators beginning with Bernstein's ones, in terms of some smoothness modules (see [13]).

1.2 Iterates of a sequence of operators

Assume now that $(L_n)_{n\geq 1}$ is an approximation process in X, in the sense that

$$L_n(x) \to x \quad \text{as } n \to \infty \text{ for every } x \in X.$$

For such a sequence of operators, one has

Theorem 2. *Assume that every L_n is a contraction operator with a Lipschitz constant $\lambda_n < 1$ and x^* is the unique fixed point of all operators L_n. Then*

(a) $\lambda_n \to 1$ *as* $n \to \infty$;

(b) *For each $x \in X$, a sequence of iterates $\left(L_n^{k_n}(x)\right)_{n\geq 1}$ converges to x^* if $(k_n)_{n\geq 1}$ is such*

$$\frac{\lambda_n^{k_n}}{1-\lambda_n} \leq C, \quad \text{for all } n \geq 1 \tag{3}$$

and some constant C.

Proof. (a) Let λ be any limit point of the sequence $(\lambda_n)_{n\geq 1}$. Obviously $\lambda \leq 1$. Passing eventually to a subsequence, we may assume that $\lambda_n \to \lambda$. Choose two different element $x, y \in X$. From

$$d(L_n(x), L_n(y)) \leq \lambda_n d(x, y),$$

letting $n \to \infty$ and using $L_n(x) \to x$ and $L_n(y) \to y$, we deduce that $d(x, y) \leq \lambda d(x, y)$. Since $d(x, y) > 0$, we find $1 \leq \lambda$. Then $\lambda = 1$ and the proof of statement (a) is finished.

(b) Using (1) one has

$$d\left(L_n^{k_n}(x), x^*\right) \leq \frac{\lambda_n^{k_n}}{1-\lambda_n} d(L_n(x), x).$$

Here $d(L_n(x), x) \to 0$ as $n \to \infty$, while from (3), the front coefficients are uniformly bounded. As a result $d\left(L_n^{k_n}(x), x^*\right) \to 0$ as $n \to \infty$, which is our statement. □

Remark 1. *(1^0) By virtue of (a), if $k_n \to \infty$, then the limit of coefficient $\frac{\lambda_n^{k_n}}{1-\lambda_n}$ is $\frac{1^\infty}{+0} = +\infty \cdot 1^\infty$. Hence a necessary condition for (3) to hold is that $1^\infty = 0$, i.e., $\lambda_n^{k_n} \to 0$.*

(2^0) Condition (3) is only a sufficient condition for the convergence of the iterates $L_n^{k_n}$, not necessarily the best for particular sequences of operators. For example, in case of Bernstein's operators, when $\lambda_n = 1 - 2^{1-n}$, condition (3) returns to $k_n/2^n \to \infty$ as $n \to \infty$. However, by the classical result of Kelinsky-Rivlin [11], the convergence is guaranteed if $k_n/n \to \infty$.

1.3 Example: Iterates of Brass's operators

Let $n \in \mathbb{N} \setminus \{0\}$ and let $\mu_1, \mu_2, \ ..., \ \mu_n \in \mathbb{N}$ such that

$$\mu_1 + 2\mu_2 + \ ... + n\mu_n = n.$$

Consider the polynomials

$$p_{n\nu}(x) = \sum \begin{pmatrix} \mu_1 \\ \eta_1 \end{pmatrix} \begin{pmatrix} \mu_2 \\ \eta_2 \end{pmatrix} \cdots \begin{pmatrix} \mu_n \\ \eta_n \end{pmatrix} x^{\eta} (1-x)^{\mu-\eta}, \quad \nu = 0, 1, ..., n,$$

where $\mu = \sum\limits_{i=1}^{n} \mu_i, \ \eta = \sum\limits_{i=1}^{n} \eta_i$, and the summation is done in relation to all systems of nonnegative integer numbers $(\eta_1, \eta_2, \ ..., \ \eta_n)$ for which

$$\eta_1 + 2\eta_2 + \ ... + n\eta_n = \nu.$$

Define the operator

$$P_n^{(\mu_1,\mu_2,...,\mu_n)}(f)(x) = \sum_{\nu=0}^{n} f\left(\frac{\nu}{n}\right) p_{n\nu}(x) \qquad (f \in C[0,1], \ x \in [0,1]). \qquad (4)$$

and consider the class $\mathcal{M}$ of all operators L_n which are convex combinations of operators of the form (4) which leave invariant the functions 1 and x. Hence

$$L_n = \sum \gamma(\mu_1, \mu_2, ..., \mu_n) P_n^{(\mu_1,\mu_2,...,\mu_n)},$$

where the coefficients $\gamma(\mu_1, \mu_2, ..., \mu_n)$, in finite number, are nonnegative and with the sum equal to one, and $L_n(1) = 1, \ L_n(x) = x$.

We note that Bernstein's operators are of type (4), more exactly

$$B_n = P_n^{(n,0,...,0)}, \quad B_1 = P_n^{(0,...,0,1)}.$$

The class $\mathcal{M}$ also contains other classical operators such as the operators of Cheney-Sharma [8], Mühlbach [12] and Stancu [16].

Lemma 1. *Any operator* $L := P_n^{(\mu_1,\mu_2,...,\mu_n)}$ *from* $\mathcal{M}$ *is a contraction operator on* $X_{\alpha,\beta}$ *with the Lipschitz constant* $1 - 2^{1-\mu}$ *and has the fixed point* $f^*(x) = \alpha + (\beta - \alpha)x$.

Proof. Indeed, for any $f, g \in X_{\alpha,\beta}$, since $L(1) = 1$, one has

$$\begin{aligned} \|L(f) - L(g)\| &\leq \|f-g\| (L(1)(x) - p_{n0}(x) - p_{nn}(x)) \\ &= \|f-g\| (1 - p_{n0}(x) - p_{nn}(x)) \\ &\leq \left(1 - \min_{x\in[0,1]} (p_{n0}(x) + p_{nn}(x))\right) \|f-g\|. \end{aligned}$$

Also, for $\nu = 0$, one has $\eta = 0$ and so $p_{n0}(x) = (1-x)^{\mu}$, while if $\nu = n$, then $\eta = \nu$ and so $p_{nn}(x) = x^{\mu}$. Hence

$$\min_{x\in[0,1]} (p_{n0}(x) + p_{nn}(x)) = \min_{x\in[0,1]} ((1-x)^{\mu} + x^{\mu}) = \frac{1}{2^{\mu-1}}.$$

Then

$$\|L(f) - L(g)\| \leq \left(1 - \frac{1}{2^{\mu-1}}\right) \|f - g\|.$$

The fact that f^* is the fixed point of L in $X_{\alpha,\beta}$ is a simple consequence of the two properties $L(1) = 1,\ L(x) = x.$ □

The above lemma guarantees that any operator from the class $\mathcal{M}$ is a contraction operator and in $X_{\alpha,\beta}$ has the unique fixed point f^*. Thus Theorem 1 applies for any operator from the class $\mathcal{M}$, and we have

Theorem 3. *If $L \in \mathcal{M}$, then $L^k(f) \to f^*$ as $k \to \infty$ for every $f \in X_{\alpha,\beta}$.*
Additionally, formula (2) is true and

$$\left\|L^k(f) - f^*\right\| \leq 2^{\mu-1}\left(1 - \frac{1}{2^{\mu-1}}\right) \|L(f) - f\| \quad (k \geq 1,\ f \in X_{\alpha,\beta}).$$

For a subclass of $\mathcal{M}$, namely that of the so called operators of Bernstein type, the result given by Theorem 3 was proved by using completely different techniques, by Albu [4] and myself [14].

Obviously, Theorem 2 also applies to sequences $(L_n)_{n\geq 1}$ of operators in $\mathcal{M}$, which are assumed to be approximation processes. A better result was obtained in [14] for the subclass of Bernstein type operators by using the extremality property of the classical Bernstein operators in that subclass, namely the relations

$$B_n(f) \leq L_n(f) \leq B_1(f),$$

for every nonnegative and convex function $f \in C[0,1]$.

2 Iterates of multidimensional operators

We begin this section with the matrix version of Theorem 1.

Theorem 4. *Let $(X_i, d_i),\ i = 1, 2, ..., p$ be metric spaces and let $\mathbf{L} : X = X_1 \times X_2 \times ... \times X_p \to X$, $\mathbf{L} = (\mathbf{L}_1, \mathbf{L}_2, ..., \mathbf{L}_p)$, where $\mathbf{L}_i : X \to X_i$, be such that*

(i) *For each i,*

$$d_i(\mathbf{L}_i(x), \mathbf{L}_i(y)) \leq \sum_{j=1}^{p} \gamma_{ij} d_j(x_j, y_j) \tag{5}$$

for all $x = (x_1, x_2, ..., x_p),\ y = (y_1, y_2, ..., y_p) \in X$ and some nonnegative numbers γ_{ij} for which the matrix $\Gamma = [\gamma_{ij}]_{1\leq i,j\leq p}$ is convergent to zero in the sense that its power Γ^k tends to the zero matrix as $k \to \infty$.

(ii) $\mathbf{L}$ *has the (unique) fixed point x^*.*

Then for each $x \in X$, the sequence $\left(\mathbf{L}^k(x)\right)$ of the iterates of $\mathbf{L}$ calculated at x is convergent in X to x^*.

Proof. Denoting

$$\mathbf{d}(x,y)=\begin{bmatrix} d_1(x_1,y_1) \\ \dots \\ d_p(x_p,y_p)\end{bmatrix} \quad (x,y\in X),$$

we may rewrite (5) in the matrix way as

$$\mathbf{d}(\mathbf{L}(x),\mathbf{L}(y))\leq \Gamma \mathbf{d}(x,y). \tag{6}$$

Then

$$\mathbf{d}\left(\mathbf{L}^k(x),x^*\right)=\mathbf{d}\left(\mathbf{L}^k(x),\mathbf{L}^k(x^*)\right)\leq \Gamma^k\mathbf{d}(x,x^*)$$

and the result follows since $\Gamma^k\to 0$ as $k\to\infty$. □

If $X_i,\ i=1,2,...,p,$ are complete metric spaces, then inequality (6) showing that $\mathbf{L}$ is a Perov contraction on X implies the existence and uniqueness of the fixed point x^* of $\mathbf{L}$ and moreover that the following estimate holds for all $k\geq 1$ and $x\in X$:

$$\mathbf{d}\left(\mathbf{L}^k(x),x^*\right)\leq (I-\Gamma)^{-1}\Gamma^k\mathbf{d}(\mathbf{L}(x),x). \tag{7}$$

Example 1. Let $X_i=D,\ i=1,2,...,p,$ where D is a closed convex subset of a Banach space, and $L_{ij}:D\to D$ $(1\leq i,j\leq p)$ be λ_{ij}-Lipschitz continuous, i.e.,

$$\|L_{ij}(x_j)-L_{ij}(y_j)\|\leq \lambda_{ij}\|x_j-y_j\| \quad (x_j,y_j\in D).$$

Let $\sigma_{ij}\geq 0$ $(1\leq i,j\leq p)$ be such that $\sum_{j=1}^{p}\sigma_{ij}=1$ for $i=1,2,...,p$. Define the operator $\mathbf{L}:D^p\to D^p,\ \mathbf{L}=(\mathbf{L}_1,\mathbf{L}_2,...,\mathbf{L}_p)$ by

$$\mathbf{L}_i(x)=\sum_{j=1}^{p}\sigma_{ij}L_{ij}(x_j), \qquad x=(x_1,x_2,...,x_p),\ i=1,2,...,p.$$

If the matrix $\Gamma:=[\sigma_{ij}\lambda_{ij}]_{1\leq i,j\leq p}$ is convergent to zero, then the operator $\mathbf{L}$ is a Perov contraction on D^p. Consequently, it has a unique fixed point x^*, its iterates $\mathbf{L}^k(x)$ converge to x^* and estimate (7) holds. In particular, if $\lambda_{ij}<1$ for all i and j, matrix Γ is convergent to zero as shown in [1]. Indeed, letting $\lambda:=\max\{\lambda_{ij}:\ 1\leq i,j\leq p\}$, one has $\lambda<1$ and $\Gamma\leq\lambda M$, where $M:=[\sigma_{ij}]_{1\leq i,j\leq p}$ and the powers of M are dominated by the matrix U having all entries equal to 1 (this is proved by induction). Then

$$\Gamma^k=\lambda^kM^k\leq\lambda^kU\to 0 \ \text{ as } \ k\to\infty,$$

as claimed. In this case, each operator L_{ij} has a unique fixed point x_{ij}^*. In case that for each i, one has that $x_{ij}^*=:x_i^*$ for all j, then $x^*=\left(x_1^*,x_2^*,...,x_p^*\right)$. Such a case occurs if the operators L_{ij} are classical Bernstein operators (of different degrees), or other operators of Bernstein type, and $D=\{f\in C[0,1]:\ f(0)=\alpha,\ f(1)=\beta\}$, when the fixed point of $\mathbf{L}$ is $\mathbf{f}^*:=(f^*,f^*,...,f^*)$.

Theorem 2 extends to multidimensional operators as follows.

Theorem 5. *Let $(\mathbf{L}_n)_{n\geq 1}$ be a sequence of Perov contraction operators with Lipschitz matrices Γ_n convergent to zero, and let x^* is the unique fixed point of all operators $\mathbf{L}_n$. Assume that*

$$\mathbf{L}_n(x) \to x \quad \text{as } n\to\infty \ \text{ for every } x\in X. \tag{8}$$

Then

(a) *The element from the diagonal of Γ_n tend to 1 as $n\to\infty$;*

(b) *For each $x\in X$, a sequence of iterates $\left(\mathbf{L}_n^{k_n}(x)\right)_{n\geq 1}$ converges to x^* if $(k_n)_{n\geq 1}$ is such that the matrices*

$$(I-\Gamma_n)^{-1}\Gamma_n^{k_n}, \quad n\geq 1$$

are bounded (componentwise).

Proof. (a) It is known that the elements from the diagonal of a matrix which is convergent to zero are strictly less than one. Letting $\Gamma_n = \left[(\lambda_n)_{ij}\right]_{1\leq i,j\leq p}$ one has $(\lambda_n)_{ii} < 1$ for $i=1,2,...,p$. For any i, let λ_{ii} be a limit point of the sequence $((\lambda_n)_{ii})$. Then $\lambda_{ii}\leq 1$. Passing eventually to a subsequence, we may assume that $(\lambda_n)_{ii}\to\lambda_{ii}$. Now we take two element $x,y\in X$ having the components equal to zero except x_i and y_i which are chosen different. From

$$\mathbf{d}(\mathbf{L}_n(x),\mathbf{L}_n(y)) \leq \Gamma_n\mathbf{d}(x,y),$$

looking at the i-th component, one has

$$d_i((\mathbf{L}_n)_i(x),(\mathbf{L}_n)_i(y)) \leq (\lambda_n)_{ii}\, d_i(x_i,y_i).$$

Here we let $n\to\infty$ and use $(\mathbf{L}_n)_i(x)\to x_i$ and $(\mathbf{L}_n)_i(y)\to y_i$, to deduce that $d_i(x_i,y_i)\leq\lambda_{ii}d_i(x_i,y_i)$. Since $d_i(x_i,y_i)>0$, we find $1\leq\lambda_{ii}$. Then $\lambda_{ii}=1$ and the proof of statement (a) is finished.

(b) Using (7) one has

$$\mathbf{d}\left(\mathbf{L}_n^{k_n}(x),x^*\right) \leq (I-\Gamma_n)^{-1}\Gamma_n^{k_n}\mathbf{d}(\mathbf{L}_n(x),x),$$

whence the conclusion of (b) is immediate. □

The next example gives a scheme of construction of sequences of multidimensional operators which are approximation processes in the sense of (8).

Example 2. Let $(\mathbf{L}_n)_{n\geq 1}$ be a sequence of operators of the type of those from the previous example, that is

$$(\mathbf{L}_n)_i(x) = \sum_{j=1}^{p}(\sigma_n)_{ij}(L_n)_{ij}(x_j), \qquad i=1,2,...,p,\ n\geq 1.$$

It is easily seen that the sequence $(\mathbf{L}_n)_{n\geq 1}$ is an approximation process on D^p, i.e., $\mathbf{L}_n(x)\to x$ as $n\to\infty$ for every $x\in D^p$, if the following conditions are satisfied:

(i) for every i, the sequence $((L_n)_{ii})_{n\geq 1}$ is an approximation process on D;

(ii) for every (i,j) with $i \neq j$, and for each $x_j \in D$, the sequence $\left((L_n)_{ij}(x_j)\right)_{n\geq 1}$ is bounded;

(ii) for every (i,j) with $i \neq j$, $(\sigma_n)_{ij} \to 0$ as $n \to \infty$.

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