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SEMI-DISCRETE VORONOVSKAYA-TYPE THEOREM FOR POSITIVE LINEAR OPERATORS BASED ON HERMITE INTERPOLATION WITH TWO DOUBLE KNOTS

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Dedicated to Professor Radu Păltănea on the occasion of his 70th anniversary

Abstract

Since the classical asymptotic theorems of Voronovskaya-type for positive and linear operators are in fact based on the Taylor's formula which is a very particular case of Lagrange-Hermite interpolation formula, in the recent paper Gal [3], I have obtained semi-discrete quantitative Voronovskaya-type theorems based on other Lagrange-Hermite interpolation formulas. These include Lagrange interpolation on two and three simple knots and Hermite interpolation on two knots, one simple and the other one double. In the present paper we obtain a semi-discrete quantitative Voronovskaya-type theorems based on Hermite interpolation on two double knots.

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1 Introduction

Let us consider the well-known Bernstein polynomials defined by $B_n(f)(x) = \sum_{k=0}^{n} p_{n,k}(x) f\left(\frac{k}{n}\right)$, with $p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}$, $f \in C[0,1]$, $x \in [0,1]$, $n \in \mathbb{N}$.

One of the most important result in approximation theory is the asymptotic Voronovskaya's result on Bernstein polynomials obtained in [15] :

Theorem 1.1. *If* $f \in C^2[0,1]$*, then*

$$\lim_{n\to\infty} n[B_n(f)(x) - f(x)] = \frac{x(1-x)}{2} \cdot f''(x),$$

uniformly on $[0,1]$*. Here* $C^p[0,1]$ *denotes the space of all real functions having a continuous derivative of order* $p \in \mathbb{N} \cup \{0\}$ *on* $[0,1]$.

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In [1], Bernstein gave the following generalization :

Theorem 1.2. *If $p \in \mathbb{N}$ is even and $f \in C^p[0,1]$, then*

$$\lim_{n\to\infty} n^{p/2}\left(B_n(f)(x) - \sum_{r=0}^{p} B_n((\cdot - x)^r)(x)\cdot \frac{f^{(r)}(x)}{r!}\right) = 0,$$

uniformly on $[0,1]$.

This result was proved for all $p \in \mathbb{N}$ by Gavrea and Ivan [4] and Tachev [13].

In [10], Mamedov extended Theorem 1.2 to positive linear operators, as follows.

Theorem 1.3. *Suppose that $p \in \mathbb{N}$ is even, $f \in C^p[0,1]$ and let $L_n : C[0,1] \to C[0,1]$, $n \in \mathbb{N}$, be a sequence of positive linear operators preserving the constants and satisfying*

$$L_n((\cdot - x)^{p+2j})(x) = o(L_n((\cdot - x)^p)(x)), \text{ as } n \to \infty,$$

for some $x \in [0,1]$ and for at least one integer $j > 0$.

Then, denoting

$$R_p(L_n(f))(x) = L_n(f)(x) - \sum_{r=0}^{p} L_n((\cdot - x)^r)(x)\cdot \frac{f^{(r)}(x)}{r!},$$

we have

$$R_p(L_n(f))(x) = o(L_n((\cdot - x)^p)(x), \text{ as } n \to \infty.$$

Here $a_n = o(b_n)$ means that there exists $c_n \to 0$ such that $a_n = c_n b_n$, $n \in \mathbb{N}$.

The first quantitative estimates in Theorem 1.1 were obtained for $f \in C^3[0,1]$ by Ditzian-Ivanov in [2], for $f \in C^4[0,1]$ by Gonska-Rasa in [8], pointwise estimates in terms of the modulus of continuity by Videnskij in [14], p. 19, Theorem 15.2 and in terms of the least concave majorant by Gonska-Pitul-Rasa in [7].

Also, quantitative estimates in Theorem 1.3 were obtained in terms of the least concave majorant and a K-functional by Gonska in [6], Theorem 3.2 and by Gavrea-Ivan in [5].

A general characteristic of all the above results is that their proofs are based on the Taylor's formula with remainder. But because the Taylor's formula is nothing else that a particular Lagrange-Hermite interpolation formula with only one multiple knot, it is natural to seek for asymptotic formulas based on Lagrange-Hermite interpolation formula with several simple or multiple knots.

In the recent paper Gal [3], I have obtained semi-discrete quantitative Voronovskaya-type theorems based on other Lagrange-Hermite interpolation formulas, like Lagrange interpolation on two and three simple knots and Hermite interpolation on two knots, one simple and the other one double. In the present paper we obtain a semi-discrete quantitative Voronovskaya-type theorem based on Hermite interpolation on two double knots.

Section 2 contains some preliminaries on Lagrange-Hermite interpolation. In Section 3 we obtain a semi-discrete quantitative Voronovskaya-type theorem based on Hermite interpolation on two double knots.

2 Preliminaries on Lagrange-Hermite interpolation

The Lagrange-Hermite interpolation polynomial attached to a function $f : [a,b] \to \mathbb{R}$ and to the knots $a_i \in [a,b]$, $i = 0,...,m$, each a_k multiply of order r_k+1, $k = 0,...,m$, is the unique polynomial H_p of degree p, where $p+1 = \sum_{k=0}^{m}(r_k+1)$, satisfying the conditions (Hermite [9])

$$H_p^{(j)}(f)(a_k) = f^{(j)}(a_k),\ k = 0,...,m \ \text{ and } j = 0,...,r_k.$$

It is known that we can write (see Stancu [11])

$$H_p(f)(t)$$
$$= \sum_{k=0}^{m}\sum_{j=0}^{r_k} h_{k,j}(t)\cdot f^{(j)}(a_k) = \sum_{k=0}^{m} u_k(t)\sum_{j=0}^{r_k}\frac{(t-a_k)^j}{j!}\left(\frac{f(t)}{u_k(t)}\right)^{(j)}_{t=a_k}, \tag{2.1}$$

where

$$h_{k,j}(t) = \frac{(t-a_k)^j}{j!}\left[\sum_{s=0}^{r_k-j}\frac{(t-a_k)^s}{s!}\cdot\left(\frac{1}{u_k(t)}\right)^{(s)}_{t=a_k}\right]\cdot u_k(t), \tag{2.2}$$

and

$$u_k(t) = (t-a_0)^{r_0+1}\cdot ...\cdot(t-a_{k-1})^{r_{k-1}+1}(t-a_{k+1})^{r_{k+1}+1}\cdot ...\cdot(t-a_m)^{r_m+1}. \tag{2.3}$$

Denoting the remainder

$$R_p(f)(t) = f(t) - H_p(f)(t),$$

also it is known (see, e.g., Stancu [12], p. 121) that if f has a derivative of order $n+1$ on the interval $[a,b]$, then there exists a ξ in the interval which is the convex hull of the points $t, a_0, ..., a_m$, such that

$$R_p(f)(t) = \frac{\Pi_{k=0}^{m}(t-a_k)^{r_k+1}}{(p+1)!}\cdot f^{(p+1)}(\xi). \tag{2.4}$$

In what follows, let us choose, for example $a_0 = x$ arbitrary (the choice of any another knot for x does not loose the generality) while the other knots are considered fixed. Then, by writing in a separate way in (2.1) and (2.2) the index $k = 0$, we easily obtain

$$\begin{aligned} H_p(f)(t) &= \sum_{j=0}^{r_0} h_{0,j}(t)\cdot f^{(j)}(x) + \sum_{k=1}^{m}\sum_{j=0}^{r_k} h_{k,j}(t)\cdot f^{(j)}(a_k) \\ &= \sum_{j=0}^{r_0}\frac{f^{(j)}(x)}{j!}\cdot\left[\sum_{s=0}^{r_0-j}\frac{u_0(t)(e_1-xe_0)^{s+j}(t)}{s!}\cdot\left(\frac{1}{u_0(t)}\right)^{(s)}_{t=x}\right] + P_n(t), \end{aligned} \tag{2.5}$$

where $e_1(t) = t$, $e_0(t) = 1$ and

$$P_n(t) = \sum_{k=1}^{m}\sum_{j=0}^{r_k}\frac{f^{(j)}(a_k)}{j!}\cdot\left[\sum_{s=0}^{r_k-j}\frac{u_k(t)(t-a_k)^{s+j}}{s!}\cdot\left(\frac{1}{u_k(t)}\right)^{(s)}_{t=a_k}\right]. \tag{2.6}$$

If $r_k = 1$, for all $k = 0, ..., m$, then $p = 2m + 1$ and we get the Lagrange-Hermite interpolation polynomial with double knots given by the formula

$$\begin{aligned} H_{2m+1}(f)(t) = &\sum_{k=0}^{m} \frac{u_k(t)}{u_k(a_k)} \left[1 - (t - a_k) \cdot \frac{u'(a_k)}{u_k(a_k)}\right] f(a_k) \\ &+ \sum_{k=0}^{m} (t - a_k) \cdot \frac{u_k(t)}{u_k(a_k)} \cdot f'(a_k), \end{aligned} \tag{2.7}$$

If $m = 1$, then we can write (see, e.g., [11], p. 118) by the formula

$$\begin{aligned} &H_{r_0+r_1+1}(f)(t) \\ &= \left(\frac{t - a_1}{a_0 - a_1}\right)^{r_1+1} \cdot \sum_{k=0}^{r_0} \frac{(t - a_0)^k}{k!} \left[\sum_{j=0}^{r_0-k} \binom{r_1 + j}{j} \left(\frac{t - a_0}{a_1 - a_0}\right)^j\right] f^{(k)}(a_0) \\ &+ \left(\frac{t - a_0}{a_1 - a_0}\right)^{r_0+1} \cdot \sum_{k=0}^{r_1} \frac{(t - a_1)^k}{k!} \left[\sum_{j=0}^{r_1-k} \binom{r_0 + j}{j} \left(\frac{t - a_1}{a_0 - a_1}\right)^j\right] f^{(k)}(a_1), \end{aligned} \tag{2.8}$$

and the remainder is

$$R_{r_0+r_1+1}(f)(t) = \frac{(t - a_0)^{r_0+1}(t - a_1)^{r_1+1}}{(r_0 + r_1 + 2)!} \cdot f^{(r_0+r_1+2)}(\xi), \tag{2.9}$$

with ξ belonging to the interval determined by the convex hull of t, a_0, a_1.

3 Main result

In this section we obtain a semi-discrete quantitative Voronosvkaya-type theorem for the particular case of Lagrange-Hermite interpolation formula on two double knots.

Thus, let us consider the formulas (2.8) and (2.9) for $r_0 = r_1 = 1$, $a_0 = x$, $a_1 = y$, that is the case of Hermite-Fejer interpolation formula on two double knots x and y.

By (2.8) we get

$$\begin{aligned} H_3(f)(t) = &\left(\frac{t - y}{x - y}\right)^2 \left\{f(x) \left[1 + 2\frac{t - x}{y - x}\right] + (t - x)f'(x)\right\} \\ &+ \left(\frac{t - x}{y - x}\right)^2 \left\{f(y) \left[1 + 2\frac{t - y}{x - y}\right] + (t - y)f'(y)\right\} \end{aligned}$$

and by (2.9) we have

$$R_3(f)(t) = \frac{(t - x)^2 \cdot (t - y)^2}{4!} \cdot f^{(4)}(\xi),$$

with $|\xi - x| \leq \max\{|t - x|, |y - x|\} \leq |t - x| + |y - x|$.

Therefore, we can state the following result.

Theorem 3.1. *Let us denote* $e_1(t) = t$. *If* $f \in C^4[0,1]$ *and* $L_n : C[0,1] \to C[0,1]$, $n \in \mathbb{N}$, *is a sequence of positive linear operators preserving the constants, then, for all* $n \in \mathbb{N}$ *and* $x, y \in [0,1]$, $x \neq y$, *we have*

$$\begin{aligned}
&\Bigg| |L_n(f)(x) - \frac{L_n((e_1-y)^2)(x)f(x)}{(x-y)^2} + 2\frac{L_n((e_1-y)^2(e_1-x))(x)f(x)}{(x-y)^3} \\
&- \frac{L_n((e_1-x)(e_1-y)^2)(x)f'(x)}{(x-y)^2} - \frac{L_n((e_1-x)^2)(x)f(y)}{(y-x)^2} \\
&+ 2\frac{L_n((e_1-x)^2(e_1-y))(x)f(y)}{(y-x)^3} \\
&- \frac{L_n((e_1-x)^2(e_1-y))(x)f'(y)}{(y-x)^2} - \frac{L_n((e_1-x)^2(e_1-y)^2)(x)}{24}f^{(4)}(x)\Bigg| \\
\leq&\omega\left(f^{(4)}; \frac{L_n(|e_1-x|^3(e_1-y)^2)(x)}{L_n((e_1-x)^2(e_1-y)^2)(x)} + |x-y|\right) \\
&\cdot \frac{L_n((e_1-x)^2(e_1-y)^2)(x)}{12}.
\end{aligned}$$

Proof. We use the formula $f(t) - H_3(f)(t) = R_3(f)(t)$. Thus, firstly, for all $t, y \in [0,1]$, we can write

$$\begin{aligned}
&\Bigg| f(t) - \frac{(t-y)^2 f(x)}{(x-y)^2} + 2\frac{(t-y)^2(t-x)f(x)}{(x-y)^3} - \frac{(t-x)(t-y)^2 f'(x)}{(x-y)^2} \\
&- \frac{(t-x)^2 f(y)}{(y-x)^2} + 2\frac{(t-x)^2(t-y)f(y)}{(y-x)^3} - \frac{(t-x)^2(t-y)f'(y)}{(y-x)^2} \\
&- \frac{(t-x)^2(t-y)^2}{24} f^{(4)}(x)\Bigg| \\
=&\frac{(t-x)^2(t-y)^2}{24} \cdot |f^{(4)}(\xi) - f^{(4)}(x)| \\
\leq&\frac{(t-x)^2(t-y)^2}{24} \cdot \omega(f^{(4)}; |t-x| + |x-y|) \\
\leq&(t-x)^2(t-y)^2\left(1 + \frac{1}{\delta} \cdot (|t-x| + |x-y|)\right) \cdot \frac{\omega(f^{(4)}; \delta)}{24} \\
=&\frac{\omega(f^{(4)}; \delta)}{24} \cdot \left\{(t-x)^2(t-y)^2 + (t-y)^2\frac{|t-x|^3 + (t-x)^2 \cdot |x-y|}{\delta}\right\}.
\end{aligned}$$

Applying above L_n to both members, by the general inequality $|L_n(F)(x)| \leq$

$L_n(|F|)(x)$, it follows

$$\Bigg|L_n(f)(x) - \frac{L_n((e_1-y)^2)(x)f(x)}{(x-y)^2} + 2\frac{L_n((e_1-y)^2(e_1-x))(x)f(x)}{(x-y)^3}$$
$$- \frac{L_n((e_1-x)(e_1-y)^2)(x)f'(x)}{(x-y)^2} - \frac{L_n((e_1-x)^2)(x)f(y)}{(y-x)^2}$$
$$+ 2\frac{L_n((e_1-x)^2(e_1-y))(x)f(y)}{(y-x)^3}$$
$$- \frac{L_n((e_1-x)^2(e_1-y))(x)f'(y)}{(y-x)^2} - \frac{L_n((e_1-x)^2(e_1-y)^2)(x)}{24}f^{(4)}(x)\ |$$
$$\leq \frac{\omega(f^{(4)};\delta)}{24} \cdot L_n((e_1-x)^2(e_1-y)^2)(x)$$
$$\cdot \left[1 + \frac{1}{\delta}\left(\frac{L_n(|e_1-x|^3(e_1-y)^2)(x)}{L_n((e_1-x)^2(e_1-y)^2)(x)} + |x-y|\right)\right]$$

Choosing above $\delta = \frac{L_n(|e_1-x|^3(e_1-y)^2)(x)}{L_n((e_1-x)^2(e_1-y)^2)(x)} + |x-y|$, we get the estimate in the statement. □

Remark 3.2. *For $y \to x$ (and by using the Cauchy-Schwarz formula for positive linear operators preserving constants), in the right-hand side of the estimate in the above Theorem 3.1 we easily get the quantity*

$$\omega\left(f^{(4)}; \frac{L_n(|e_1-x|^5)(x)}{L_n((e_1-x)^4)(x)}\right) \cdot \frac{L_n((e_1-x)^4)(x)}{12}$$
$$\leq \omega\left(f^{(4)}; \frac{\sqrt{L_n((e_1-x)^6)(x)}}{\sqrt{L_n((e_1-x)^4)(x)}}\right) \cdot \frac{L_n((e_1-x)^4)(x)}{12}.$$

On the other hand, since for $y \to x$, $H_3(f)$ becomes the Taylor polynomial at the point x, due to the continuity of any positive linear operator, the left-hand side in Theorem 3.1 will become

$$\left|L_n(f)(x) - \sum_{i=0}^{4} L_n((e_1-x)^i)(x) \cdot \frac{f^{(i)}(x)}{i!}\right|,$$

recapturing in essence Theorem 3.2 in [6] for $q = 4$.

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References

[1] Bernstein, S. N., *Complément à l'article de E. Voronovskaya "Détermination de la forme asymptotique de l'approximation des fonctions par les polynômes de M. Bernstein"*, C. R. Acad. Sci. URSS, **1932** (1932), 86-92.

[2] Ditzian, Z., Ivanov, K. G., *Strong converse inequalities*, J. d'Analyse Math. **61** (1993), 61-111. https://doi.org/10.1007/BF02788839.

[3] Gal, S. G., *Semi-discrete quantitative Voronovskaya-type theorems for positive linear operators*, Results Math. **75** (2020), Article number 117, 17 pages. https://doi.org/10.1007/s00025-020-01236-x

[4] Gavrea, I., Ivan, M., *An answer to a conjecture on Bernstein operators*, J. Math. Anal. Appl. **390** (2012), 86-92. https://doi.org/10.1016/j.jmaa.2012.01.021

[5] Gavrea, I., Ivan, M., *The Bernstein Voronovskaja-type theorems for positive linear approximation operators*, J. Approx. Theory **192** (2015), 291-296. https://doi.org/10.1016/j.jat.2014.12.008

[6] Gonska, H., *On the degree of approximation in Voronovskaja's theorem*, Stud. Univ. Babeş-Bolyai, Math. **52** (2007), 103-115.

[7] Gonska, H., Pitul, P., Raşa, I., *On Peano's form of the Taylor remainder, Voronovskaja's theorem and the commutator of positive linear operators*, In: Numerical Analysis and Approximation Theory (Proc. Int. Conf. Cluj-Napoca,(O. Agratini and P. Blaga eds.), Casa Cărţii de Ştiinţă, Cluj-Napoca, 2006, 55-80.

[8] Gonska, H., Raşa, I., *The limiting semigroup of the Bernstein iterates: degree of convergence*, Acta Math. Hung. **111** (2006), 119-130. https://doi.org/10.1007/s10474-006-0038-4

[9] Hermite, C., *Sur la formule d'interpolation de Lagrange*, J. Reine Angew. Math., **84** (1878), 70-79.

[10] Mamedov, R. G., *The asymptotic value of the approximation of multiply differentiable functions by positive linear operators*, Dokl. Akad. Nauk SSSR, **146** (1962), 1013-1016. https://www.mathnet.ru/eng/dan27086

[11] Stancu, D. D., *Sur la formule d'interpolation d'Hermite et quelques applications de celle-ci* (in Romanian), Acad. R. P. Romîne, Fil. Cluj, Stud. Cerc. Mat. **8** (1957), 339-355.

[12] Stancu, D. D., *Course (with Problems) in Numerical Analysis* (in Romanian), "Babeş-Bolyai" University Press, Cluj-Napoca, 1977.

[13] Tachev, G. T., *The complete asymptotic expansion for Bernstein operators*, J. Math. Anal. Appl. **385** (2012), 1179-1183. https://doi.org/10.1016/j.jmaa.2011.07.042

[14] Videnskij, V. S., *Linear positive operators of finite rank*, (Russian), Leningrad, "A.I. Gerzen" State Pedagogical Institute 1985.

[15] Voronovskaja, E., *D'etermination de la forme asymptotique de l'approximation des fonctions par les polynômes de M. Bernstein*, C. R. Acad. Sci. URSS, **1932** (1932), 79-85.