Bulletin of the *Transilvania* University of Braşov
Series III: Mathematics and Computer Science, Vol. 3(65), No. 2 - 2023, 121-128
https://doi.org/10.31926/but.mif.2023.3.65.2.10

ON GEOMETRICALLY CONVEX FUNCTIONS WITH SOME APPLICATIONS

Nicuşor MINCULETE[*,1] and Hamid Reza MORADI[2]

Dedicated to Professor Radu Păltănea on the occasion of his 70th anniversary

Abstract

This paper aims to present several new properties of geometrically convex functions similar to some those known properties for convex functions. Finally, we show some improvements of Young's inequality.

2010 *Mathematics Subject Classification:* 26A51, 26D15
Key words: geometrically convex function, Hermite-Hadamard inequality, Young's inequality.

1 Introduction

Many types of convexity are studied in specialized literature (see [1], [2]). Among them is also geometrically convex (see [3], [6], [7], [8]). Recall that the function $f:(0,\infty)\to(0,\infty)$ is said to be geometrically convex, if for any $a,b>0$,

$$f\left(a^{\lambda}b^{\mu}\right)\le f(a)^{\lambda}f(b)^{\mu};\ \ (\lambda,\mu\ge 0 \text{ and } \lambda+\mu=1)\,. \tag{1}$$

In other words, for any $0\le t\le 1$,

$$f\left(a^{1-t}b^{t}\right)\le f^{1-t}\left(a\right)f^{t}\left(b\right). \tag{2}$$

The exponential function $\exp(x)$ is an example of a geometrically convex function. The proof consists in using the inequality of Young, $a^{1-t}b^{t}\le(1-t)a+tb$, for every $a,b\in(0,\infty)$ and for all $t\in[0,1]$, thus

$$\exp\left(a^{1-t}b^{t}\right)\le\exp\left((1-t)a+tb\right)=\exp^{1-t}(a)\exp^{t}(b).$$

[1*] *Corresponding author*, Faculty of Mathematics and Informatics, *Transilvania* University of Braşov, Romania, e-mail: minculete.nicusor@unitbv.ro
[2]Department of Mathematics, Mashhad Branch, Islamic Azad University, Mashhad, Iran, e-mail: hrmoradi@mshdiau.ac.ir

Thus, studying the geometrically convex functions is important because they can provide new characterizations of Young's inequality.

In [5], Niculescu uses these functions while calling them the GG-convex or the multiplicatively convex functions. The same author investigated the class of multiplicatively convex functions showing that every polynomial $P(x)$ with nonnegative coefficients is a geometrically convex function on $[0,\infty)$.

Next, we will say geometrically convex function for a function with the property (1) or (2) because this notion has been widely employed in recent papers (see [9], [10], [11]).

We see that if $\lambda=\mu=\frac{1}{2}$, then inequality (1) becomes

$$f\left(\sqrt{ab}\right)\le\sqrt{f\left(a\right)f\left(b\right)}. \tag{3}$$

A continuous function $f:(0,\infty)\to(0,\infty)$ is geometrically convex function if and only if inequality (3) holds for every $a,b\in(0,\infty)$ (see [5]). Suppose that $f:(0,\infty)\to(0,\infty)$ is geometrically convex function, then the function $F:\mathbb{R}\to\mathbb{R}$ defined by $F(x)=\log\circ f\circ\exp(x)$ is a convex function (see [5]). It is not hard to see that if f is increasing and log-convex, then f is a geometrically convex function. If function $f:(0,\infty)\to(0,\infty)$ is geometrically convex and w_i are positive scalars such that $\sum\limits_{i=1}^{n}w_i=1$, then we have a Jensen-type inequality

$$f\left(\prod_{i=1}^{n}t_i^{w_i}\right)\le\prod_{i=1}^{n}f^{w_i}\left(t_i\right)$$

for all $t_i\in(0,\infty)$.

2 Some properties of the geometrically convex functions

The Hermite-Hadamard inequality states that for any convex function f, the inequality

$$f\left(\frac{a+b}{2}\right)\le\int\limits_0^1 f\left(\left(1-t\right)a+tb\right)dt=\frac{1}{b-a}\int\limits_a^b f(t)dt\le\frac{f\left(a\right)+f\left(b\right)}{2}$$

holds. If the function f on $(0,\infty)$ is geometrically convex, then $\ln f\left(e^x\right)$ is convex on $\mathbb{R}$. This means,

$$\ln f\left(e^{\frac{a+b}{2}}\right)\le\int\limits_0^1\ln f\left(e^{(1-t)a+tb}\right)dt\le\ln\sqrt{f\left(e^a\right)f\left(e^b\right)}.$$

Therefore, we get the following:

$$f\left(e^{\frac{a+b}{2}}\right)\le\exp\left(\int\limits_0^1\ln f\left(e^{(1-t)a+tb}\right)dt\right)\le\sqrt{f\left(e^a\right)f\left(e^b\right)}.$$

If we replace a by $\ln a$ and b by $\ln b$, then we have

$$f\left(\sqrt{ab}\right) \le \exp\left(\int_0^1 \ln f\left(a^{1-t}b^t\right) dt\right) \le \sqrt{f(a) f(b)}. \tag{4}$$

Remark 1. *The inequality* (4) *is equivalent to*

$$f\left(\sqrt{ab}\right) \le \exp\left(\frac{1}{\ln b - \ln a}\int_{\ln a}^{\ln b} \ln f\left(e^t\right) dt\right) \le \sqrt{f(a) f(b)}.$$

From [7, Theorem 6], we found that if $f : (0,\infty) \to (0,\infty)$ is a geometrically convex function, then for any $a, b > 0$

$$f\left(a^{1-\nu}b^{\nu}\right) \le \left(\frac{f\left(\sqrt{ab}\right)}{\sqrt{f(a) f(b)}}\right)^{2r} f^{1-\nu}(a) f^{\nu}(b), \tag{5}$$

and

$$f^{1-\nu}(a) f^{\nu}(b) \le \left(\frac{\sqrt{f(a) f(b)}}{f\left(\sqrt{ab}\right)}\right)^{2R} f\left(a^{1-\nu}b^{\nu}\right). \tag{6}$$

Let $f : (0,\infty) \to (0,\infty)$ be a function. Using the technique from [4], we introduce the following expression:

$$\Delta_\nu(f)(a,b) := \frac{f^{\nu}(a) f^{1-\nu}(b)}{f\left(a^{\nu}b^{1-\nu}\right)},$$

where $a, b \in (0,\infty)$ and $\nu \in \mathbb{R}$. It is easy to see that if f is a geometrically convex function and $\nu \in [0,1]$, then we obtain that $\Delta_\nu(f)(a,b) \ge 1$, for all $a, b \in (0,\infty)$. We find the following properties:

$$\Delta_\nu(f)(a,a) = \Delta_0(f)(a,b) = \Delta_1(f)(a,b) = 1$$

and

$$\Delta_\nu(f)(a,b) = \Delta_{1-\nu}(f)(b,a),$$

for every $a, b \in (0,\infty)$.

Lemma 1. *Let $f : (0,\infty) \to (0,\infty)$ be a function and $a, b \in (0,\infty)$. If $\nu \in \mathbb{R}$, then the following equalities hold:*

$$\Delta_\nu(f)(a,b) = \Delta_{2\nu}(f)\left(\sqrt{ab}, b\right) \cdot \Delta_{1/2}^{2\nu}(f)(a,b) \tag{7}$$

and

$$\Delta_\nu(f)(a,b) = \Delta_{2\nu-1}(f)\left(a, \sqrt{ab}\right) \cdot \Delta_{1/2}^{2-2\nu}(f)(a,b). \tag{8}$$

Proof. Using the definition of $\Delta_\nu(f)(a,b)$, by regrouping the terms, we receive

$$\begin{aligned}
&\Delta_{2\nu}(f)\left(\sqrt{ab},b\right)\cdot\Delta_{1/2}^{2\nu}(f)(a,b)\\
&=\frac{f^{2\nu}(\sqrt{ab})f^{1-2\nu}(b)}{f\left(\sqrt{ab}^{2\nu}b^{1-2\nu}\right)}\frac{f^{\nu}(a)f^{1-\nu}(b)}{f^{2\nu}\left(\sqrt{ab}\right)}\\
&=\frac{f^{\nu}(a)f^{1-\nu}(b)}{f\left(a^{\nu}b^{1-\nu}\right)}=\Delta_\nu(f)(a,b),
\end{aligned}$$

which signifies the first relation of the statement. In the same method, we have

$$\begin{aligned}
&\Delta_{2\nu-1}(f)\left(a,\sqrt{ab}\right)\cdot\Delta_{1/2}^{2-2\nu}(f)(a,b)\\
&=\frac{f^{2\nu-1}(a)f^{2-2\nu}(\sqrt{ab})}{f\left(a^{2\nu-1}\sqrt{ab}^{2-2\nu}\right)}\frac{f^{1-\nu}(a)f^{1-\nu}(b)}{f^{2-2\nu}\left(\sqrt{ab}\right)}\\
&=\frac{f^{\nu}(a)f^{1-\nu}(b)}{f\left(a^{\nu}b^{1-\nu}\right)}=\Delta_\nu(f)(a,b),
\end{aligned}$$

which implies the second relation of the statement. □

Next, we study de case when $\nu\notin(0,1)$ and $f:(0,\infty)\to(0,\infty)$ is a geometrically convex function.

Lemma 2. *Let $f:(0,\infty)\to(0,\infty)$ be a geometrically convex function. If $\nu\in\mathbb{R}\setminus(0,1)$, then the following inequality holds:*

$$\Delta_\nu(f)(a,b)\le 1$$

for all $a,b\in(0,\infty)$.

Proof. We study two cases: I) If $\nu\le 0$, then we get

$$\Delta_\nu(f)(a,b)=\frac{f^{\nu}(a)f^{1-\nu}(b)}{f\left(a^{\nu}b^{1-\nu}\right)}=\frac{f^{1-\nu}(b)}{f^{-\nu}(a)f\left(a^{\nu}b^{1-\nu}\right)}$$

$$=\left[\frac{f(b)}{f^{\frac{-\nu}{1-\nu}}(a)f^{1/(1-\nu)}\left(a^{\nu}b^{1-\nu}\right)}\right]^{1-\nu}\le\left[\frac{f(b)}{f\left(a^{\frac{-\nu}{1-\nu}}\left(a^{\nu}b^{1-\nu}\right)^{1/(1-\nu)}\right)}\right]^{1-\nu}=\left[\frac{f(b)}{f(b)}\right]^{1-\nu}=1.$$

II) If $\nu\ge 1$, then, using the fact that f is geometrically convex, we deduce

$$\Delta_\nu(f)(a,b)=\frac{f^{\nu}(a)f^{1-\nu}(b)}{f\left(a^{\nu}b^{1-\nu}\right)}=\frac{f^{\nu}(a)}{f^{\nu-1}(b)f\left(a^{\nu}b^{1-\nu}\right)}$$

$$=\left[\frac{f(a)}{f^{\frac{\nu-1}{\nu}}(b)f^{1/\nu}\left(a^{\nu}b^{1-\nu}\right)}\right]^{\nu}\le\left[\frac{f(a)}{f\left(b^{\frac{\nu-1}{\nu}}\left(a^{\nu}b^{1-\nu}\right)^{1/\nu}\right)}\right]^{\nu}=\left[\frac{f(a)}{f(a)}\right]^{\nu}=1.$$

Therefore, the inequality of the statement is true. □

Remark 2. *If $f : (0, \infty) \to (0, \infty)$ is a geometrically convex function with $\nu \in \mathbb{R} - (0, 1)$, then the following inequality holds:*

$$f^{\nu}(a) f^{1-\nu}(b) \geq f\left(a^{\nu} b^{1-\nu}\right)$$

for all $a, b \in (0, \infty)$.

Proposition 1. *Let $f : (0, \infty) \to (0, \infty)$ be a geometrically convex function. If $\nu \in [0, 1]$, then the following inequality holds:*

$$\Delta_{1/2}^{2r}(f)(a, b) \leq \Delta_{\nu}(f)(a, b) \leq \Delta_{1/2}^{2R}(f)(a, b) \tag{9}$$

for all $a, b \in (0, \infty)$, where $r = \min\{\nu, 1 - \nu\}$ and $R = \max\{\nu, 1 - \nu\}$.

Proof. For $\nu \in [0, \frac{1}{2}]$, we have $2\nu \in [0, 1]$ and $2\nu - 1 \in [-1, 0]$, so, we show that $\Delta_{2\nu}(f)\left(\sqrt{ab}, b\right) \geq 1$ and using Lemma 2 we have inequality $\Delta_{2\nu-1}(f)\left(a, \sqrt{ab}\right) \leq 1$. From equalities (7) and (8), we obtain

$$\Delta_{1/2}^{2\nu}(f)(a, b) \leq \Delta_{\lambda}(f)(a, b) \leq \Delta_{1/2}^{2(1-\nu)}(f)(a, b). \tag{10}$$

For $\nu \in [\frac{1}{2}, 1]$, in the same way, we prove that

$$\Delta_{1/2}^{2(1-\nu)}(f)(a, b) \leq \Delta_{\nu}(f)(a, b) \leq \Delta_{1/2}^{2\nu}(f)(a, b). \tag{11}$$

Consequently, the inequality of the statement is true when combining inequalities (10) and (11). □

Remark 3. *The inequalities from Proposition 1 become,*

$$\left(\frac{\sqrt{f(a) f(b)}}{f\left(\sqrt{ab}\right)}\right)^{2r} \leq \frac{f^{\nu}(a) f^{1-\nu}(b)}{f\left(a^{\nu} b^{1-\nu}\right)} \leq \left(\frac{\sqrt{f(a) f(b)}}{f\left(\sqrt{ab}\right)}\right)^{2R},$$

which in fact are inequalities (5) and (6).

The gamma function $\Gamma(x)$ is defined as $\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt$. Since Γ is increasing and log-*convex on $[2, \infty)$, then Γ is a geometrically convex function on $[2, \infty)$. Therefore we have*

$$\left(\frac{\sqrt{\Gamma(a) \Gamma(b)}}{\Gamma\left(\sqrt{ab}\right)}\right)^{2r} \leq \frac{\Gamma^{\nu}(a) \Gamma^{1-\nu}(b)}{\Gamma\left(a^{\nu} b^{1-\nu}\right)} \leq \left(\frac{\sqrt{\Gamma(a) \Gamma(b)}}{\Gamma\left(\sqrt{ab}\right)}\right)^{2R}$$

for all $a, b \in [2, \infty)$ and $\nu \in [0, 1]$, where $r = \min\{\nu, 1 - \nu\}$ and $R = \max\{\nu, 1 - \nu\}$.

Theorem 1. *Suppose that $f : (0, \infty) \to (0, \infty)$ is a geometrically convex function. If $\nu \in [0, \frac{1}{2}]$, then the following inequality holds:*

$$\Delta_{1/2}^{2\nu}(f)(a, b) \cdot \Delta_{1/2}^{2\min\{2\nu, 1-2\nu\}}(f)\left(\sqrt{ab}, b\right) \leq \Delta_{\nu}(f)(a, b)$$

$$\leq \Delta_{1/2}^{2\nu}(f)(a,b) \cdot \Delta_{1/2}^{2\max\{2\nu,1-2\nu\}}(f)\left(\sqrt{ab},b\right) \tag{12}$$

and if $\nu \in [\frac{1}{2},1]$*, then the inequality*

$$\Delta_{1/2}^{2(1-\nu)}(f)(a,b) \cdot \Delta_{1/2}^{2\min\{2\nu-1,2-2\nu\}}(f)\left(a,\sqrt{ab}\right) \leq \Delta_{\nu}(f)(a,b)$$

$$\leq \Delta_{1/2}^{2(1-\nu)}(f)(a,b) \cdot \Delta_{1/2}^{2\max\{2\nu-1,2-2\nu\}}(f)\left(a,\sqrt{ab}\right)$$

holds.

Proof. For $\nu \in [0,\frac{1}{2}]$, we have $2\nu \in [0,1]$ and replacing a by $\sqrt{ab}$ in inequality (9), we deduce

$$\Delta_{1/2}^{2\min\{2\nu,1-2\nu\}}(f)\left(\sqrt{ab},b\right) \leq \Delta_{2\nu}(f)\left(\sqrt{ab},b\right)$$

$$\leq \Delta_{1/2}^{2\max\{2\nu,1-2\nu\}}(f)\left(\sqrt{ab},b\right). \tag{13}$$

Consequently, we show the first inequality of the statement by combining equality (7) with inequality (13). For $\nu \in [\frac{1}{2},1]$, we have $2\nu-1 \in [0,1]$ and replacing b by $\sqrt{ab}$ in inequality (9), we deduce

$$\Delta_{1/2}^{2\min\{2\lambda-1,2-2\lambda\}}(f)\left(a,\sqrt{ab}\right) \leq \Delta_{2\nu-1}(f)\left(a,\sqrt{ab}\right)$$

$$\leq \Delta_{1/2}^{2\max\{2\nu-1,2-2\nu\}}(f)\left(a,\sqrt{ab}\right). \tag{14}$$

Consequently, combining equality (8) with inequality (14), we establish the second inequality of the statement. In fact, this inequality is obtained by interchanging ν with $1-\nu$ and a with b in inequality (13). □

Remark 4. *For* $\nu \in [0,\frac{1}{2}]$*, then inequality* (12) *becomes,*

$$\left(\frac{\sqrt{f(a)f(b)}}{f\left(\sqrt{ab}\right)}\right)^{2\nu}\left(\frac{\sqrt{f\left(\sqrt{ab}\right)f(b)}}{f\left(\sqrt[4]{ab^3}\right)}\right)^{2r'} \leq \frac{f^{\nu}(a)f^{1-\nu}(b)}{f\left(a^{\nu}b^{1-\nu}\right)}$$

$$\leq \left(\frac{\sqrt{f(a)f(b)}}{f\left(\sqrt{ab}\right)}\right)^{2\nu}\left(\frac{\sqrt{f\left(\sqrt{ab}\right)f(b)}}{f\left(\sqrt[4]{ab^3}\right)}\right)^{2R'},$$

where $r' = \min\{2\nu,1-2\nu\}$ *and* $R' = \max\{2\nu,1-2\nu\}$*. For* $\nu \in [\frac{1}{2},1]$*, then inequality* (13) *becomes,*

$$\left(\frac{\sqrt{f(a)f(b)}}{f\left(\sqrt{ab}\right)}\right)^{2-2\nu}\left(\frac{\sqrt{f(a)f\left(\sqrt{ab}\right)}}{f\left(\sqrt[4]{a^3b}\right)}\right)^{2r''} \leq \frac{f^{\nu}(a)f^{1-\nu}(b)}{f\left(a^{\nu}b^{1-\nu}\right)}$$

$$\leq \left(\frac{\sqrt{f(a)\,f(b)}}{f\left(\sqrt{ab}\right)}\right)^{2-2\nu}\left(\frac{\sqrt{f(a)\,f\left(\sqrt{ab}\right)}}{f\left(\sqrt[4]{a^3b}\right)}\right)^{2R''},$$

where $r'' = \min\{2\nu-1, 2-2\nu\}$ *and* $R'' = \max\{2\nu-1, 2-2\nu\}$, *these refines inequalities* (5) *and* (6).

If we take instead of f the exponential function $\exp(x)$, then we get the following improvements of the inequality of Young:

$$\nu(\sqrt{a}-\sqrt{b})^2+r'(\sqrt[4]{ab}-\sqrt{b})^2 \leq \nu a+(1-\nu)b-a^{\nu}b^{1-\nu} \leq \nu(\sqrt{a}-\sqrt{b})^2+R'(\sqrt[4]{ab}-\sqrt{b})^2 \tag{15}$$

for every $a, b \in (0,\infty)$ and for all $\nu \in [0, \frac{1}{2}]$, where $r' = \min\{2\nu, 1-2\nu\}$ and $R' = \max\{2\nu, 1-2\nu\}$ and

$$\begin{aligned}&(1-\nu)(\sqrt{a}-\sqrt{b})^2 + r''(\sqrt{a}-\sqrt[4]{ab})^2\\ &\leq (1-\nu)a+\nu b-a^{1-\nu}b^{\nu} \leq (1-\nu)(\sqrt{a}-\sqrt{b})^2 + R''(\sqrt{a}-\sqrt[4]{ab})^2\end{aligned} \tag{16}$$

for every $a, b \in (0,\infty)$ and for all $\nu \in [\frac{1}{2}, 1]$, where $r'' = \min\{2\nu-1, 2-2\nu\}$ and $R'' = \max\{2\nu-1, 2-2\nu\}$.

If if we take $\frac{a}{b} = t \geq 0$, then inequalities (15) and (16) becomes

$$\nu(\sqrt{t}-1)^2 + r'(\sqrt[4]{t}-1)^2 \leq \nu t+1-\nu-t^{\nu} \leq \nu(\sqrt{t}-1)^2 + R'(\sqrt[4]{t}-1)^2$$

for every $t \in (0,\infty)$ and for all $\nu \in [0, \frac{1}{2}]$, where $r' = \min\{2\nu, 1-2\nu\}$ and $R' = \max\{2\nu, 1-2\nu\}$, and

$$(1-\nu)(\sqrt{t}-1)^2+r''(\sqrt{t}-\sqrt[4]{t})^2 \leq (1-\nu)t+\nu-t^{1-\nu} \leq (1-\nu)(\sqrt{t}-1)^2+R''(\sqrt{t}-\sqrt[4]{t})^2$$

for every $t \in (0,\infty)$ and for all $\nu \in [\frac{1}{2}, 1]$, where $r'' = \min\{2\nu-1, 2-2\nu\}$ and $R'' = \max\{2\nu-1, 2-2\nu\}$.

For $p \geq 2$ and $\frac{1}{p}+\frac{1}{q} = 1$ we replace a by a^p and b by b^q in (15), thus, we deduce

$$\frac{1}{p}(a^{\frac{p}{2}}-b^{\frac{q}{2}})^2+r'(a^{\frac{p}{4}}b^{\frac{q}{4}}-b^{\frac{q}{2}})^2 \leq \frac{a^p}{p}+\frac{b^q}{q}-ab \leq \frac{1}{p}(a^{\frac{p}{2}}-b^{\frac{q}{2}})^2+R'(a^{\frac{p}{4}}b^{\frac{q}{4}}-b^{\frac{q}{2}})^2 \tag{17}$$

for every $a, b \in (0,\infty)$, where $r' = \min\{\frac{2}{p}, \frac{2}{q}-1\}$ and $R' = \max\{\frac{2}{p}, \frac{2}{q}-1\}$.

For $1 < p \leq 2$ and $\frac{1}{p}+\frac{1}{q} = 1$ we replace a by a^p and b by b^q in (16), thus, we deduce

$$\frac{1}{q}(a^{\frac{p}{2}}-b^{\frac{q}{2}})^2+r''(a^{\frac{p}{2}}-a^{\frac{p}{4}}b^{\frac{q}{4}})^2 \leq \frac{a^p}{p}+\frac{b^q}{q}-ab \leq \frac{1}{q}(a^{\frac{p}{2}}-b^{\frac{q}{2}})^2+R''(a^{\frac{p}{2}}-a^{\frac{p}{4}}b^{\frac{q}{4}})^2$$

for every $a, b \in (0,\infty)$, where $r'' = \min\{\frac{2}{p}-1, \frac{2}{q}\}$ and $R'' = \max\{\frac{2}{p}-1, \frac{2}{q}\}$. In fact, this inequality is obtained by interchanging p with q and a with b in inequality (17).

Notice that the above inequalities help us to improve Bernoulli's inequality and Hölder's inequality.

Acknowledgments: The authors express their deepest gratitude to the anonymous reviewers for their important comments and suggestions, which significantly improved our article.

References

[1] Dragomir, S.S., *Bounds for the normalised Jensen functional*, Bull. Austral. Math. Soc. **3** (2006), 471–478.

[2] Furuichi, S., Moradi, H.R., Zardadi, A. *Some new Karamata type inequalities and their applications to some entropies*, Rep. Math. Phys. **84** (2019), no. 2, 201–214.

[3] Işcan, I., *Some new Hermite-Hadamard type inequalities for geometrically convex functions*, Mathematics and Statistics, **1** (2013), no. 2, 86-91.

[4] Minculete, N. *On several inequalities related to convex functions*, accepted for publication in J. Math. Ineq., 2023.

[5] Niculescu, C.P., *Convexity according to the geometric mean*, Math. Inequal. Appl. **3** (2000), no. 2, 155–167.

[6] Sababheh, M., Moradi, H.R., Furuichi, S., *Integrals refining convex inequalities*, Bull. Malays. Math. Sci. Soc. **43** (2020), 2817–2833 .

[7] Sababheh, M., Moradi, H.R., Furuichi, S., *Operator inequalities via geometric convexity*, Math. Inequal. Appl. **22** (2019), no. 4, 1215–1231.

[8] Taghavi, A., Darvish, V., Nazari, H.M., Dragomir, S.S., *Hermite-Hadamard type inequalities for operator geometrically convex functions*, Monatshefte für Mathematik, **181** (2016), 187—203.

[9] Xi, B.Y., Bai, R.F., Qi, F., *Hermite–Hadamard type inequalities for the m– and (α, m)–geometrically convex functions*, Aequat. Math. **84** (2012), 261–269.

[10] Zhang, T.-Y., Ji, A.-P., Qi, F., *On integral inequalities of Hermite-Hadamard type for s-geometrically convex functions*, Abstr. Appl. Anal. **2012** (2012), Article number 560586.

[11] Zhang, X.-M., *Geometrically Convex Functions*, Anhui University Press, Hefei (2004) (in Chinese).