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ON A HILFER TYPE FRACTIONAL INTEGRO-DIFFERENTIAL INCLUSION

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Dedicated to Professor Radu Păltănea on the occasion of his 70th anniversary

Abstract

A boundary value problem associated to a Hilfer generalized proportional fractional integro-differential inclusion is studied. The existence of solutions is established in the case when the set-valued map has nonconvex values.

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1 Introduction

The recent literature is full of papers devoted to the study of systems governed by fractional order derivatives. The main reason is that the models taking into account fractional derivatives are more realistic than the models realized with classical derivatives (see [2, 7, 11, 13, 15] etc.).

A generalization of both Riemann-Liouville and Caputo fractional derivatives was introduced by Hilfer in [9]. In fact, this derivative interpolates between Riemann-Liouville and Caputo derivatives. Several properties and applications of Hilfer fractional derivative may be found in [10]. Recently, in [12], an extension of this derivative was proposed. Namely, the ψ-Hilfer generalized proportional fractional derivative of a function with respect to another function. Several properties of this derivative were studied in [12].

In this note we consider the following problem

$$D_H^{\alpha,\beta,\sigma,\psi}x(t) \in F(t,x(t),V(x)(t)) \quad a.e.\ ([a,b]) \tag{1}$$

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with nonlocal integral boundary conditions of the form

$$x(a) = 0, \quad x(b) = \sum_{j=1}^{m} \eta_j x(\xi_j) + \sum_{i=1}^{n} \zeta_i I^{\varphi_i,\sigma,\psi} x(\theta_i), \tag{2}$$

where $D_H^{\alpha,\beta,\sigma,\psi}$ denotes the ψ-Hilfer generalized proportional fractional derivative operator of order $\alpha \in (1,2]$ and type $\beta \in [0,1]$, respectively, $\sigma \in (0,1]$, $\xi_j, \theta_i \in (a,b)$, $\eta_j, \zeta_i \in \mathbb{R}$, $j = \overline{1,m}, i = \overline{1,n}$, $I^{\varphi_i,\sigma,\psi}$ is the generalized proportional fractional integral operator of order $\varphi_i > 0$, $F : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a set-valued map and $V : C([a,b],\mathbb{R}) \to C([a,b],\mathbb{R})$ is a nonlinear Volterra integral operator defined by $V(x)(t) = \int_a^t k(t,s,x(s))ds$ with $k(.,.,.) : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ a given function.

Our study is motivated by a recent paper [14]. Namely, in [14] several existence results for problem (1)-(2) may be found in the case when F does not depends on the last variable. All the results in [14] are proved by using certain suitable theorems from fixed point theory.

The goal of this note is to obtain the existence of solutions for problem (1)-(2) in the case when the set-valued map F has nonconvex values but it is assumed to be Lipschitz in the second and third variable. Our result is based on Filippov's techniques ([9]); namely, the existence of solutions is obtained by starting from a given "quasi" solution. In addition, the result provides an estimate between the "quasi" solution and the solution obtained.

Our result improve an existence theorem in [14] in the case when the right-hand side is Lipschitz in the second variable. Moreover, our result may be viewed as a generalization to the case when the right-hand side contains a nonlinear Volterra integral operator. Even if the method we use here is known in the theory of differential inclusions (e.g., [3, 4, 5, 6] etc.) it is largely ignored by the authors that are dealing with such problems in favor of fixed point approaches, most probably, because it is much easier to handle the applications of classical fixed point theorems.

The paper is organized as follows: in Section 2 we recall some preliminary results that we need in the sequel and in Section 3 we prove our main results.

2 Preliminaries

Let (X,d) be a metric space. Recall that the Pompeiu-Hausdorff distance of the closed subsets $A, B \subset X$ is defined by

$$d_H(A,B) = \max\{d^*(A,B), d^*(B,A)\}, \ d^*(A,B) = \sup\{d(a,B); a \in A\},$$

where $d(x,B) = \inf_{y \in B} d(x,y)$.

Let $I = [a,b]$, we denote by $C(I,\mathbb{R})$ the Banach space of all continuous functions from I to $\mathbb{R}$ with the norm $||x(.)||_C = \sup_{t \in I} |x(t)|$ and $L^1(I,\mathbb{R})$ is the Banach space of integrable functions $u(.) : I \to \mathbb{R}$ endowed with the norm $||u(.)||_1 = \int_0^T |u(t)|dt$.

In what follows $\psi(.) \in C^1(I,\mathbb{R})$ such that $\psi'(t) > 0 \ \forall \, t \in I$.

Definition 1. *Let $\sigma \in (0,1]$ and $\alpha \in \mathbb{R}_+$. The generalized proportional fractional integral of order α of $f(.) \in L^1(I,\mathbb{R})$ with respect to $\psi(.)$ is defined by*

$$I^{\alpha,\sigma,\psi} f(t) = \frac{1}{\sigma^\alpha \Gamma(\alpha)} \int_a^t e^{\frac{\sigma-1}{\sigma}(\psi(t)-\psi(s))}(\psi(t)-\psi(s))^{\alpha-1}\psi'(s)f(s)ds,$$

where Γ is the (Euler's) Gamma function defined by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt$.

Remark 1. *If $\sigma = 1$, $\psi(t) = t$ the above definition yields the Riemann-Liouville fractional integral, if $\sigma = 1$, $\psi(t) = \ln t$ the previous definition gives the Hadamard fractional integral and if $\sigma = 1$, $\psi(t) = \frac{t^\rho}{\rho}$, $\rho > 0$, Definition 1 covers the Katugampola fractional integral.*

Definition 2. *Let $\sigma \in (0,1]$ and $\alpha \in \mathbb{R}_+$. The generalized proportional fractional derivative of order α of $f(.) \in C(I,\mathbb{R})$ with respect to $\psi(.)$ is defined by*

$$D^{\alpha,\sigma,\psi} f(t) = \frac{1}{\sigma^{n-\alpha}\Gamma(n-\alpha)} D^n(\int_a^t e^{\frac{\sigma-1}{\sigma}(\psi(t)-\psi(s))}(\psi(t)-\psi(s))^{n-\alpha+1}\psi'(s)f(s)ds),$$

where $n = [\alpha]+1$, $[\alpha]$ is the integer part of $\alpha \in \mathbb{R}$.

Definition 3. *Let $f(.), \psi(.) \in C^n(I,\mathbb{R})$ such that $\psi(t), \psi'(t) > 0$ $\forall$ $t \in I$. The ψ-Hilfer generalized proportional fractional derivative operator of order α and type β, respectively, σ with respect to $\psi(.)$ is defined by*

$$D_H^{\alpha,\beta,\sigma,\psi} f(t) = (I^{\beta(n-\alpha),\sigma,\psi}(D^{n,\sigma,\psi})I^{(1-\beta)(n-\alpha),\sigma,\psi} f)(t),$$

where $n-1 < \alpha < n$, $\beta \in [0,1]$, $\sigma \in (0,1]$ and $n \in \mathbb{N}$.

In what follows $\alpha \in (1,2]$ and $\gamma = \alpha + \beta(2-\alpha)$.

Lemma 1. ([14]) *Let $h(.) \in C(I,\mathbb{R})$ and*

$$\Lambda = \frac{e^{\frac{\sigma-1}{\sigma}(\psi(b)-\psi(a))}(\psi(b)-\psi(a))^{\gamma-1}}{\sigma^{\gamma-1}\Gamma(\gamma)} - \sum_{j=1}^m \eta_j \frac{e^{\frac{\sigma-1}{\sigma}(\psi(\xi_j)-\psi(a))}(\psi(\xi_j)-\psi(a))^{\gamma-1}}{\sigma^{\gamma-1}\Gamma(\gamma)}$$
$$- \sum_{i=1}^n \zeta_i \frac{e^{\frac{\sigma-1}{\sigma}(\psi(\theta_i)-\psi(a))}(\psi(\theta_i)-\psi(a))^{\gamma+\varphi_i-1}}{\sigma^{\gamma+\varphi_i-1}\Gamma(\gamma+\varphi_i)} \neq 0.$$

Then, the solution of problem $D_H^{\alpha,\beta,\sigma,\psi} x(t) = h(t)$ with boundary conditions (2) is given by

$$x(t) = I^{\alpha,\sigma,\psi}h(t) + \frac{e^{\frac{\sigma-1}{\sigma}(\psi(b)-\psi(a))}(\psi(b)-\psi(a))^{\gamma-1}}{\Lambda\sigma^{\gamma-1}\Gamma(\gamma)}[\sum_{j=1}^m \eta_j I^{\alpha,\sigma,\psi}h(\xi_j)+$$
$$\sum_{i=1}^n \zeta_i I^{\alpha+\varphi_i,\sigma,\psi}h(\theta_i) - I^{\alpha,\sigma,\psi}h(t)], \quad t \in [a,b]. \tag{3}$$

By definition a function $x(.) \in C(I,\mathbb{R})$ is called a solution of a problem (1)-(2) if there exists $h(.) \in L^1(I,\mathbb{R})$ such that $h(t) \in F(t,x(t),V(x)(t))$ *a.e.* (I) and $x(.)$ is given by (3).

Remark 2. *If we denote* $c(t)=\frac{e^{\frac{\sigma-1}{\sigma}(\psi(b)-\psi(a))}(\psi(b)-\psi(a))^{\gamma-1}}{\Lambda\sigma^{\gamma-1}\Gamma(\gamma)}$ *and*

$$G(t,s)=\frac{1-c(t)}{\sigma^\alpha\Gamma(\alpha)}e^{\frac{\sigma-1}{\sigma}(\psi(t)-\psi(s))}(\psi(t)-\psi(s))^{\alpha-1}\psi'(s)\chi_{[a,t](s)}+$$
$$\frac{c(t)}{\sigma^\alpha\Gamma(\alpha)}e^{\frac{\sigma-1}{\sigma}(\psi(t)-\psi(s))}(\psi(t)-\psi(s))^{\alpha-1}\psi'(s)\sum_{j=1}^m\eta_j\chi_{[a,\xi_j]}(s)+$$
$$c(t)^{\frac{\sigma-1}{\sigma}(\psi(t)-\psi(s))}\psi'(s)\sum_{i=1}^n\frac{\zeta_i}{\sigma^{\alpha+\varphi_i}\Gamma(\alpha+\varphi_i)}(\psi(t)-\psi(s))^{\alpha+\varphi_i-1}\chi_{[a,\theta_i]}(s)$$

where $\chi_A(.)$ *denotes the characteristic function of the set* A*, then the solution* $x(.)$ *in (3) may be put as* $x(t)=\int_a^b G(t,s)h(s)ds$.

Moreover, if we assume that there exists $M_0>0$ *such that* $0<\psi'(t)\leq M_0$ $\forall\, t\in I$*, it follows that* $\forall\, t,s\in I$*, we have*

$$|G(t,s)|\leq(1+\frac{(\psi(b)-\psi(a))^{\gamma-1}}{|\Lambda|\sigma^{\gamma-1}\Gamma(\gamma)})\frac{1}{\sigma^\alpha\Gamma(\alpha)}(\psi(b)-\psi(a))^{\alpha-1}M_0+$$
$$\frac{(\psi(b)-\psi(a))^{\alpha+\gamma-2}}{|\Lambda|\sigma^{\alpha+\gamma-1}\Gamma(\alpha)\Gamma(\gamma)}M_0\sum_{j=1}^m|\eta_j|+\frac{(\psi(b)-\psi(a))^{\alpha+\gamma-2}}{|\Lambda|\sigma^{\alpha+\gamma-1}\Gamma(\gamma)}M_0\sum_{i=1}^n|\eta_j|\frac{|\zeta_i|(\psi(b)-\psi(a))^{\varphi_i}}{\sigma^{\varphi_i}\Gamma(\alpha+\varphi_i)}=:M$$

Also, we need a variant of Kuratowski and Ryll-Nardzewski selection theorem concerning measurable set-valued maps.

Lemma 2. ([1]) *Consider* X *a separable Banach space,* B *is the closed unit ball in* X*,* $H:I\to\mathcal{P}(X)$ *is a set-valued map with nonempty closed values and* $g:I\to X, L:I\to\mathbb{R}_+$ *are measurable functions. If*

$$H(t)\cap(g(t)+L(t)B)\neq\emptyset\quad a.e.(I),$$

then the set-valued map $t\to H(t)\cap(g(t)+L(t)B)$ *has a measurable selection.*

3 The results

In order to prove our results we need the following hypotheses.

Hypothesis 1. i) $F(.,.):I\times\mathbb{R}\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ *has nonempty closed values and is* $\mathcal{L}(I)\otimes\mathcal{B}(\mathbb{R}\times\mathbb{R})$ *measurable.*

ii) *There exists* $L(.)\in L^1(I,(0,\infty))$ *such that, for almost all* $t\in I, F(t,.,.)$ *is* $L(t)$*-Lipschitz in the sense that*

$$\mathrm{d}_H(F(t,x_1,y_1),F(t,x_2,y_2))\leq L(t)(|x_1-x_2|+|y_1-y_2|)\quad\forall\ x_1,x_2,y_1,y_2\in\mathbb{R}.$$

iii) $k(.,.,.):I\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ *is a function such that* $\forall x\in\mathbb{R},\ (t,s)\to k(t,s,x)$ *is measurable.*

iv) $|k(t,s,x)-k(t,s,y)|\leq L(t)|x-y|\quad a.e.\ (t,s)\in I\times I,\quad\forall\, x,y\in\mathbb{R}.$

We use next the following notations

$$M(t):=L(t)(1+\int_a^b L(u)du),\quad t\in I,\quad K_0=\int_a^b M(t)dt.$$

Theorem 1. *Let* $\alpha \in (1,2]$, $\beta \in [0,1]$, $\sigma \in (0,1]$ *and assume that there exists* $M_0 > 0$ *such that* $0 < \psi'(t) \leq M_0$ $\forall\, t \in I$. *Assume that Hypothesis 1 is satisfied and* $MK_0 < 1$. *Let* $y(.) \in C(I,\mathbb{R})$ *be such that* $y(a) = 0$, $y(b) = \sum_{j=1}^{m} \eta_j y(\xi_j) + \sum_{i=1}^{n} \zeta_i I^{\varphi_i,\sigma,\psi} y(\theta_i)$ *and such that there exists* $p(.) \in L^1(I,\mathbb{R}_+)$ *with* $d(D_H^{\alpha,\beta,\sigma,\psi} y(t), F(t, y(t), V(y)(t))) \leq p(t)$ *a.e.* (I).

Then there exists $x(.) : I \to \mathbb{R}$ *a solution of problem (1)-(2) satisfying for all* $t \in I$

$$|x(t) - y(t)| \leq \frac{M}{1 - MK_0} ||p(.)||_1.$$

Proof. The set-valued map $t \to F(t, y(t), V(y)(t))$ is measurable with closed values and

$$F(t, y(t), V(y)(t)) \cap \{D_H^{\alpha,\beta,\sigma,\psi} y(t) + p(t)[-1,1]\} \neq \emptyset \quad a.e.\ (I).$$

It follows from Lemma 2 that there exists a measurable selection $h_1(t) \in F(t, y(t), V(y)(t))$ *a.e.* (I) such that

$$|h_1(t) - D_H^{\alpha,\beta,\sigma,\psi} y(t)| \leq p(t) \quad a.e.\ (I). \tag{4}$$

Define $x_1(t) = \int_a^b G(t,s) h_1(s) ds$ and one has

$$|x_1(t) - y(t)| \leq M \int_a^b p(t) dt.$$

We construct two sequences $x_n(.) \in C(I,\mathbb{R})$, $h_n(.) \in L^1(I,\mathbb{R})$, $n \geq 1$ with the following properties

$$x_n(t) = \int_a^b G(t,s) h_n(s) ds, \quad t \in I, \tag{5}$$

$$h_n(t) \in F(t, x_{n-1}(t), V(x_{n-1})(t)) \quad a.e.\,(I), \tag{6}$$

$$|h_{n+1}(t) - h_n(t)| \leq L(t)(|x_n(t) - x_{n-1}(t)| + \int_a^b L(s)|x_n(s) - x_{n-1}(s)| ds)\ a.e.\ (I). \tag{7}$$

If this is done, then from (4)-(7) we have for almost all $t \in I$

$$|x_{n+1}(t) - x_n(t)| \leq M(MK_0)^n \int_a^b p(t) dt \quad \forall n \in \mathbb{N}.$$

Indeed, assume that the last inequality is true for $n-1$ and we prove it for n. One has

$$\begin{aligned} &|x_{n+1}(t) - x_n(t)| \leq \int_a^b |G(t,t_1)|.|h_{n+1}(t_1) - h_n(t_1)| dt_1 \leq \\ &M \int_a^b L(t_1)[|x_n(t_1) - x_{n-1}(t_1)| + \int_a^{t_1} L(s)|x_n(s) - x_{n-1}(s)| ds] dt_1 \leq \\ &M \int_a^b L(t_1)(1 + \int_a^{t_1} L(s) ds) dt_1 . M^n K_0^{n-1} \int_a^b p(t) dt = M(MK_0)^n \int_a^b p(t) dt. \end{aligned}$$

Therefore $\{x_n(.)\}$ is a Cauchy sequence in the Banach space $C(I,\mathbb{R})$, hence converging uniformly to some $x(.) \in C(I,\mathbb{R})$. Hence, by (7), for almost all $t \in I$, the sequence $\{h_n(t)\}$ is Cauchy in $\mathbb{R}$. Let $h(.)$ be the pointwise limit of $h_n(.)$.

At the same time, one has

$$\begin{aligned}&|x_n(t)-y(t)|\leq |x_1(t)-y(t)|+\sum_{i=1}^{n-1}|x_{i+1}(t)-x_i(t)|\leq\\ &M\int_a^b p(t)dt+\sum_{i=1}^{n-1}(M\int_a^b p(t)dt)(MK_0)^i\leq \frac{M\int_a^b p(t)dt}{1-MK_0}.\end{aligned} \tag{8}$$

On the other hand, from (4), (7) and (8) we obtain for almost all $t\in I$

$$\begin{aligned}&|h_n(t)-D_H^{\alpha,\beta,\sigma,\psi}y(t)|\leq \sum_{i=1}^{n-1}|h_{i+1}(t)-h_i(t)|+|h_1(t)-D_H^{\alpha,\beta,\sigma,\psi}y(t)|\leq\\ &L(t)\frac{M\int_a^b p(t)dt}{1-MK_0}+p(t).\end{aligned}$$

Hence the sequence $h_n(.)$ is integrably bounded and therefore $h(.)\in L^1(I,\mathbb{R})$.

Using Lebesgue's dominated convergence theorem and taking the limit in (5), (6) we deduce that $x(.)$ is a solution of (1)-(2). Finally, passing to the limit in (8) we obtained the desired estimate on $x(.)$.

It remains to construct the sequences $x_n(.), h_n(.)$ with the properties in (5)-(7). The construction will be done by induction.

Since the first step is already realized, assume that for some $N\geq 1$ we already constructed $x_n(.)\in C(I,\mathbb{R})$ and $h_n(.)\in L^1(I,\mathbb{R})$, $n=1,2,...N$ satisfying (5), (7) for $n=1,2,...N$ and (6) for $n=1,2,...N-1$. The set-valued map $t\to F(t,x_N(t),V(x_N)(t))$ is measurable. Moreover, the map $t\to L(t)(|x_N(t)-x_{N-1}(t)|+\int_a^t L(s)|x_N(s)-x_{N-1}(s)|ds)$ is measurable. By the lipschitzianity of $F(t,.)$ we have that for almost all $t\in I$

$$\begin{aligned}&F(t,x_N(t),V(x_N)(t))\cap\{h_N(t)+L(t)(|x_N(t)-x_{N-1}(t)|+\\ &\int_a^t L(s)|x_N(s)-x_{N-1}(s)|ds)[-1,1]\}\neq\emptyset.\end{aligned}$$

Lemma 2 yields that there exists a measurable selection $h_{N+1}(.)$ of $F(.,x_N(.), V(x_N)(.))$ such that for almost all $t\in I$

$$|h_{N+1}(t)-h_N(t)|\leq L(t)(|x_N(t)-x_{N-1}(t)|+\int_a^t L(s)|x_N(s)-x_{N-1}(s)|ds).$$

We define $x_{N+1}(.)$ as in (3.5) with $n=N+1$. Thus $f_{N+1}(.)$ satisfies (6) and (7) and the proof is complete. □

Corollary 1. *Let $\alpha\in(1,2]$, $\beta\in[0,1]$, $\sigma\in(0,1]$ and assume that there exists $M_0>0$ such that $0<\psi'(t)\leq M_0\ \forall\, t\in I$. Assume that Hypothesis 1 is satisfied, $d(0,F(t,0,0)\leq L(t)$ a.e. (I) and $MK_0<1$. Then there exists $x(.)$ a solution of problem (1)-(2) satisfying for all $t\in I$ $|x(t)|\leq \frac{M}{1-MK_0}||L(.)||_1$.*

Proof. It is enough to take $y(.)=0$ and $p(.)=L(.)$ in Theorem 1. □

If F does not depend on the last variable, Hypothesis 1 becames

Hypothesis 2. i) *$F(.,.): I\times\mathbb{R}\to\mathcal{P}(\mathbb{R})$ has nonempty closed values and is $\mathcal{L}(I)\otimes\mathcal{B}(\mathbb{R})$ measurable.*

ii) *There exists $L(.) \in L^1(I,(0,\infty))$ such that, for almost all $t \in I$, $F(t,.)$ is $L(t)$-Lipschitz in the sense that*

$$d_H(F(t,x_1),F(t,x_2)) \leq L(t)|x_1 - x_2| \quad \forall\, x_1,x_2 \in \mathbb{R}.$$

Denote $L_0 = \int_a^b L(t)dt$.

Corollary 2. *Let $\alpha \in (1,2]$, $\beta \in [0,1]$, $\sigma \in (0,1]$ and assume that there exists $M_0 > 0$ such that $0 < \psi'(t) \leq M_0\ \forall\, t \in I$. Assume that Hypothesis 2. is satisfied, $d(0,F(t,0) \leq L(t)$ a.e. (I) and $ML_0 < 1$. Then there exists $x(.)$ a solution of the fractional differential inclusion*

$$D_H^{\alpha,\beta,\sigma,\psi}x(t) \in F(t,x(t)) \quad a.e.\ (I), \tag{9}$$

with boundary conditions (2) satisfying for all $t \in I$

$$|x(t)| \leq \frac{ML_0}{1-ML_0}. \tag{10}$$

Remark 3. *In the particular case when $\sigma = 1$, $\psi(t) = t$ and $\eta_j = 0$, $j = \overline{1,m}$ then Theorem 1 yields Theorem 3.8 in [6].*

A similar result to the one in Corollary 2 may be found in [1]; namely, Theorem 7. The proof of Theorem 7 in [1] is done by using the set-valued contraction principle. Our approach improves the hypothesis concerning the set-valued map in [1]. More exactly, we do not require for the values of F to be compact as in [1] and we do not require that the Lipschitz constant of F to be a mapping from $C(I,\mathbb{R})$ as in [1]. Moreover, Theorem 7 in [1] does not contains a priori bounds for solutions as in (10).

As an example, let us consider the problem

$$\begin{aligned} &D_H^{\frac{3}{2},\frac{1}{2},\frac{3}{4},\psi_0}x(t) \in [-\tfrac{3}{16}(1+\sqrt{\tfrac{21}{17}})\tfrac{|x(t)|}{1+|x(t)|},0] \cup [0,\\ &\tfrac{3}{16}(1+\sqrt{\tfrac{21}{17}})\tfrac{|\int_1^t x(s)ds|}{1+(\frac{1}{4\sqrt{15}}-\frac{1}{16})|\int_1^t x(s)ds|}], \quad a.e.\ ([\tfrac{1}{9},\tfrac{13}{9}]) \end{aligned} \tag{11}$$

with nonlocal integral boundary conditions as in [14]; namely,

$$\begin{aligned} &x(\tfrac{1}{9}) = 0, \quad x(\tfrac{13}{9}) = \tfrac{1}{22}x(\tfrac{1}{3}) + \tfrac{3}{44}x(\tfrac{7}{9}) + \tfrac{7}{66}x(\tfrac{11}{9}) + \tfrac{9}{101}I^{\frac{2}{5},\frac{3}{4},\psi_0}x(\tfrac{5}{9}) + \\ &\tfrac{11}{123}I^{\frac{7}{5},\frac{3}{4},\psi_0}x(\tfrac{10}{9}). \end{aligned} \tag{12}$$

In this case, $\alpha = \frac{3}{2}$, $\beta = \frac{1}{2}$, $\sigma = \frac{3}{4}$, $a = \frac{1}{9}$, $b = \frac{13}{9}$, $m = 3$, $n = 2$, $\eta_1 = \frac{1}{22}$, $\eta_2 = \frac{3}{44}$, $\eta_3 = \frac{7}{66}$, $\xi_1 = \frac{1}{3}$, $\xi_2 = \frac{7}{9}$, $\xi_3 = \frac{11}{9}$, $\zeta_1 = \frac{9}{101}$, $\zeta_2 = \frac{11}{123}$, $\varphi_1 = \frac{2}{5}$, $\varphi_2 = \frac{7}{5}$, $\theta_1 = \frac{5}{9}$, $\theta_2 = \frac{10}{9}$, $\psi_0(t) = t^2+1$ and $\gamma = \frac{7}{4}$.

Define $F(.,.) : I \times \mathbb{R} \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ by

$$F(t,x,y) = [-\frac{3}{16}(1+\sqrt{\frac{21}{17}})\frac{|x|}{1+|x|},0] \cup [0,\frac{3}{16}(1+\sqrt{\frac{21}{17}})\frac{|y|}{1+|y|}]$$

and $k(.,.,.) : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $k(t,s,x) = \frac{3}{16}(1+\sqrt{\frac{21}{17}})x$.

Since

$$\sup\{|u|; \quad u \in F(t,x,y)\} \le \frac{3}{16}(1+\sqrt{\frac{21}{17}}) \quad \forall\, t \in I,\ x,y \in \mathbb{R},$$

$$d_H(F(t,x_1,y_1),F(t,x_2,y_2)) \le \frac{3}{16}(1+\sqrt{\frac{21}{17}})(|x_1-x_2|+|y_1-y_2|)$$

$\forall\, x_1,x_2,y_1,y_2 \in \mathbb{R}$, in this situation $L(t) \equiv \frac{3}{16}(1+\sqrt{\frac{21}{17}})$ and $K_0 = \frac{1}{4}(1+\sqrt{\frac{21}{17}})(1+\frac{1}{4}(1+\sqrt{\frac{21}{17}})) < \frac{1}{17}$.

By standard computations (e.g., [14]) $M \approx 16,99$; therefore, $MK_0 < 1$. So, we may apply Corollary 2 in order to obtain the existence of a solution for problem (11)-(12).

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