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A WEAK SOLUTION TO A KIRKOFF TYPE PROBLEM WITH DISCONTINUOUS NONLINEARITIES

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Dedicated to Professor Radu Păltănea on the occasion of his 70th anniversary

Abstract

This paper is devoted to the study of a class of Kirchhoff-type problems with discontinuous nonlinearities with Neumann boundary data. Here, by employing the topological degree methods for the abstract Hammerstein equation, we establish the existence of at least one solution.

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1 Introduction and hypotheses

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ be a bounded open set with smooth boundary $\partial\Omega$, and p be a real number such that $2 < p < \infty$. The main purpose of this paper is to demonstrate the existence of weak solutions for the following Neumann boundary value problems with discontinuous nonlinearities of the Kirchhoff type

$$\begin{cases} -M\Big(\displaystyle\int_\Omega \Theta(x,\nabla u)dx\Big) div(\sigma(x,\nabla u)) + u \in -\big[\,\underline{\psi}(x,u), \overline{\psi}(x,u)\big] & \text{in} \quad \Omega, \\ \displaystyle\sum_{i=1}^N \sigma(x,\partial_i u)\cdot \eta_i = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1}$$

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where η_i are the components of the outer normal unit vector, $\sigma(x,\xi) : \Omega\times\mathbb{R}^N \longrightarrow \mathbb{R}^N$ is a Carathéodory vector-valued function, such that $\sigma(x,\xi) = \nabla_\xi\Theta(x,\xi)$, where $\Theta(x,\xi) : \Omega\times\mathbb{R}^N \longrightarrow \mathbb{R}$. Suppose that σ and Θ satisfy the following hypotheses, for a. e. in $x\in\Omega$ and all $\xi,\xi'\in\mathbb{R}^N$, $(\xi\neq\xi')$.

(A_1) $\quad\Theta(x,0)=0,$

(A_2) $\quad\sigma(x,\xi)\cdot\xi\geq\alpha\big|\xi\big|^p,$

(A_3) $\quad|\sigma(x,\xi)|\leq\beta\left(k(x)+|\xi|^{p-1}\right),$

(A_4) $\quad\big[\sigma(x,\xi)-\sigma(x,\xi')\big]\cdot(\xi-\xi')>0,$

where $\alpha,\ \beta$ are some positive constants and $k(x)$ is a positive function in $L^{p'}(\Omega)(p'$ is the conjugate exponent of $p)$.
(M_0) $M:\mathbb{R}^+\to\mathbb{R}^+$ is continuous and non-decreasing function, for which there exist two positive constant m_0 and m_1 such that $m_0\leq M(t)\leq m_1$ for all $t\in[0,+\infty[$.
Furthermore, The functions $\psi:\Omega\times\mathbb{R}\to\mathbb{R}$ is a possibly discontinuous function, we "fill the discontinuity gaps" of ψ, replacing ψ by an interval $\big[\underline{\psi}(x,u),\overline{\psi}(x,u)\big]$, where

$$\underline{\psi}(x,s)=\liminf_{\eta\to s}\psi(x,\eta)=\lim_{\delta\to0^+}\inf_{|\eta-s|<\delta}\psi(x,\eta),$$

$$\overline{\psi}(x,s)=\limsup_{\eta\to s}\psi(x,\eta)=\lim_{\delta\to0^+}\sup_{|\eta-s|<\delta}\psi(x,\eta).$$

Assume that $\psi:\Omega\times\mathbb{R}\to\mathbb{R}$ is a real-valued function such that

(H_1) $\overline{\psi}$ and $\underline{\psi}$ are super-positionally measurable (i.e, $\overline{\psi}(\cdot,u(\cdot))$ and $\underline{\psi}(\cdot,u(\cdot))$ are measurable on Ω for every measurable function $u:\Omega\to\mathbb{R}$).

(H_2) ψ fulfil the growth condition:

$$|\psi(x,s)|\leq b(x)+c|s|^{p/p'},$$

for almost all $x\in\Omega$ and all $s\in\mathbb{R}$, where $b\in L^{p'}(\Omega)$, c is a positive constant.

The study of these discontinuous nonlinearities problems has grown considerably in recent years due to their presence in the modeling of several physical and biological problems such as the Elenbass equation, population density and dynamics, the obstacle problem and the seepage surface problem. The reader can consult [6, 12, 19, 20] and the references therein. Note that the problem (1) is nonlocal due to the presence of the term $M\Big(\displaystyle\int_\Omega\Theta(x,\nabla u)dx\Big)$. Moreover, our problem has no variational structure because the nonlinear term ψ is discontinuous. This creates serious mathematical difficulties, especially preventing the use of variational methods. This makes the study of such a problem particularly interesting.

In order to overcome the discontinuous difculty, we will transform this Neumann boundary value problems with discontinuous nonlinearities into a new one governed by a Hammerstein equation. Then, we shall employ the topological degree theory developed by Kim in [24] for a class of weakly upper semi-continuous locally bounded set-valued operators of (S_+) type in the framework of real reflexive separable Banach spaces, based on the Berkovits-Tienari degree [8]. The topological degree theory is constructed the first time by Leray-Schauder [26] in their study of the nonlinear equations for compact perturbations of the identity in infinite-dimensional Banach spaces. Furthermore, Browder [9] has developed a topological degree for operators of class (S_+) in reflexive Banach spaces, see also [32]. Among many examples, we refer the reader to do classical works [15, 37] for more details.

In this regard, the problem (1) is a generalization of a model proposed by Kirchhoff [25] in 1883 in the study of the oscillations of stretched strings and plates, More precisely, Kirchhoff established a model model given by the equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \Big(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L \Big|\frac{\partial u}{\partial x}\Big|^2 dx\Big)\frac{\partial^2 u}{\partial x^2} = 0 \tag{2}$$

where $\rho,\ \rho_0,\ h,\ E,\ L$ are all constants, this equation is an extension of the classical d'Alembert's wave equation for free vibrations of elastic strings by considering the effect of a change in the length of the string during the vibration. However, after the famous Lions article [28], this type of problem has attracted the attention of several authors and since that time dozens of articles have been published. We can cite in particular the works of Chipot [13, 14], Corrêa et al. [16, 17] and their references.

We now provide an overview of the results presented in this article, when $\Theta(x,\xi) = \dfrac{1}{p}|\xi|^p$, we get $\sigma(x,\xi) = |\xi|^{p-2}|\xi|$, where $p \geq 2$, then, we obtain the p-Laplace operator. Many problems of Kirchhoff type have been studied, we refer to [23, 27, 29].

In the simplest case $M \equiv 1$ and when we take $-(b(x)|u|^{p-2}u+\lambda H(x,u,\nabla u))+u$ instead of $\big[\underline{\psi}(x,u),\overline{\psi}(x,u)\big]$. In [2], Abbassi et al. demonstrated the existence of a weak solution to the problem

$$\begin{cases} -div\,\sigma(x,u,\nabla u) = b(x)|u|^{p-2}u + \lambda H(x,u,\nabla u) & \text{in} \quad \Omega, \\ \displaystyle\sum_{i=1}^{N} \sigma(x,\partial_i u)\cdot \eta_i = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Where $\sigma(x,u,\nabla u)$ is a Carathéodory's function that tests certain hypothesis. The function $H(x,u,\nabla u)$ is also a Carathéodory's function satisfies only the growth condition. By using the topological degree method. On this topic, we mention the works [1, 4, 30, 35].

In case $p = p(x)$, some interesting studies of problem like (1) by degree theory methods can be found in [10, 11, 36]. Moreover, many problems related to the Kirchhoff term have been studied by a number of authors by employing different techniques as the method of sub-supersolution, fixed point theory, variational methods, genus theory and approximation techniques. For obtain the existence results, we mention the works [7, 18, 21, 34].

On the other hand, when $\sigma(x,s,\xi) = |\xi|^{p_i-2}\cdot\xi$ for all $i \in \{1,\cdots,N\}$, where $\overrightarrow{p} = \{p_1,\ldots,p_N\}$ with p_i are real numbers, $p_i \geq 2$, we get the anisotropic $\overrightarrow{p}$-Laplace operator, we refer to the recent paper by Figueiredo and Silva [22] have established the existence of a nonnegative solution for nonlinear anisotropic elliptic equations. Some related results can be found in [3, 5].
This paper is organized as follows: In the next section, we recall some basic definitions and preliminary results. Section 3 is focused on some auxiliary lemmas. In section 4, we present and prove our main results.

2 Necessary facts on the topological degree

In this section we recall some fundamental definitions and theorems about topological degree theory which are useful for our aim, and we refer the reader to [1, 2, 15, 24] for more details.
Let $\Omega \subset \mathbb{R}^N$ $(N \geq 1)$ be a bounded open set, $p \in \mathbb{R}$ such that $2 < p < \infty$. We will work in the Sobolev space $W^{1,p}(\Omega)$ endowed with the norm

$$\|u\| = \Big(\int_\Omega |u|^p + |\nabla u|^p dx\Big)^{1/p}.$$

The norm in $L^p(\Omega)$ will be denoted by $\|u\|_p = \Big(\int_\Omega |u|^p dx\Big)^{1/p}$.
Let X be a real separable reflexive Banach space with dual X^* and with continuous dual pairing $\langle\cdot,\cdot\rangle$ between X^* and X in this order. The symbol $\rightharpoonup$ stands for weak convergence.
Let Y be another real Banach space.

Definition 1. *[33]*

1. *We say that the set-valued operator* $\mathrm{F} : \Omega \subset X \to 2^{\mathrm{Y}}$ *is bounded, if* F *maps bounded sets into bounded sets.*

2. *We say that the set-valued operator* $\mathrm{F} : \Omega \subset X \to 2^{\mathrm{Y}}$ *is locally bounded at the point* $u \in \Omega$, *iff there is a neighborhood* V *of* u *such that the set* $F(\mathrm{V}) = \bigcup_{u\in V} Fu$ *is bounded.*

Definition 2. *[33] The set-valued operator* $\mathrm{F} : \Omega \subset X \to 2^{\mathrm{Y}}$ *is called*

1. *upper semicontinuous (u.s.c.) at the point* u, *iff, for any open neighborhood* V *of the set* Fu, *there is a neighbhorhood* U *of the point* u *such that* $\mathrm{F}(U) \subseteq V$. *We say that* F *is upper semicontinuous (u.s.c) if it is u.s.c at every* $u \in X$.

2. *weakly upper semicontinuous (w.u.s.c.), if* $F^{-1}(U)$ *is closed in* X *for all weakly closed set* U *in* Y.

Definition 3. *[33] Let* Ω *be a nonempty subset of* X, *the sequence* $(u_n)_{n\geq 1} \subseteq \Omega$ *and* $F : \Omega \subset X \to 2^{X^*}\backslash\emptyset$. *Then, the set-valued operator* F *is*

1. *of type* (S_+), *if* $u_n \rightharpoonup u$ *in* X *and for each sequence* (h_n) *in* X^* *with* $h_n \in Fu_n$ *such that*
$$\limsup_{n\to\infty}\langle h_n, u_n - u\rangle \leq 0,$$
we get $u_n \to u$ *in* X*;*

2. *quasi-monotone, if* $u_n \rightharpoonup u$ *in* X *and for each sequence* (w_n) *in* X^* *such that* $w_n \in Fu_n$ *yield*
$$\liminf_{n\to\infty}\langle w_n, u_n - u\rangle \geq 0.$$

Definition 4. *[33] Let* Ω *be a nonempty subset of* X *such that* $\Omega \subset \Omega_1$, $(u_n)_{n\geq 1} \subseteq \Omega$ *and* $T : \Omega_1 \subset X \to X^*$ *be a bounded operator. Then, the set-valued operator* $F : \Omega \subset X \to 2^X \backslash \emptyset$ *is of type* $(S_+)_T$, *if*

$$\begin{cases} u_n \rightharpoonup u \text{ in } X, \\ Tu_n \rightharpoonup y \text{ in } X^*, \end{cases}$$

and for any sequence (h_n) *in* X *with* $h_n \in Fu_n$ *such that*

$$\limsup_{n\to\infty}\langle h_n, Tu_n - y\rangle \leq 0,$$

we have $u_n \to u$ *in* X*;*

Next, we consider the following sets :

$$\begin{aligned} \mathcal{F}_1(\Omega) &:= \{F : \Omega \to X^* \mid F \text{ is bounded, demicontinuous} \\ &\qquad\qquad\qquad\qquad \text{and satifies condition}(S_+)\}, \\ \mathcal{F}_T(\Omega) &:= \{F : \Omega \to 2^X \mid F \text{ is locally bounded, w.u.s.c.} \\ &\qquad\qquad\qquad\qquad \text{and satifies condition}(S_+)_T\} \end{aligned}$$

For any $\Omega \subset D_F$ and each bounded operator $T : \Omega \to X^*$, where D_F denotes the domain of F.

Remark 1. *We say that the operator* T *is an essential inner map of* F, *if* $T \in \mathcal{F}_1(\overline{G})$.

Lemma 1. *[24, Lemma 1.4] Let* X *be a real reflexive Banach space and* $G \subset X$ *is a bounded open set. Assume that* $T \in \mathcal{F}_1(\overline{G})$ *is continuous and* $S : D_S \subset X^* \to 2^X$ *weakly upper semicontinuous and locally bounded with* $T(\overline{G}) \subset D_s$. *Then the following alternative holds*

1. *If* S *is quasi-monotone, yield* $I + SoT \in \mathcal{F}_T(\overline{G})$, *where* I *denotes the identity operator.*

2. *If* S *is of type* (S_+), *yield* $SoT \in \mathcal{F}_T(\overline{G})$.

Definition 5. *[24] Let $T : \overline{G} \subset X \to X^*$ is to be a bounded operator, a homotopy $H : [0,1] \times \overline{G} \to 2^X$ is called of type $(S_+)_T$, if for every sequence (t_k, u_k) in $[0,1] \times \overline{G}$ and each sequence (a_k) in X with $a_k \in H(t_k, u_k)$ such that*

$$u_k \rightharpoonup u \in X,\ t_k \to t \in [0,1],\ Tu_k \rightharpoonup y \ \text{ in } \quad X^* \text{ and } \quad \limsup_{k\to\infty}\langle a_k, Tu_k - y\rangle \leq 0,$$

we get $u_k \to u$ in X.

Lemma 2. *[24] Let X be a real reflexive Banach space and $G \subset X$ is a bounded open set, $T : \overline{G} \to X^*$ is continuous and bounded. If F, S are bounded and of class $(S_+)_T$, then an affine homotopy $H : [0,1] \times \overline{G} \to 2^X$ giving by*

$$H(t,u) := (1-t)Fu + tSu, \quad \text{for } (t,u) \in [0,1] \times \overline{G},$$

is of type $(S_+)_T$.

Now, we introduce the topological degree for a class of locally bounded, w.u.s.c. and satifies condition $(S_+)_T$ for more details see [24].

Theorem 1. *[24] Let*

$$L = \big\{(F, G, g) \ | \ G \in \mathcal{O},\ T \in \mathcal{F}_1(\overline{G}),\ F \in \mathcal{F}_T(\overline{G}),\ g \notin F(\partial G)\big\},$$

where $\mathcal{O}$ denotes the collection of all bounded open set in X. There exists a unique (Hammerstein type) degree function

$$d : L \longrightarrow \mathbb{Z}$$

such that the following alternative holds:

1. *(* Normalization*) For each $g \in G$, we have $d(I, G, g) = 1$.*
2. *(* Domain Additivity*) Let $F \in \mathcal{F}_T(\overline{G})$. We have*

   $$d(F, G, g) = d(F, G_1, g) + d(F, G_2, g),$$

   with G_1, $G_2 \subseteq G$ disjoint open such that $g \notin F(\overline{G}\backslash(G_1 \cup G_2))$.
3. *(* Homotopy invariance*) If $H : [0,1]\times\overline{G} \to X$ is a bounded admissible affine homotopy with a common continuous essential inner map and g: $[0,1] \to X$ is a continuous path in X such that $g(t) \notin H(t, \partial G)$ for all $t \in [0,1]$, then the value of $d(H(t,\cdot), G, g(t))$ is constant for any $t \in [0,1]$.*
4. *(* Solution Property*) if $d(F, G, g) \neq 0$, then the equation $g \in Fu$ has a solution in G.*

3 Some useful lemmas

The following lemmas allow us to transform this discontinuous nonlinear elliptic problems (1) with Neumann boundary condition into a new one governed by a Hammerstein equation.

Lemma 3. *[3] Let $g \in L^r(\Omega)$ and $g_n \subset L^r(\Omega)$ such that $\|g_n\|_r \leq C,\ 1 < r < \infty$, If $g_n(x) \to g(x)$ a.e. in Ω then $g_n \rightharpoonup g$ weakly in $L^r(\Omega)$.*

Lemma 4. *[3] Assume that (A_2)-(A_4) hold, let $(u_n)_n$ be a sequence in $W^{1,p}(\Omega)$ such that $u_n \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$ and*

$$\int_\Omega \big[\sigma(x, \nabla u_n) - \sigma(x, \nabla u)\big]\nabla(u_n - u)dx \longrightarrow 0, \tag{3}$$

then $u_n \longrightarrow u$ strongly in $W^{1,p}(\Omega)$.

Proof. From (8)

$$D_n \to 0 \quad \text{in } L^1(\Omega),$$

where $D_n = \big[\sigma(x, \nabla u_n) - \sigma(x, \nabla u)\big]\big(\nabla u_n - \nabla u\big)$.
By using $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega)$, we can find a subsequence still denoted by u_n such that

$$\begin{cases} u_n \to u & \text{a.e. in } \Omega, \\ D_n \to 0 & \text{a.e. in } \Omega. \end{cases}$$

Then, there exists a subset B of Ω, of zero measure, such that for $x \in \Omega \backslash B,\ |u(x)| < \infty,\ |\nabla u(x)| < \infty,\ |k(x)| < \infty,\ u_n(x) \to u(x),\ D_n(x) \to 0$.
If we put $\xi_n = \nabla u_n,\ \ \xi = \nabla u$, we get

$$\begin{aligned} D_n(x) &= \big[\sigma(x, \xi_n) - \sigma(x, \xi)\big].\big(\xi_n - \xi\big) \\ &= \sigma(x, \xi_n)\xi_n + \sigma(x, \xi)\xi - \sigma(x, \xi_n)\xi - \sigma(x, \xi)\xi_n \\ &\geq \alpha|\xi_n|^p + \alpha|\xi|^p - \beta\big(k(x) + |\xi_n|^{p-1}\big)|\xi| - \beta\big(k(x) + |\xi|^{p-1}\big)|\xi_n| \\ &\geq \alpha|\xi_n|^p - C_x[1 + |\xi_n|^{p-1} + |\xi_n|], \end{aligned}$$

where C_x is a constant which depends only on x. As $u_n(x) \to u(x)$ a.e. in Ω, we infer $|u_n(x)| \leq P_x$, where $P_x > 0$ is some constant. Then by a standard argument $|\xi_n|$ is bounded uniformly with respect to n, we deduce that

$$D_n(x) \geq |\xi_n|^p\Big(\alpha - \frac{C_x}{|\xi_n|^p} - \frac{C_x}{|\xi_n|} - \frac{C_x}{|\xi_n|^{p-1}}\Big).$$

If $|\xi_n| \to \infty$ (for a subsequence), then $D_n(x) \to \infty$ which makes it absurd. Let now ξ^* be a cluster point of ξ_n. We have $|\xi^*| < \infty$ and by using the continuity of a we have

$$\big[\sigma(x, \xi_n) - \sigma(x, \xi)\big]\big(\xi^* - \xi\big) = 0.$$

According to (A_3), we get $\xi^* = \xi$, which implies that

$$\nabla u_n(x) \longrightarrow \nabla u(x) \qquad \text{a.e. in } \Omega.$$

Since the sequence $\big(\sigma(x,\nabla u_n)\big)$ is bounded in $\big(L^{p'}(\Omega)\big)^N$, and $\sigma(x,\nabla u_n)$ converge to $\sigma(x,\nabla u)$ a.e. in Ω, in view of Lemma 3, we obtain

$$\sigma(x,\nabla u_n) \rightharpoonup \sigma(x,\nabla u) \quad \text{in } (L^{p'}(\Omega))^N \quad \text{a.e. in } \Omega. \tag{4}$$

We take $\overline{y}_n = \sigma(x,\nabla u_n)\nabla u_n$ and $\overline{y} = \sigma(x,\nabla u)\nabla u$. We can write

$$\overline{y}_n \to \overline{y} \quad \text{in } L^1(\Omega).$$

From (A_1), we have

$$\alpha|\nabla u_n|^p \le \sigma(x,\nabla u_n)\nabla u_n.$$

Let $z_n = |\nabla u_n|^p,\ z = |\nabla u|^p,\ y_n = \dfrac{\overline{y}_n}{\alpha}$, and $y = \dfrac{\overline{y}}{\alpha}$.
Thanks to Fatou's lemma,

$$\int_\Omega 2\,y\,dx \le \liminf_{n\to\infty}\int_\Omega y + y_n - |z_n - z|dx,$$

i.e., $0 \le -\limsup\limits_{n\to\infty}\displaystyle\int_\Omega |z_n - z|\,dx$. Then

$$0 \le \liminf_{n\to\infty}\int_\Omega |z_n - z|\,dx \le \limsup_{n\to\infty}\int_\Omega |z_n - z|\,dx \le 0,$$

this implies

$$\nabla u_n \longrightarrow \nabla u \quad \text{in} \quad \big(L^p(\Omega)\big)^N. \tag{5}$$

From the compact embedding of $W^{1,p}(\Omega)$ into $L^p(\Omega)$, we have $\|u_n\|_p \to \|u\|_p$ as $n\to\infty$, from where $\|u_n\| \longrightarrow \|u\|$ (see (5)). As $W^{1,p}(\Omega)$ satisfies the Kadec-Klee property [31, Remark 2.47(a),(c)], it follows from $u_n \rightharpoonup u$ and $\|u_n\| \longrightarrow \|u\|$, the convergence $u_n \longrightarrow u$ in $W^{1,p}(\Omega)$ as $n\to\infty$, which completes the proof. □

New, let us consider the following functional

$$\mathcal{E}(u) = \widehat{M}\Big(\int_\Omega \Theta(x,\nabla u)dx\Big), \qquad \text{for all } u\in W^{1,p}(\Omega)$$

where $\widehat{M}:\ [0,+\infty[\longrightarrow [0,+\infty[$ be the primitive of the function M, defined by

$$\widehat{M}(t) = \int_0^t M(\xi)d\xi.$$

It is well known that $\mathcal{E}$ is well defined and continuously Gâteaux differentiable whose Gâteaux derivatives at point $u\in W^{1,p}(\Omega)$ is the functional $\mathcal{E}'(u)$ in $\big(W^{1,p}(\Omega)\big)^*$ setting by

$$\langle \mathcal{E}'(u), v\rangle = \langle Fu, v\rangle, \qquad \text{for all } u,v\in W^{1,p}(\Omega)$$

where the operator F acting from $W^{1,p}(\Omega)$ to its dual $\big(W^{1,p}(\Omega)\big)^*$ is defined by

$$\langle Fu, v\rangle = M\Big(\int_\Omega \Theta(x,\nabla u)dx\Big)\int_\Omega \sigma(x,\nabla u)\nabla v dx \tag{6}$$

for all $u,v\in W^{1,p}(\Omega)$.

Lemma 5. *Suppose that* (M_0),$(A_1)-(A_4)$ *hold, then*

(i) F *is bounded, strictly monotone, coercive, continuous operator.*

(ii) F *is of type* (S_+).

Proof. $i)$ It is obvious that F is continuous, because F is the Fréchet derivative of $\mathcal{E}$.
Now, we show that the operator F is bounded.
Let $u, v \in W^{1,p}(\Omega)$, by the Hölder's inequality and (M_0), we obtain

$$\begin{aligned} < Fu, v > &= M\Big(\int_\Omega \Theta(x, \nabla u)dx\Big) \int_\Omega \sigma(x, \nabla u)\nabla v dx \\ &\le m_1 \int_\Omega \sigma(x, \nabla u)\nabla v dx \\ &\le m_1 \Big(\int_\Omega |\sigma(x, \nabla u)|^{p'}dx\Big)^{1/p'} \Big(\int_\Omega |\nabla v|^p dx\Big)^{1/p}. \end{aligned}$$

Thanks to the growth condition (A_2), we can easily show that $\Big(\int_\Omega \left|\sigma(x, \nabla u)\right|^{p'}dx\Big)^{1/p'}$ is bounded for all u in $W^{1,p}(\Omega)$. Therefore

$$\langle Fu, v\rangle \le const \Big(\int_\Omega |\nabla v|^p dx\Big)^{1/p},$$

as a result the operator F is bounded.
Next, we prove that F is strictly monotone operator.
For that, we consider the functional $L : W^{1,p}(\Omega) \to \mathbb{R}$ setting by

$$L(u) = \int_\Omega \Theta(x, \nabla u)dx \quad \text{for all} \quad u \in V,$$

so $L \in C^1(W^{1,p}(\Omega), \mathbb{R})$ and

$$\langle L'(u), v\rangle = \int_\Omega \sigma(x, \nabla u)\nabla v dx \qquad \text{for all } \ u, v \in V.$$

By using (A_4), we obtain for any $u, v \in W^{1,p}(\Omega)$ with $u \neq v$

$$\langle L'(u) - L'(v), u - v\rangle > 0$$

which implies that L' is strictly monotone. Thus, by [37, Proposition 25.10], L is strictly convex. Furthermore, as M is nondecreasing, then $\widehat{M}$ is convex in $[0, +\infty[$. So, for any $u, v \in X$ with $u \neq v$, and every $s, t \in (0, 1)$ with $s + t = 1$, we have

$$\widehat{M}(L(su + tv)) < \widehat{M}(sL(u) + tL(v)) \le s\widehat{M}(L(u)) + t\widehat{M}(L(v)).$$

This proves that $\mathcal{E}$ is strictly convex, since $\mathcal{E}'(u) = F(u)$ in $\big(W^{1,p}(\Omega)\big)^*$, we infer that F is strictly monotone in $W^{1,p}(\Omega)$.

It remains to show that the operator F is coercive.
Let $v \in W^{1,p}(\Omega)$, according to (A_2) and (M_0), we obtain

$$
\begin{aligned}
\frac{\langle Fv, v\rangle}{\|v\|} &= \frac{M\Big(\int_\Omega \Theta(x,\nabla u)dx\Big)\int_\Omega \sigma(x,\nabla u)\nabla v dx}{\|v\|} \\
&\geq \alpha\, m_0 \frac{\int_\Omega |\nabla v|^p dx + \int_\Omega |v|^p dx - \int_\Omega |v|^p dx}{\|v\|} \\
&\geq \alpha\, m_0 \|v\|^{p-1} - \alpha\, m_0 \frac{\|v\|_p}{\|v\|} \\
&\geq \alpha\, m_0 \|v\|^{p-1} - C. \quad (\text{ Due to } W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^p(\Omega))
\end{aligned}
$$

which means that $\dfrac{\langle Fv, v\rangle}{\|v\|} \to \infty \quad \text{as} \quad \|v\| \to \infty.$
Therefore F is coercive.
$ii)-$ We verify that the operator F is of type (S_+).
Let $(u_n)_n$ be a sequence in $W^{1,p}(\Omega)$ such that

$$
\begin{cases} u_n \rightharpoonup u & \text{in } \ W^{1,p}(\Omega) \\ \limsup\limits_{n\to\infty}\langle Au_n, u_n - u\rangle \leq 0. \end{cases} \tag{7}
$$

We will show that $u_n \to u$ in $W^{1,p}(\Omega)$.
On the one hand, in fact $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$, so $(u_n)_n$ is a bounded sequence in $W^{1,p}(\Omega)$, then there exist a subsequence still denoted by $(u_n)_n$ such that $u_n \rightharpoonup u$ in $W^{1,p}(\Omega)$, under the strict monotonicity of F we get

$$
0 = \limsup_{n\to\infty}\langle Fu_n - Fu,\ u_n - u\rangle = \lim_{n\to\infty}\langle Fu_n - Fu,\ u_n - u\rangle. \tag{8}
$$

Then

$$
\lim_{n\to\infty}\langle Fu_n,\ u_n - u\rangle = 0,
$$

which means

$$
\lim_{n\to\infty} M\Big(\int_\Omega \Theta(x,\nabla u_n)dx\Big)\int_\Omega \sigma(x,\nabla u_n)\nabla(u_n - u)dx = 0. \tag{9}
$$

On the other hand, by (A_1) we have for any $x \in \Omega$ and $\xi \in \mathbb{R}^n$

$$
\Theta(x,\xi) = \int_0^1 \frac{d}{ds}\Theta(x, s\xi)ds = \int_0^1 \sigma(x, s\xi)\xi ds.
$$

By combining (A_3), Fubini's theorem and Young's inequality we have

$$\begin{aligned}\int_\Omega \Theta(x,\nabla u_n)dx &= \int_\Omega\int_0^1 \sigma(x,s\nabla u_n)\nabla u_n ds\,dx\\ &= \int_0^1\int_\Omega \sigma(x,s\nabla u_n)\nabla u_n dx\,ds\\ &\le \int_0^1 \Big[C_{p'}\int_\Omega \big|\sigma(x,s\nabla u_n)\big|^{p'}dx + C_p\int_\Omega |\nabla u_n|^p\Big]ds\\ &\le C_1 + C'\int_0^1\int_\Omega |s\,\nabla u_n|^p dx\,ds + C_p\|u_n\|^p\\ &\le C_1 + C_2\int_\Omega |\nabla u_n|^p dx + C_p\|u_n\|^p\\ &\le C\|u_n\|^p.\end{aligned}$$

Then, we infer that $\Big(\int_\Omega (\Theta(x,\nabla u_n)dx\Big)_{n\ge1}$ is bounded.
As M is continuous, up to a subsequence there is $t_0 \ge 0$ such that

$$M\Big(\int_\Omega (\Theta(x,\nabla u_n)dx\Big) \longrightarrow M(t_0) \ge m_0 \qquad \text{as} \qquad n\to\infty. \tag{10}$$

From (9) and (10), we get

$$\lim_{n\to\infty}\int_\Omega \sigma(x,\nabla u_n)\nabla(u_n-u)dx = 0.$$

In light of Lemma 4, we obtain

$$u_n \longrightarrow u \qquad \text{strongly in } \; W^{1,p}(\Omega),$$

which implies that F is of type (S_+), which completes the proof. □

Proposition 1. *[12, Proposition 1] For any fixed $x\in\Omega$, the functions $\overline{\psi}(x,s)$ and $\underline{\psi}(x,s)$ are upper semicontinuous (u.s.c.) functions on $\mathbb{R}^N$.*

Lemma 6. *Let $\Omega\subset\mathbb{R}^N$ $(N\ge1)$ be a bounded open set with smooth boundary. The operator $A : W^{1,p}(\Omega)\to \big(W^{1,p}(\Omega)\big)^*$ giving by*

$$\langle Au, v\rangle = -\int_\Omega uv dx \quad \textit{for} \quad u,v\in W^{1,p}(\Omega)$$

is compact.

Proof. Firstly, since the embedding $i : L^p(\Omega)\to L^{p'}(\Omega)$ is continuous.
Secondly, as the embedding $I : W^{1,p}(\Omega)\to L^p(\Omega)$ is compact, it is known that the adjoint operator $I^* : L^{p'}(\Omega)\to\big(W^{1,p}(\Omega)\big)^*$ is also compact.
Hence, $A = I^*\,o\,i\,o\,I$ is compact. □

Let us define the following operator $\mathcal{N}$ acting from $W^{1,p}(\Omega)$ into $2^{\left(W^{1,p}(\Omega)\right)^*}$ by

$$\mathcal{N}u = \Big\{\varphi \in \left(W^{1,p}(\Omega)\right)^* \ \setminus \ \exists\, h \in L^{p'}(\Omega);\ \underline{\psi}(x,u(x)) \leq h(x) \leq \overline{\psi}(x,u(x)) \ \text{ a.e. } x \in \Omega \ \text{ and } \ \langle \varphi, v\rangle = \int_\Omega h\, v\, dx, \quad \forall v \in W^{1,p}(\Omega)\Big\}.$$

Lemma 7. *If the assumptions* (H_1) *and* (H_2) *hold, then the set-valued operator* $\mathcal{N}$ *is bounded, upper semicontinuous (u.s.c.) and compact.*

Proof. Let $\Lambda : L^p(\Omega) \to 2^{L^{p'}(\Omega)}$ be a set-valued operator defined as follows

$$\Lambda u = \Big\{h \in L^{p'}(\Omega) \setminus \quad \underline{\psi}(x,u(x)) \leq h(x) \leq \overline{\psi}(x,u(x)) \quad a.e.\ x \in \Omega\Big\}.$$

Let $u \in W^{1,p}(\Omega)$, by the growth condition (H_2) we get

$$\max\big\{ \,|\underline{\psi}(x,s)|\,;\, |\overline{\psi}(x,s)|\, \big\} \leq b(x) + c\,|s|^{p/p'}.$$

It follows that

$$\int_\Omega |\,\overline{\psi}(x,u(x))|^{p'} dx \leq 2^{p'}\Big(\int_\Omega |b(x)|^{p'} dx + c^{p'} \int_\Omega |u(x)|^p dx\Big).$$

A same inequality is shown for $\underline{\psi}(x,s)$, as a result the set-valued operator Λ is bounded on $W^{1,p}(\Omega)$.
It remains to show that Λ is upper semicontinuous (u.s.c.), i.e.,

$$\forall \varepsilon > 0,\ \exists \delta > 0, \quad \|u - u_0\|_p < \delta \ \Rightarrow\ \Lambda u \subset \Lambda u_0 + B_\varepsilon,$$

with B_ε is the ε-ball in $L^{p'}(\Omega)$.
Come to an end, given $u_0 \in L^p(\Omega)$, let us consider the sets

$$G_{m,\varepsilon} = \bigcap_{t\in\mathbb{R}^N} K_t,$$

where

$$K_t = \Big\{x \in \Omega,\ \text{if } |t - u_0(x)| < \frac{1}{m}, \text{ then } [\underline{\psi}(x,t), \overline{\psi}(x,t)] \subset \,]\underline{\psi}(x,u_0(x)) - \frac{\varepsilon}{R}, \overline{\psi}(x,u_0(x)) + \frac{\varepsilon}{R}[\Big\},$$

m is an integer, $|t| = \max\limits_{1\leq i\leq N} |t_i|$ and R is a constant to be determined in the following pages. In view of Proposition 1, we define the sets of points as follows

$$G_{m,\varepsilon} = \bigcap_{r\in\mathbb{R}^N_a} K_r,$$

where $\mathbb{R}_a^N$ denotes the set of all rational grids in $\mathbb{R}^N$. For any $r=(r_1,\cdots,r_N)\in\mathbb{R}_a^N$,

$$\begin{aligned}K_r=\Big\{x\in\Omega \mid u_0(x)\in C\prod_{i=1}^{N}]r_i-\frac{1}{m},r_i+\frac{1}{m}[\Big\}\\ \cup\Big\{x\in\Omega \mid u_0(x)\in\prod_{i=1}^{N}]r_i-\frac{1}{m},r_i+\frac{1}{m}[\Big\}\\ \cap\Big\{x\in\Omega \mid \overline{\psi}(x,r)<\overline{\psi}(x,u_0(x))+\frac{\varepsilon}{R}\text{ and }\underline{\psi}(x,r)>\underline{\psi}(x,u_0(x))-\frac{\varepsilon}{R}\},\end{aligned}$$

so that K_r and $G_{m,\varepsilon}$ therefore are measurable. It is clear that

$$G_{1,\varepsilon}\subset G_{2,\varepsilon}\subset\cdots$$

By virtue of Proposition 1,

$$\bigcup_{m=1}^{\infty}G_{m,\varepsilon}=\Omega,$$

hence there exists an integer m_0 such that

$$m(G_{m_0,\varepsilon})>m(\Omega)-\frac{\varepsilon}{R}. \tag{11}$$

But for any $\varepsilon>0$, there exists $\eta=\eta(\varepsilon)>0$, such that $m(T)<\eta$ yield

$$2^{p'}\int_T 2|b(x)|^{p'}+c'(2^p+1)|u_0(x)|^p dx<\Big(\frac{\varepsilon}{3}\Big)^{p'}, \tag{12}$$

thanks to $b\in L^{p'}(\Omega)$ and $u_0\in L^p(\Omega)$.
Let now

$$0<\delta<\min\Big\{\frac{1}{m_0}\Big(\frac{\eta}{2}\Big)^{1/p},\frac{1}{2}\Big(\frac{\varepsilon}{6C}\Big)^{p'/p}\Big\}, \tag{13}$$

$$R>\max\Big\{\frac{2\varepsilon}{\eta},3\left(m(\Omega)\right)^{1/p'}\Big\}. \tag{14}$$

Assume that $\|u-u_0\|_p<\delta$ and define the set $G=\{x\in\Omega\setminus|u(x)-u_0(x)|\geq\frac{1}{m_0}\}$, we obtain

$$m(G)<(m_0\delta)^p<\frac{\eta}{2}. \tag{15}$$

If $x\in G_{m_0,\varepsilon}\backslash G$, then, for any $h\in\Lambda u$,

$$|u(x)-u_0(x)|<\frac{1}{m_0}$$

and

$$h(x)\in\Big]\underline{\psi}(x,u_0(x))-\frac{\varepsilon}{R},\overline{\psi}(x,u_0(x))+\frac{\varepsilon}{R}\Big[.$$

Let

$$\begin{aligned} K^0 &= \Big\{x\in\Omega;\quad h(x)\in\big[\,\underline{\psi}(x,u_0(x)),\overline{\psi}(x,u_0(x))\big]\Big\},\\ K^- &= \Big\{x\in\Omega;\quad h(x)<\underline{\psi}(x,u_0(x))\Big\},\\ K^+ &= \Big\{x\in\Omega;\quad h(x)>\overline{\psi}(x,u_0(x))\Big\},\end{aligned}$$

and

$$w(x)=\begin{cases} \overline{\psi}(x,u_0(x)), & \text{for} \quad x\in K^+;\\ h(x)\ , & \text{for} \quad x\in K^0;\\ \underline{\psi}(x,u_0(x)), & \text{for} \quad x\in K^-.\end{cases}$$

Hence $w\in\Lambda u_0$ and

$$|w(x)-h(x)|<\frac{\varepsilon}{R}\qquad\text{for all}\quad x\in G_{m_0,\varepsilon}\backslash G. \tag{16}$$

According to (14) and (16), we get

$$\int_{G_{m_0,\varepsilon}\backslash G}|w(x)-h(x)|^{p'}dx<\Big(\frac{\varepsilon}{R}\Big)^{p'}m(\Omega)<\Big(\frac{\varepsilon}{3}\Big)^{p'}. \tag{17}$$

Assume that V is a coset in Ω of $G_{m_0,\varepsilon}\backslash G$, then $V=(\Omega\backslash G_{m_0,\varepsilon})\cup(G_{m_0,\varepsilon}\cap G)$ and

$$m(V)\le m(\Omega\backslash G_{m_0,\varepsilon})+m(G_{m_0,\varepsilon}\cap G)<\frac{\varepsilon}{R}+m(G)<\eta,$$

thanks to (11), (14) and (15). From (H_2), (12) and (13), we have

$$\begin{aligned}\int_V|w(x)-h(x)|^{p'}dx &\le \int_V |w(x)|^{p'}+|h(x)|^{p'}dx\\ &\le 2^{p'}\big(\int_V|b(x)|^{p'}+c^{p'}|u_0(x)|^p+|b(x)|^{p'}+c^{p'}|u(x)|^p dx\big)\\ &\le 2^{p'}\big(\int_V 2|b(x)|^{p'}+c^{p'}(2^p+1)|u_0(x)|^p+2^p\,c^{p'}|u(x)-u_0(x)|^p dx\big)\\ &\le 2^{p'}\int_V 2|b(x)|^{p'}+c^{p'}(2^p+1)|u_0(x)|^p dx+2^{p+p'}\,c^{p'}\int_V|u(x)-u_0(x)|^p dx\\ &\le \big(\frac{\varepsilon}{3}\big)^{p'}+2^{p+p'}\,c^{p'}\delta^p\le 2\Big(\frac{\varepsilon}{3}\Big)^{p'}\le\varepsilon^{p'}.\end{aligned} \tag{18}$$

From (17) and (18), we have $\|w-h\|_{p'}<\varepsilon$.
Therefore Λ is upper semicontinuous (u.s.c.).
Hence $\mathcal{N}=I^*o\Lambda oI$ is clearly bounded, upper semicontinuous (u.s.c.) and compact. $\square$

4 Existence of a weak solution

In this section, we will give our main result. The proof are based on the topological degree introduced in section 2.

Definition 6. *We say that* $u \in W^{1,p}(\Omega)$ *is a weak solution of* (1), *if there exists an element* $\varphi \in \mathcal{N}u$ *verifying*

$$M\Big(\int_\Omega \Theta(x,\nabla u)dx\Big)\int_\Omega \sigma(x,\nabla u)\nabla v dx+\int_\Omega uvdx+\langle\varphi,v\rangle=0, \text{ for all } v\in W^{1,p}(\Omega).$$

Now, we present our main result.

Theorem 2. *If hypotheses* $(A_1)-(A_4)$,(M_0), (H_1) *and* (H_2) *hold. Then, the problem* (1) *admits at least weak solutions* u *in* $W^{1,p}(\Omega)$.

Proof. Let $F,\ A : W^{1,p}(\Omega) \to \big(W^{1,p}(\Omega)\big)^*$ and $\mathcal{N}$: $W^{1,p}(\Omega) \to 2^{\left(W^{1,p}(\Omega)\right)^*}$ be given in Section 2. This implies that $u \in W^{1,p}(\Omega)$ is a weak solution of (1) if and only if

$$Fu \in -Su, \tag{19}$$

where $\quad S := A + \mathcal{N} : W^{1,p}(\Omega) \to 2^{\left(W^{1,p}(\Omega)\right)^*}$.
Thanks to properties of the operator F given in Lemma 5 and by the Minty-Browder Theorem on monotone operators in [37, Theorem 26 A], the inverse operator $T := F^{-1} : \big(W^{1,p}(\Omega)\big)^* \to W^{1,p}(\Omega)$ is of type (S_+), continuous and bounded. Additionally, from Lemma 6 the operator S is quasi-monotone, upper semicontinuous (u.s.c.) and bounded.
As a result, the equation (19) is equivalent to the abstract Hammerstein equation

$$u = Tv \quad \text{and} \quad v \in -SoTv. \tag{20}$$

We will apply the degree theory introduced in Section 2 to solve equations (20). To do this, we first prove the following Lemma.

Lemma 8. *the following set*

$$B := \Big\{\ v \in \big(W^{1,p}(\Omega)\big)^*,\ \textit{such that } v \in -tSoTv \quad \textit{for some} \quad t \in [0,1]\Big\}$$

is bounded.

Proof. Let $v \in B$, so, $v + ta = 0$ for some $t \in [0,1]$, with $a \in SoTv$. Taking $u := Tv$, we can write $a = Au + \varphi \in Su$, where $\varphi \in \mathcal{N}u$, namely,

$$\langle\varphi, u\rangle = \int_\Omega h(x)u(x)dx,$$

for some $h \in L^{p'}(\Omega)$ with $\underline{\psi}(x,u(x)) \le h(x) \le \overline{\psi}(x,u(x))$ for almost all $x \in \Omega$.
Based on (H_2), (A_2), the Young's inequality, the compact embedding $W^{1,p}(\Omega) \hookrightarrow\hookrightarrow$

$L^p(\Omega)$ and the continuous embedding $L^p(\Omega) \hookrightarrow L^2(\Omega)$, we obtain

$$\begin{aligned}
\|Tv\|^p &= \|u\|^p = \int_\Omega |u|^p\,dx + \int_\Omega |\nabla u|^p dx \\
&\leq \int_\Omega |u|^p dx + M\Big(\int_\Omega \Theta(x,\nabla u)\,dx\Big)\int_\Omega \sigma(x,\nabla u)\nabla v dx = \int_\Omega |u|^p dx + \frac{1}{\alpha}\langle v, Tv\rangle \\
&\leq \int_\Omega |u|^p dx + \frac{t}{\alpha}|\langle a, Tv\rangle| = \int_\Omega |u|^p\,dx + \frac{t}{\alpha}\int_\Omega |(u+h)u| dx \\
&\leq \int_\Omega |u|^p\,dx + \frac{t}{\alpha}\int_\Omega |u|^2\,dx + \frac{t}{\alpha}\int_\Omega |hu| dx \\
&\leq \int_\Omega |u|^p dx + C'\frac{t}{\alpha}\Big(\int_\Omega |u|^p dx\Big)^{2/p} + C_p\frac{t}{\alpha}\Big(\int_\Omega |u|^p dx\Big)^{1/p} + C_{p'}\frac{t}{\alpha}\Big(\int_\Omega |h|^{p'}\,dx\Big)^{1/p'} \\
&\leq \int_\Omega |u|^p dx + C'\frac{t}{\alpha}\Big(\int_\Omega |u|^p dx\Big)^{2/p} + C_p\frac{t}{\alpha}\Big(\int_\Omega |u|^p dx\Big)^{1/p} \\
&\quad + 2C'_p\frac{t}{\alpha}\Big(\int_\Omega |b|^{p'} dx\Big)^{1/p'} + 2CC_{p'}\frac{t}{\alpha}\Big(\int_\Omega |u|^p dx\Big)^{1/p'} \\
&\leq \mathrm{Const}\big(\|Tv\|^p + \|Tv\|^2 + \|Tv\| + \|Tv\|^{p-1} + 1\big).
\end{aligned}$$

It follows that $\big\{Tv \ \backslash \ v \in B\ \big\}$ is bounded.
Since the operator S is bounded, by (20) that the set B is bounded in $\big(W^{1,p}(\Omega)\big)^*$. □

Under the lemma 8, we can choose a positive constant R such that

$$\|v\|_{\left(W^{1,p}(\Omega)\right)^*} < R \quad \text{for any} \quad v \in B.$$

This says that

$$v \in -tSoTv \quad \text{for each} \quad v \in \partial B_R(0) \quad \text{and each} \quad t \in [0,1],$$

where $\partial B_R(0) = \{v \in \big(W^{1,p}(\Omega)\big)^*, \text{ such that } \|v\|_{\left(W^{1,p}(\Omega)\right)^*} = R\}$.
In light of Lemma 1, we have

$$I + SoT \in \mathfrak{F}_T(\overline{B_R(0)}) \quad \text{and} \quad I = FoT \in \mathfrak{F}_T(\overline{B_R(0)}).$$

Next, we Consider the affine homotopy $H : [0,1] \times \overline{B_R(0)} \to 2^{\left(W^{1,p}(\Omega)\right)^*}$ setting by

$$H(t,v) := (1-t)Iv + t(I + SoT)v \quad \text{for} \quad (t,v) \in [0,1] \times \overline{B_R(0)}.$$

By using the normalization and homotopy invariance property of the degree d fixed in Theorem 1, we obtain

$$d(I + SoT, B_R(0), 0) = d(I, B_R(0), 0) = 1.$$

Thus, we can find a point $v \in B_R(0)$ such that

$$v \in -SoTv.$$

Which implies that $u = Tv$ is a weak solution of (1). This completes the proof. □

5 Conclusion

In this work, we have studied the existence of weak solutions for Neumann boundary value problems with discontinuous nonlinearities of the Kirchhoff type by using the topological degree theory.

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