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LOGARITHMIC COEFFICIENTS FOR A CLASS OF ANALYTIC FUNCTIONS DEFINED BY SUBORDINATION

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Dedicated to Professor Radu Păltănea on the occasion of his 70th anniversary

Abstract

In this paper we consider a class of functions $\mathcal{M}_\alpha(\varphi)$ defined by subordination, consisting of functions $f \in \mathcal{A}$ satisfying the condition

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \varphi(z),\ z \in U.$$

In the study of univalent functions, estimates on the Taylor coefficients are usually given. Another significant problem deals with the estimates of logarithmic coefficients. For the class $\mathcal{S}$ of univalent functions no sharp bounds for the modulus of the individual logarithmic coefficients are known if $n \geq 3$. For different subclasses of $\mathcal{S}$ the results are not better and in most cases only the first three initial coefficients of $\log \dfrac{f(z)}{z}$ are considered. For the class $\mathcal{M}_\alpha(\varphi)$ we obtain upper bounds for the logarithmic coefficients $\gamma_n,\ n \in \{1,2,3\}$ and also for $\Gamma_n,\ n \in \{1,2,3\}$, the logarithmic coefficients of the inverse of $\mathcal{M}_\alpha(\varphi)$. Connections with previous known results are pointed out.

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1 Introduction

Let $\mathcal{A}$ be the class of analytic functions f in $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{U}. \tag{1}$$

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The subclass of $\mathcal{A}$ consisting of univalent functions is denoted by $\mathcal{S}$. For a function $f \in \mathcal{S}$ the logarithmic coefficients γ_n are defined by the series expansion

$$\log\left(\frac{f(z)}{z}\right) = 2\sum_{n=1}^{\infty} \gamma_n(f) z^n, \quad z \in \mathbb{U}. \tag{2}$$

If there is no confusion, we use γ_n instead of $\gamma_n(f)$.

It is well known that the logarithmic coefficients play an important role in Milin conjecture [5], that for $f \in \mathcal{S}$,

$$\sum_{m=1}^{n}\sum_{k=1}^{n}\left(k|\gamma_k|^2 - \frac{1}{k}\right) \leq 0.$$

It is interesting that for the class $\mathcal{S}$ the sharp estimates of logarithmic coefficients are known only for the first two γ_1 and γ_2:

$$|\gamma_1| \leq 1, \quad |\gamma_2| \leq \frac{1}{2} + \frac{1}{e}$$

and it is not known for $n \geq 3$.

The situation is not better for the subclasses of $\mathcal{S}$ where, in most cases, only the initial coefficients of $\log \dfrac{f(z)}{z}$ are investigated.

Recently, several authors have considered the problem of finding sharp upper bounds for the logarithmic coefficients of univalent functions, (see, for example [1], [8]).

Equality (2) can be rewritten in the following form

$$\begin{aligned} 2\sum_{n=1}^{\infty} \gamma_n(f) z^n = \quad & a_2 z + a_3 z^2 + a_4 z^3 + \cdots - \frac{1}{2}(a_2 z + a_3 z^2 + a_4 z^3 + \cdots)^2 + \\ & + \frac{1}{3}(a_2 z + a_3 z^2 + a_4 z^3 + \cdots)^3 + \cdots . \end{aligned} \tag{3}$$

Equating the coefficients of z^n, for $n = 1, 2, 3$ in (3), we obtain

$$\begin{cases} 2\gamma_1 & = & a_2 \\ 2\gamma_2 & = & a_3 - \frac{1}{2}a_2^2 \\ 2\gamma_3 & = & a_4 - a_2 a_3 + \frac{1}{3}a_2^3 \end{cases} \tag{4}$$

Let F be the inverse of a function $f \in \mathcal{S}$ defined by

$$F(w) = f^{-1}(w) = w + \sum_{n=2}^{\infty} A_n w^n, \tag{5}$$

with $|w| < \dfrac{1}{4}$, from Koebe's 1/4 - theorem.

In view of (1) and (5), we have

$$\begin{cases} A_2 &= -a_2 \\ A_3 &= -a_3 + 2a_2^2 \\ A_4 &= -a_4 + 5a_2a_3 - 5a_2^3. \end{cases} \tag{6}$$

The logarithmic coefficients Γ_n $(n \in \mathbb{N})$ of F are defined by

$$\log\left(\frac{F(z)}{z}\right) = 2\sum_{n=1}^{\infty} \Gamma_n(F)w^n, \quad |w| < \frac{1}{4}. \tag{7}$$

When there is no confusion we consider $\Gamma_n(F) = \Gamma_n$.

In [9] Ponnusamy et al. obtained sharp upper bound of $|\Gamma_n(F)|$ for $f \in \mathcal{S}$ and $n \geq 1$. Furthermore, in a case of a convex function, they proved that $|\Gamma_n(F)| \leq 1/(2n)$ for $n = 1, 2, 3$.

Making use of (7), we have

$$\begin{aligned} 2\sum_{n=1}^{\infty} \Gamma_n(F)w^n = &\quad A_2w + A_3w^2 + A_4w^3 + \cdots - \frac{1}{2}(A_2w + A_3w^2 + A_4w^3 + \cdots)^2 + \\ &+ \frac{1}{3}(A_2w + A_3w^2 + A_4w^3 + \cdots)^3 + \cdots . \end{aligned} \tag{8}$$

Equating the coefficients of w^n, for $n = 1, 2, 3$, we obtain

$$\begin{cases} 2\Gamma_1 &= A_2 \\ 2\Gamma_2 &= A_3 - \frac{1}{2}A_2^2 \\ 2\Gamma_3 &= A_4 - A_2A_3 + \frac{1}{3}A_2^3 \end{cases} \tag{9}$$

Denote by $\mathcal{B}$ denote the class of analytic functions which satisfies the conditions: $\omega(0) = 0$ and $|\omega(z)| < 1, \quad z \in \mathbb{U}$. Functions in $\mathcal{B}$ are called *Schwarz* functions.

Let f and g be analytic in $\mathbb{U}$. We say that the function f is *subordinated* to the function g, denoted $f \prec g$, if there exists a function $\omega \in \mathcal{B}$, such that

$$f(z) = g(\omega(z)), \quad z \in \mathbb{U}.$$

Using the concept of subordination, Ma and Minda [4] defined the following two classes of functions

$$\mathcal{S}^*(\varphi) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \varphi(z),\ z \in \mathbb{U} \right\}$$

$$\mathcal{C}(\varphi) = \left\{ f \in \mathcal{A} : 1 + \frac{zf''(z)}{f'(z)} \prec \varphi(z),\ z \in \mathbb{U} \right\},$$

where φ is an analytic function with positive real part in $\mathbb{U}$, with $\varphi(0) = 1, \varphi'(0) > 0$ and such that $\varphi(\mathbb{U})$ is a starlike region with respect to 1 and symmetric with respect to the real axis. The classes $\mathcal{S}^*(\varphi)$ and $\mathcal{C}(\varphi)$ contain , as special cases, several well-known subclasses of starlike and convex functions.

Following Ma and Minda, we consider the following class of analytic functions defined by subordination.

Definition 1. *A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{M}_\alpha(\varphi)$, $0 \le \alpha \le 1$, if it satisfies the subordination:*

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \varphi(z). \tag{10}$$

Note that $\mathcal{M}_0(\varphi) = \mathcal{S}^*(\varphi)$ and $\mathcal{M}_1(\varphi) = \mathcal{C}(\varphi)$.

For $\varphi(z) = \dfrac{1+z}{1-z}$ the class reduces to the class $\mathcal{M}_\alpha$ of α-convex functions

$$\mathcal{M}_\alpha = \left\{ f \in \mathcal{A} : \Re\left[(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] > 0,\ z \in U \right\},$$

which was first introduced by P.T. Mocanu [6].

Like in the case of $\mathcal{M}_\alpha$, the class $\mathcal{M}_\alpha(\varphi)$ provides a continuous passage from the class $\mathcal{S}^*(\varphi)$ to the class $\mathcal{C}(\varphi)$.

In this paper we obtain upper bounds for the logarithmic coefficients γ_n of the functions in $\mathcal{M}_\alpha(\varphi)$ and in the same time for Γ_n, the logarithmic coefficients of the inverse of $\mathcal{M}_\alpha(\varphi)$. Some connections with previous known results are pointed out.

In order to prove our results, the following two lemmas will be used.

Lemma 1. *[7] Assume that ω is a Schwarz function so that $\omega(z) = \sum_{n=1}^{\infty} c_n z^n$. Then*

$$\begin{cases} |c_1| \le 1, \\ |c_n| \le 1 - |c_1|^2,\ n = 2, 3, \cdots \end{cases}$$

Lemma 2. *[10] If $\omega(z) = \sum_{n=1}^{\infty} c_n z^n \in \mathcal{B}$ then, for any real numbers q_1 and q_2, the following sharp estimate holds:*

$$|c_3 + q_1 c_1 c_2 + q_2 c_1^3| \le H(q_1; q_2);$$

where

$$H(q_1;q_2)=\begin{cases} 1 & if \quad (q_1,q_2)\in D_1\cup D_2\cup(2,1);\\ |q_2| & if \quad (q_1,q_2)\in \cup_{k=3}^{7}D_k;\\ \dfrac{2}{3}(|q_1|+1)\left(\dfrac{|q_1|+1}{3|q_1|+1+q_2}\right)^{\frac{1}{2}} & if \quad (q_1,q_2)\in D_8\cup D_9;\\ \dfrac{q_2}{3}\left(\dfrac{q_1^2-4}{q_1^2-4q_2}\right)\left(\dfrac{q_1^2-4}{3(q_2-1)}\right)^{\frac{1}{2}} & if \quad (q_1,q_2)\in D_{10}\cup D_{11}\setminus(2,1);\\ \dfrac{2}{3}(|q_1|-1)\left(\dfrac{|q_1|-1}{3|q_1|-1-q_2}\right)^{\frac{1}{2}} & if \quad (q_1,q_2)\in D_{12}. \end{cases}$$

The sets $D_k,\ k=1,2,\cdots 12$ *are defined as follows:*

$$\begin{aligned}
D_1 &= \left\{(q_1,q_2): |q_1|\le \frac{1}{2},\ |q_2|\le 1\right\};\\
D_2 &= \left\{(q_1,q_2): \frac{1}{2}\le |q_1|\le 2,\ \frac{4}{27}(|q_1|+1)^3-(|q_1|+1)\le |q_2|\le 1\right\};\\
D_3 &= \left\{(q_1,q_2): |q_1|\le \frac{1}{2},\ q_2\le -1\right\};\\
D_4 &= \left\{(q_1,q_2): |q_1|\ge \frac{1}{2},\ |q_2|\le -\frac{2}{3}(|q_1|+1)\right\};\\
D_5 &= \{(q_1,q_2): |q_1|\le 2,\ |q_2|\ge 1\};\\
D_6 &= \left\{(q_1,q_2): 2\le |q_1|\le 4,\ |q_2|\ge \frac{1}{12}(q_1^2+8)\right\};\\
D_7 &= \left\{(q_1,q_2): |q_1|\ge 4,\ |q_2|\ge \frac{2}{3}(|q_1|-1)\right\};\\
D_8 &= \left\{(q_1,q_2): \frac{1}{2}\le|q_1|\le 2,\ -\frac{2}{3}(|q_1|+1)\le |q_2|\le \frac{4}{27}(|q_1|+1)^3-(|q_1|+1)\right\};\\
D_9 &= \left\{(q_1,q_2): |q_1|\ge 2,\ -\frac{2}{3}(|q_1|+1)\le |q_2|\le \frac{2|q_1||q_1+1|}{q_1^2+2|q_1|+4}\right\};\\
D_{10} &= \left\{(q_1,q_2): 2\le |q_1|\le 4,\ \frac{2|q_1||q_1+1|}{q_1^2+2|q_1|+4}\le |q_2|\le \frac{1}{12}(q_1^2+8)\right\};\\
D_{11} &= \left\{(q_1,q_2): |q_1|\ge 4,\ \frac{2|q_1||q_1+1|}{q_1^2+2|q_1|+4}\le |q_2|\le \frac{2|q_1||q_1-1|}{q_1^2-2|q_1|+4}\right\};\\
D_{12} &= \left\{(q_1,q_2): |q_1|\ge 4,\ \frac{2|q_1||q_1-1|}{q_1^2-2|q_1|+4}\le |q_2|\le \frac{2}{3}(|q_1|-1)\right\}.
\end{aligned}$$

Unless otherwise mentioned, we assume through this paper that φ is a univalent function in $\mathbb{U}$, satisfying $\varphi(0)=1$, with series expansion of the form

$$\varphi(z)=1+B_1z+B_2z^2+B_3z^3\cdots,\quad B_1\neq 0. \tag{11}$$

2 Logarithmic coefficients for $\mathcal{M}_\alpha(\varphi)$

In this section we obtain upper bounds of the first three logarithmic coefficients for the class $\mathcal{M}_\alpha(\varphi)$.

Theorem 1. *Let f be a function in the class $\mathcal{M}_\alpha(\varphi)$. Then, the logarithmic coefficients of f satisfy the inequalities:*

$$|\gamma_1| \leq \frac{|B_1|}{2(1+\alpha)};$$

$$|\gamma_2| \leq \begin{cases} \dfrac{|B_1|}{4(1+2\alpha)} & if \quad |(1+\alpha)^2 B_2 + \alpha B_1^2| \leq (1+\alpha)^2 |B_1|; \\ \\ \dfrac{|(1+\alpha)^2 B_2 + \alpha B_1^2|}{4(1+2\alpha)(1+\alpha)^2} & if \quad |(1+\alpha)^2 B_2 + \alpha B_1^2| \geq (1+\alpha)^2 |B_1|; \end{cases}$$

and if B_1, B_2 and B_3 are real, then

$$|\gamma_3| \leq \frac{|B_1|}{6(1+3\alpha)} H(q_1, q_2);$$

where $H(q_1, q_2)$ is given by Lemma 2, with

$$\begin{cases} q_1 &= \dfrac{3\alpha}{(1+\alpha)(1+2\alpha)} B_1 + \dfrac{2B_2}{B_1}; \\ \\ q_2 &= \dfrac{3\alpha}{(1+\alpha)(1+2\alpha)} B_2 + \dfrac{\alpha(\alpha-1)}{(1+\alpha)^3(1+2\alpha)} B_1^2 + \dfrac{B_3}{B_1}. \end{cases}$$

Proof. Let $f \in \mathcal{M}_\alpha(\varphi)$. Then there exist a function $\omega \in \mathbb{B}$ with $\omega(z) = \sum\limits_{n=1}^{\infty} c_n z^n$ such that

$$\begin{aligned} &(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) = \varphi(\omega(z)) \\ &= 1 + B_1 c_1 z + (B_1 c_2 + B_2 c_1^2) z^2 + (B_1 c_3 + 2c_1 c_2 B_2 + B_3 c_1^3) z^3 + \cdots . \end{aligned} \tag{12}$$

From the Taylor expansion of f, we have:

$$\begin{aligned} &(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) = 1 + (1+\alpha)a_2 z + [2(1+2\alpha)a_3 - \\ &-(1+3\alpha)a_2^2] z^2 + \left[3(1+3\alpha)a_4 - 3(1+5\alpha)a_2 a_3 + 1 + 7\alpha)a_2^3\right] z^3 + \cdots . \end{aligned} \tag{13}$$

Equating the coefficients of z^n $(n = 1, 2, 3)$ in (12) and (13), we obtain

$$\begin{cases} (1+\alpha)a_2 & = & B_1c_1 \\ 2(1+2\alpha)a_3-(1+3\alpha)a_2^2 & = & B_1c_2+B_2c_1^2 \\ 3(1+3\alpha)a_4-3(1+5\alpha)a_2a_3+(1+7\alpha)a_2^3 & = & B_1c_3+2c_1c_2B_2+B_3c_1^3 \end{cases} \tag{14}$$

Substituting a_2 from (4) in (14), we obtain $\gamma_1 = \dfrac{B_1c_1}{2(1+\alpha)}$. Applying Lemma 1, we get $|\gamma_1| \leq \dfrac{|B_1|}{2(1+\alpha)}$.

Making use of the second equalities from both (4) and (14), after elementary computations, we obtain

$$\gamma_2 = \frac{(1+\alpha)^2B_1c_2+\left[(1+\alpha)^2B_2+\alpha B_1^2\right]c_1^2}{4(1+2\alpha)(1+\alpha)^2}.$$

An application of Lemma 1 gives

$$|\gamma_2| \leq \frac{(1+\alpha)^2|B_1|(1-|c_1|^2)+\left|(1+\alpha)^2B_2+\alpha B_1^2\right||c_1|^2}{4(1+2\alpha)(1+\alpha)^2}.$$

which yields to the required upper bound of $|\gamma_2|$.

Now, combining the last equalities from (4) and (14), we have

$$\begin{aligned} \gamma_3 = \quad & \frac{B_1}{6(1+3\alpha)}\left\{c_3+\left[\frac{3\alpha}{(1+\alpha)(1+2\alpha)}B_1+\frac{2B_2}{B_1}\right]c_1c_2+\right. \\ & \left. +\left[\frac{3\alpha}{(1+\alpha)(1+2\alpha)}B_2+\frac{\alpha(\alpha-1)}{(1+\alpha)^3(1+2\alpha)}B_1^2+\frac{B_3}{B_1}\right]c_1^3\right\}. \end{aligned}$$

Let

$$\begin{aligned} q_1 &= \frac{3\alpha}{(1+\alpha)(1+2\alpha)}B_1+\frac{2B_2}{B_1}; \\ q_2 &= \frac{3\alpha}{(1+\alpha)(1+2\alpha)}B_2+\frac{\alpha(\alpha-1)}{(1+\alpha)^3(1+2\alpha)}B_1^2+\frac{B_3}{B_1}. \end{aligned}$$

Making use of Lemma (2), from the above equality, we get

$$|\gamma_3| \leq \frac{|B_1|}{6(1+3\alpha)}|c_3+q_1c_1c_2+q_2c_1^3| \leq \frac{B_1}{6(1+3\alpha)}H(q_1;q_2).$$

□

Remark 1. *For $\alpha = 1$ we find the result obtained by E.A. Adegani et al. [1] in Theorem 2.*

In [3] Kanas et al. investigated certain subclasses of Ma-Minda type defined with the help of function

$$q_s(z) = \frac{1}{(1-z)^s} = \mathrm{e}^{-s\log(1-z)} = 1 + sz + \frac{s(s+1)}{2}z^2 + \frac{s(s+1)(s+2)}{6}z^3 + \cdots$$

where the branch of the logarithm is determined by $q_s(0) = 1$. This function maps the unit disk onto a domain bounded by the right branch of the hyperbola

$$H(s) = \left\{\rho\mathrm{e}^{\mathrm{i}\varphi} \ : \ \rho = \frac{1}{(2\cos(\varphi/s))^s}, \ |\varphi| < \frac{\pi s}{2}\right\}.$$

If we consider $\varphi(z) = q_s(z)$ $(0 < s \leq 1)$ in Theorem 1 we obtain the following result.

Theorem 2. *If $f \in \mathcal{M}_\alpha(q_s)$, then*

$$|\gamma_1| \leq \frac{s}{2(1+\alpha)};$$

$$|\gamma_2| \leq \begin{cases} \dfrac{s}{4(1+2\alpha)} & if \quad s \in \left(0, \dfrac{(1+\alpha)^2}{\alpha^2+4\alpha+1}\right]; \\[2ex] \dfrac{s^2(\alpha^2+4\alpha+1)+s(1+\alpha)^2}{8(1+2\alpha)(1+\alpha)^2} & if \quad s \in \left[\dfrac{(1+\alpha)^2}{\alpha^2+4\alpha+1}, 1\right]; \end{cases}$$

and

$$|\gamma_3| \leq \begin{cases} \dfrac{s}{6(1+3\alpha)} & if \quad s \in \left(0, \dfrac{-3(1+\alpha)^2(2\alpha^2+6\alpha+1)+\sqrt{\Delta}}{2(2\alpha^4+16\alpha^3+33\alpha^2+8\alpha+1)}\right]; \\[2ex] \dfrac{s}{6(1+3\alpha)}q_2 & if \quad s \in \left[\dfrac{-3(1+\alpha)^2(2\alpha^2+6\alpha+1)+\sqrt{\Delta}}{2(2\alpha^4+16\alpha^3+33\alpha^2+8\alpha+1)}, 1\right]; \end{cases}$$

where

$$\begin{aligned} q_2 \ &= \ s^2\left[\frac{3\alpha}{2(1+\alpha)(1+2\alpha)} + \frac{\alpha(\alpha-1)}{(1+\alpha)^3(1+2\alpha)} + \frac{1}{6}\right] \\ &\quad + s\left[\frac{3\alpha}{2(1+\alpha)(1+2\alpha)} + \frac{1}{2}\right] + \frac{1}{3}; \end{aligned}$$

and

$$\Delta = (a+1)^3(100a^5 + 796a^4 + 1888a^3 + 1252a^2 + 277a + 25).$$

Proof. The estimates for $|\gamma_1|$ and $|\gamma_2|$ are obtained directly from Theorem 1, taking $B_1 = s$ and $B_2 = \dfrac{s(s+1)}{2}$.

To obtain the estimate of $|\gamma_3|$ we will use only a part from Lemma 2, that is

$$H(q_1;q_2)=\begin{cases} 1 & if \quad (q_1,q_2)\in D_2;\\ |q_2| & if \quad (q_1,q_2)\in D_5\cup D_6.\end{cases}$$

If in Theorem 1, we take $B_1=s$, $B_2=\dfrac{s(s+1)}{2}$ and $B_3=\dfrac{s(s+1)(s+2)}{6}$ we obtain

$$|\gamma_3|\le \frac{s}{6(1+3\alpha)}H(q_1,q_2),$$

where

$$\begin{cases} q_1 &= \dfrac{3\alpha}{(1+\alpha)(1+2\alpha)}s+s+1;\\ q_2 &= s^2\left[\dfrac{3\alpha}{2(1+\alpha)(1+2\alpha)}+\dfrac{\alpha(\alpha-1)}{(1+\alpha)^3(1+2\alpha)}+\dfrac{1}{6}\right]+\\ & +s\left[\dfrac{3\alpha}{2(1+\alpha)(1+2\alpha)}+\dfrac{1}{2}\right]+\dfrac{1}{3}.\end{cases}$$

First, we consider

$$D_2=\left\{(q_1,q_2):\frac{1}{2}\le|q_1|\le 2,\ \frac{4}{27}(|q_1|+1)^3-(|q_1|+1)\le|q_2|\le 1\right\} \qquad (15)$$

It is easy to prove that inequality $\frac{1}{2}\le|q_1|\le 2$ holds for $s\in\left(0,\dfrac{(1+\alpha)(1+2\alpha)}{2\alpha^2+6\alpha+1}\right]$. Now, the first part of the second inequality in (15) is equivalent to

$$\frac{4}{27}\left[\frac{3\alpha}{(1+\alpha)(1+2\alpha)}s+s+2\right]^3-\left[\frac{3\alpha}{(1+\alpha)(1+2\alpha)}s+s+2\right]-$$
$$-s^2\left[\frac{3\alpha}{2(1+\alpha)(1+2\alpha)}+\frac{\alpha(\alpha-1)}{(1+\alpha)^3(1+2\alpha)}+\frac{1}{6}\right]-s\left[\frac{3\alpha}{(1+\alpha)(1+2\alpha)}+\frac{1}{2}\right]-\frac{1}{3}$$
$$\le 0.$$

For $\alpha=1$, the above inequality implies

$$\frac{1}{2}s^3+\frac{19}{12}s^2+\frac{5}{12}s-\frac{31}{27}\le 0,$$

and this inequality holds for $s\in(0,0.67243...]$.

The second part of the inequality in (15) is equivalent to

$$s^2\left[\frac{3\alpha}{2(1+\alpha)(1+2\alpha)}+\frac{\alpha(\alpha-1)}{(1+\alpha)^3(1+2\alpha)}+\frac{1}{6}\right]+s\left[\frac{3\alpha}{2(1+\alpha)(1+2\alpha)}+\frac{1}{2}\right]+\frac{1}{3}\le 1,$$

that is

$$\frac{1}{6(1+\alpha)^3(1+2\alpha)}[s^2(2\alpha^4+16\alpha^3+33\alpha^2+8\alpha+1)+s(6\alpha^4+30\alpha^3+45\alpha^2+ \\ +24\alpha+3)-(8\alpha^4+28\alpha^3+36\alpha^2+20\alpha+4)]\leq 0,$$

and this inequality holds for

$$s\in\left(0,\frac{-3(1+\alpha)^2(2\alpha^2+6\alpha+1)+\sqrt{\Delta}}{2(2\alpha^4+16\alpha^3+33\alpha^2+8\alpha+1)}\right], \tag{16}$$

where $\Delta=(a+1)^3(100a^5+796a^4+1888a^3+1252a^2+277a+25)$.
For $\alpha=1$, the above inequality holds for $s\in(0,0.65241...]$.

From the above results, we conclude that $(q_1,q_2)\in D_2$ for s in (16).

Next, we consider

$$D_5=\{(q_1,q_2):|q_1|\leq 2,\ |q_2|\geq 1\} \tag{17}$$

We observe that first inequality in (17), $|q_1|\leq 2$, holds for

$$s\in\left(0,\frac{(1+\alpha)(1+2\alpha)}{2\alpha^2+6\alpha+1}\right] \tag{18}$$

The second inequality in (17) is equivalent to

$$s^2\left[\frac{3\alpha}{2(1+\alpha)(1+2\alpha)}+\frac{\alpha(\alpha-1)}{(1+\alpha)^3(1+2\alpha)}+\frac{1}{6}\right]+s\left[\frac{3\alpha}{2(1+\alpha)(1+2\alpha)}+\frac{1}{2}\right]+\frac{1}{3}\geq 1,$$

which holds for

$$s\in\left[\frac{-3(1+\alpha)^2(2\alpha^2+6\alpha+1)+\sqrt{\Delta}}{2(2\alpha^4+16\alpha^3+33\alpha^2+8\alpha+1)},1\right]. \tag{19}$$

Therefore, from (18) and (19), $(q_1,q_2)\in D_5$ for

$$s\in\left(0,\frac{(1+\alpha)(1+2\alpha)}{2\alpha^2+6\alpha+1}\right]\cap\left[\frac{-3(1+\alpha)^2(2\alpha^2+6\alpha+1)+\sqrt{\Delta}}{2(2\alpha^4+16\alpha^3+33\alpha^2+8\alpha+1)},1\right],$$

that is

$$s\in\left[\frac{-3(1+\alpha)^2(2\alpha^2+6\alpha+1)+\sqrt{\Delta}}{2(2\alpha^4+16\alpha^3+33\alpha^2+8\alpha+1)},\frac{(1+\alpha)(1+2\alpha)}{2\alpha^2+6\alpha+1}\right]. \tag{20}$$

Finaly, we consider

$$D_6=\left\{(q_1,q_2):2\leq|q_1|\leq 4,\ |q_2|\geq\frac{1}{12}(q_1^2+8)\right\} \tag{21}$$

After some simple computation we observe that first inequality, $2 \leq q_1 \leq 4$, holds for

$$s \in \left(\frac{(1+\alpha)(1+2\alpha)}{2\alpha^2+6\alpha+1}, 1 \right] \tag{22}$$

The second inequality in (21) is equivalent to

$$s^2 \left[\frac{3\alpha}{2(1+\alpha)(1+2\alpha)} + \frac{\alpha(\alpha-1)}{(1+\alpha)^3(1+2\alpha)} + \frac{1}{6} \right] + s \left[\frac{3\alpha}{2(1+\alpha)(1+2\alpha)} + \frac{1}{2} \right] -$$

$$-\frac{1}{12} \left[\frac{2\alpha^2+6\alpha+1}{(1+\alpha)(1+2\alpha)} s + 1 \right]^2 - \frac{1}{3} \geq 0,$$

which holds for

$$\left[\frac{(1+\alpha)(1+2\alpha)\left[2(1+\alpha)(2\alpha^2+6\alpha+1)+\sqrt{\Delta}\right]}{4\alpha^5+40\alpha^4+100\alpha^3+46\alpha^2+7\alpha+1}, 1 \right] \tag{23}$$

where $\Delta = 3(1+\alpha)(12\alpha^5+104\alpha^4+252\alpha^3+146\alpha^2+29\alpha+3)$.
For $\alpha = 1$, the above inequality holds for $s \in [0.64352, 1]$.
Therefore, from (22) and (23), $(q_1, q_2) \in D_6$ for

$$s \in \left(\frac{(1+\alpha)(1+2\alpha)}{2\alpha^2+6\alpha+1}, 1 \right] \tag{24}$$

In view of the conditions from (20) and (24) it follows that $(q_1, q_2) \in D_5 \cup D_6$ for

$$s \in \left[\frac{-3(1+\alpha)^2(2\alpha^2+6\alpha+1)+\sqrt{\Delta}}{2(2\alpha^4+16\alpha^3+33\alpha^2+8\alpha+1)}, 1 \right].$$

□

Remark 2. *For $\alpha = 1$ we find the result obtained by A. Ebadian et al. [2] in Theorem 2.*

3 Logarithmic coefficients of the inverse of $\mathcal{M}_\alpha(\varphi)$

Motivated by the results obtained in [9], in this section, we investigate the upper bounds of the logarithmic coefficients of an inverse function of $\mathcal{M}_\alpha(\varphi)$.

Theorem 3. *Let $f \in \mathcal{M}_\alpha(\varphi)$ and let F, given by (5), be the inverse function of f. Then, the logarithmic coefficients Γ_n, $n = 1, 2, 3$, of F satisfy the inequalities:*

$$|\Gamma_1| \leq \frac{|B_1|}{2(1+\alpha)};$$

$$|\Gamma_2| \leq \begin{cases} \dfrac{|B_1|}{4(1+2\alpha)} & if \quad |(1+\alpha)^2 B_2 - (2+3\alpha)B_1^2| \\ & \qquad \leq (1+\alpha)^2|B_1|; \\ \dfrac{|(1+\alpha)^2 B_2 - (2+3\alpha)B_1^2|}{4(1+2\alpha)(1+\alpha)^2} & if \quad |(1+\alpha)^2 B_2 - (2+3\alpha)B_1^2| \\ & \qquad \geq (1+\alpha)^2|B_1|; \end{cases}$$

and if B_1, B_2 and B_3 are real, then

$$|\Gamma_3| \leq \frac{|B_1|}{6(1+3\alpha)} H(q_1, q_2);$$

where $H_1(q_1, q_2)$ is given by Lemma 2, with

$$\begin{cases} q_1 &= \dfrac{2B_2}{B_1} - \dfrac{3(3+7\alpha)}{2(1+\alpha)(1+2\alpha)} B_1; \\ q_2 &= \dfrac{29\alpha^2 + 34\alpha + 9}{2(1+\alpha)^3(1+2\alpha)} B_1^2 - \dfrac{3(3+7\alpha)}{2(1+\alpha)(1+2\alpha)} B_2 + \dfrac{B_3}{B_1}. \end{cases}$$

Proof. Let $f \in \mathcal{M}_\alpha(\varphi)$. Proceeding as in the proof of Theorem 1, we have the equalities (14).

Next, replacing A_2, A_3 and A_4 from (6) in (9), we obtain

$$\begin{cases} 2\Gamma_1 &= -a_2 \\ 4\Gamma_2 &= -2a_3 + 3a_2^2 \\ 6\Gamma_3 &= -3a_4 + 12a_2a_3 - 10a_2^3 \end{cases} \tag{25}$$

Substituting a_2 from (25) in (14), we obtain $\Gamma_1 = \dfrac{B_1c_1}{-2(1+\alpha)}$ and applying Lemma 1, we get $|\Gamma_1| \leq \dfrac{|B_1|}{2(1+\alpha)}$.

Making use of the second equalities from both (25) and (14), and after elementary computations, we have

$$\Gamma_2 = \frac{(1+\alpha)^2 B_1 c_2 + \left[(1+\alpha)^2 B_2 - (2+3\alpha)B_1^2\right] c_1^2}{-4(1+2\alpha)(1+\alpha)^2}.$$

Taking into account Lemma 1, we obtain

$$|\Gamma_2| \leq \frac{(1+\alpha)^2|B_1|(1-|c_1|^2) + |(1+\alpha)^2 B_2 - (2+3\alpha)B_1^2||c_1|^2}{4(1+2\alpha)(1+\alpha)^2},$$

from which, with $|c_1| \leq 1$, we obtain the required upper bound. In order to obtain the estimate of $|\Gamma_3|$, we use the last equalities from (25) and (14), and we get

$$\Gamma_3 = \frac{B_1}{-6(1+3\alpha)}\left\{c_3 + \left[\frac{2B_2}{B_1} - \frac{3(3+7\alpha)B_1}{2(1+\alpha)(1+2\alpha)}\right]c_1c_2 + \right.$$
$$\left. + \left[\frac{29\alpha^2+34\alpha+9}{2(1+\alpha)^3(1+2\alpha)}B_1^2 - \frac{3(3+7\alpha)}{2(1+\alpha)(1+2\alpha)}B_2 + \frac{B_3}{B_1}\right]c_1^3\right\}.$$

Consider $q_1 = \frac{2B_2}{B_1} - \frac{3(3+7\alpha)B_1}{2(1+\alpha)(1+2\alpha)}$ and $q_2 = \frac{29\alpha^2+34\alpha+9}{2(1+\alpha)^3(1+2\alpha)}B_1^2 - \frac{3(3+7\alpha)}{2(1+\alpha)(1+2\alpha)}B_2 + \frac{B_3}{B_1}$. An application of Lemma (2) yields

$$|\Gamma_3| \leq \frac{|B_1|}{6(1+3\alpha)}|c_3 + q_1c_1c_2 + q_2c_1^3| \leq \frac{B_1}{6(1+3\alpha)}H(q_1;q_2).$$

□

For $\alpha = 1$ we obtain the following corollary:

Corollary 1. *Let $f \in \mathcal{C}(\varphi)$ and let F, given by (5), be the inverse function of f. Then, the logarithmic coefficients $\Gamma_n,\ n = 1,2,3,$ of F satisfy the inequalities:*

$$|\Gamma_1| \leq \frac{|B_1|}{4};$$

$$|\Gamma_2| \leq \begin{cases} \dfrac{|B_1|}{12} & if \quad |4B_2 - 5B_1^2| \leq 4|B_1|; \\[2ex] \dfrac{|4B_2 - 5B_1^2|}{48} & if \quad |4B_2 - 5B_1^2| \geq 4|B_1|; \end{cases}$$

and if B_1, B_2 and B_3 are real, then

$$|\Gamma_3| \leq \frac{|B_1|}{24}H(q_1,q_2);$$

where $H_1(q_1,q_2)$ is given by Lemma 2, with

$$\begin{cases} q_1 &= \dfrac{2B_2}{B_1} - \dfrac{5}{2}B_1; \\[2ex] q_2 &= \dfrac{3}{2}B_1^2 - \dfrac{5}{2}B_2 + \dfrac{B_3}{B_1}. \end{cases}$$

Remark 3. *For $\varphi(z) = \frac{1+(1-2\alpha)z}{1-z}$ in the above corollary, we find the result obtained by S. Ponnusamy et al. [9] in Theorems 10,11 and 13.*

Conclusions.

In this paper we consider a class $\mathcal{M}_\alpha(\varphi)$ of analytic functions defined by subordination. For the class $\mathcal{M}_\alpha(\varphi)$ we investigated the upper bounds for the logarithmic coefficients $\gamma_n,\ n \in \{1,2,3\}$ and also for $\Gamma_n,\ n \in \{1,2,3\}$, the logarithmic coefficients for the inverse of $\mathcal{M}_\alpha(\varphi)$. The results obtained in this paper could be a subject of further investigation related to logarithmic coefficients γ_n and Γ_n for $n \geq 4$.

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