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NUMERICAL APPROXIMATIONS AND ASYMPTOTIC LIMITS OF SOME NONLINEAR PROBLEMS

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Dedicated to Professor Radu Păltănea on the occasion of his 70th anniversary

Abstract

In the present work, a numerical approach is dedicated to the approximation to the solutions of a time-independent nonlinear Schrödinger equation in a mixed case provided with numerical tests on the asymptotic limits of the solution according to some parameters. A finite difference discretization with calibrations is applied leading to a quasi-linear algebraic system. where the its solvability is investigated as well as its stability and convergence via Von Neumann method. Some numerical experiments are developed to validate the result, and to test the effect of some parameters on the asymptotic limit of the problem.

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1 Introduction

The goal of this work is to apply a numerical method to transform the continuous problem expressed by means of the nonlinear Schrödinger equation (1)

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below to a discrete system. The numerical quasi linear system will be applied for the study of two types of asymptotic limits. The first is relative to the focusing-defocusing parameter ε when it goes to 0. Being equal to 0, we get a superlinear convex NLS equation, where the solution may be expressed as a soliton type particle. The second asymptotic limit is relative to the power parameter q of the sublinear, concave term $|u|^{q-1}u$ by letting it to the limit value 1. In this limit case, we get an extended extension of the well-known Brezis-Nirenberg problem, and the focusing-defocusing parameter ε may be compared to the eigenvalues of the Laplacian. It also permits to discuss the existence of positive solutions by means of the Pohozaev inequality.

Let $0 < q < 1 < p$ and $\varepsilon > 0$ be fixed real numbers, and denote

$$f_\varepsilon(u) = |u|^{p-1}u + \varepsilon|u|^{q-1}u.$$

Consider from now on the Initial Boundary Value NLS problem

$$\begin{cases} iu_t + u_{xx} + f_\varepsilon(u) = 0, \quad t \geq 0 \;\; L_2 \geq x \geq L_1, \\ u(0,x) = u_0(x), \quad L_1 \leq x \leq L_2, \\ u_x(t,L_1) = u_x(t,L_2) = 0, \quad t \geq 0, \end{cases} \tag{1}$$

with $L_1 < L_2$ in $\mathbb{R}$, $u = u(t,x)$ and $u_0 = u_0(x)$ are complex valued functions.

To deal with the goal of our study, we construct a numerical scheme by adapting a calibrating finite difference scheme leading to a discrete version of the problem (1). The discrete version is written on the form of quasi-linear system $U^n = F(U^{n-1}, U^{n-2})$, in the time step n. U^n is the discrete solution at the time n, and F is a function defined on the vector space $\mathbb{R}^M \times \mathbb{R}^M$, where M is the length of the vector U^n obtained from the space discretization grid.

In section 2, the discrete scheme will be explained. In section 3, the solvability of the scheme is established. We then discuss in section 4, the stability and the consistency of the discrete scheme via the concept of local truncation error. The Von Neumann method is applied for stability and convergence by means of the Fourier transform. Finally, some concrete numerical experimentations are developed to confirm the theoretical findings.

To investigate the problem of asymptotic limits, comparisons in the sense of error estimates relatively to $\varepsilon \to 0$ and $q \to 1$ are studied to evaluate the closeness or not of the limits to the solutions for $\varepsilon = 0$ and $q = 1$. This joins in some sense [4] where some similar nonlinear problems are studied. A majority of studies have been focusing on the one-term NLS equation

$$iu_t + u_{xx} \pm |u|^{p-1}u = 0. \tag{2}$$

The nonlinear term is superlinear convex, or sublinear concave separately. Few studies are done on the mixed case especially the numerical solutions. This is reasonable, especially for the cubic Schrödinger equation $p = 3$ which is related to many physical contexts. This separated or one nonlinear term equations is known as an asymptotic limit for the case of dispersive wave envelope with slow propagation in nonlinear mediums. This is one reasonable motivation of the present work. For more details the readers can also refer to [8, 9, 19, 21, 24, 25] and the references therein.

2 The discretization scheme

This section is concerned with the development of the discrete scheme to be applied on the continuous problem (1) to obtain a semi-discrete and quasi-linear version.

Consider for $k \in \mathbb{N}$, the time intervals $[t^k, t^{k+1}]$, where

$$t^k = kl \text{ and } l = \Delta t = t^{k+1} - t^k.$$

l is known as the time step. Consider similarly a space grid is obtained by letting for $M \in \mathbb{N}$ fixed,

$$\Omega_h = \{x_m = L_0 + mh, \;\; m \in \{0, ..., M+1\}\}, \;\; h = \Delta x = \frac{L_2 - L_1}{M+1}.$$

h is known as usual the space step. We thus split the space domain $[L_1, L_2]$ into sub-domains $[x_m, x_{m+1}]$. We associate u_m^k to the net function $u(t^k, x_m)$ and U_m^k to the numerical solution.

Define the space W_h of functions on Ω_h vanishing at zero. For $U \in W_h$, we introduce respectively the discrete inner product and the discrete L^2-norm in W_h

$$< U, V >_h = h \sum_{m=0}^{M+1} U_m V_m, \quad \|U\|_{h,2} = (U, U)_h^{1/2} = \left(h \sum_{M=0}^{M+1} U_m \right)^{1/2}$$

The discrete problem is based on the following approximations due to the time and space differentiation,

$$\delta_m^k U = \frac{U_m^{k+1} - U_m^{k-1}}{2l}, \quad \Delta_m^k U = \frac{U_{m+1}^k - 2U_m^k + U_{m-1}^k}{h^2},$$

$$(U_t)_m^k = \lambda \delta_{m-1}^k U + (1 - 2\lambda)\delta_m^k U + \lambda \delta_{m+1}^k U,$$

$$(U_x)_m^k = \frac{U_{m+1}^k - U_{m-1}^k}{2h},$$

and

$$(U_{xx})_m^k = \mu \Delta_m^{k+1} U + (1 - 2\mu)\Delta_m^k U + \mu \Delta_m^{k-1} U.$$

where $\lambda, \mu \in]0, 1[$. Denote next for $q \in \mathbb{R}$,

$$g_q(u) = \varepsilon |u|^{q-1} u,$$

and consider for the nonlinear part the approximation

$$f_\varepsilon(U_m^k) = \widetilde{g}_p \left(\nu_1 U_m^k + (1 - \nu_1) U_m^{k-1} \right) + \varepsilon |U_m^{k-1}|^{q-1} U_m^{k-1},$$

where $\nu_1 \in [0, 1]$ and

$$\widetilde{g}_p = \max_m \left| U_m^0 \right|^{p-1}.$$

We discretize problem (1) by

$$i\,(U_t)_m^k + (U_{xx})_m^k + f_\varepsilon(U_m^k) = 0, \tag{3}$$

with the boundary/initial values

$$\begin{cases} U_m^0 = u(0,x_m) = u_0(x_m), \quad 0 \le m \le N+1, \\ U_m^1 = U_m^0 + il\Big(u_0''(x_m) + f(u_0(x_m))\Big), \quad 0 \le m \le N+1, \\ U_1^k = U_{-1}^k \qquad \text{and} \qquad U_M^k = U_{M+2}^k, \quad \forall\ k. \end{cases} \tag{4}$$

Developing equations (3)-(4), we obtain

$$\begin{aligned} a_1U_{m-1}^{k+1} + a_2U_m^{k+1} + a_1U_{m+1}^{k+1} &= b_1U_{m-1}^k + b_2U_m^k + b_1U_{m+1}^k \\ &+ c_1U_{m-1}^{k-1} + c_2U_m^{k-1} + c_1U_{m+1}^{k-1} \\ &- 2lg_q(U_m^{k-1}), \end{aligned} \tag{5}$$

for $1 \le m \le M$, and

$$\begin{cases} a_2U_0^{k+1} + 2a_1U_1^{k+1} = b_2U_0^k + 2b_1U_1^k + c_2U_0^{k-1} + 2c_1U_1^{k-1} - 2lg_q(U_0^{k-1}), \\ 2a_1U_M^{k+1} + a_2U_{M+1}^{k+1} = 2b_1U_M^k + b_2U_{M+1}^k + 2c_1U_M^{k-1} + c_2U_{M+1}^{k-1} - 2lg_q(U_M^{k-1}), \end{cases} \tag{6}$$

where the coefficients a_i, b_i and c_i are defined by

$$a_1 = 2\mu\sigma + i\lambda, \quad a_2 = -4\mu\sigma + (1-2\lambda)i,$$
$$b_1 = -2\sigma(1-2\mu), \quad b_2 = 4\sigma(1-2\mu) - 2\widetilde{g}_p\nu_1 l,$$
$$c_1 = -2\mu\sigma + i\lambda \quad \text{and} \quad c_2 = 4\mu\sigma - 2\widetilde{g}_p(1-\nu_1)l + i(1-2\lambda)$$

where $\sigma = \dfrac{l}{h^2}$.

3 Solvability of the discrete method

The purpose here is to show the solvability of (5)-(6). The following result. will be proved in the present section.

Theorem 1. *The discrete system (4)-(5)-(6) is uniquely solvable.*

Proof of Theorem 1. Denote $U^k = (U_0^k, U_1^k, ..., U_{M+1}^k)^t$. The discrete system (5)-(6) can be transformed to the form

$$AU^{k+1} = BU^k + CU^{k-1} - 2lD^k, \tag{7}$$

where A, B and C are the $(N+2, N+2)$-matrices defined by

$$A = \begin{pmatrix} a_2 & 2a_1 & 0 & \dots & \dots & 0 \\ a_1 & a_2 & a_1 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & a_1 & a_2 & a_1 \\ 0 & \dots & \dots & 0 & 2a_1 & a_2 \end{pmatrix}, \quad B = \begin{pmatrix} b_2 & 2b_1 & 0 & \dots & \dots & 0 \\ b_1 & b_2 & b_1 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & b_1 & b_2 & b_1 \\ 0 & \dots & \dots & 0 & 2b_1 & b_2 \end{pmatrix}$$

and finally

$$C = \begin{pmatrix} c_2 & 2c_1 & 0 & \dots & \dots & 0 \\ c_1 & c_2 & c_1 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & c_1 & c_2 & c_1 \\ 0 & \dots & \dots. & 0 & 2c_1 & c_2 \end{pmatrix},$$

and D^k is the $(N+2)$-vector whom coordinates are

$$D_j^k = g_q(U_j^{k-1}).$$

Denote $Det_{M+2}(A)$ the determinant of the matrix A. Then, from [15]-[18], the following recursive equation holds.

$$Det_{M+2}(A) - a_2 Det_{M+1}(A) + 2a_1^2 Det_N(A) = 0.$$

We now divide the proof into cases.
Case 1. $\mu = 0$ and $\lambda = \lambda_0 = \dfrac{\sqrt{2}-1}{2}$. Standard computations yield that

$$Det_{M+2}(A) = (N+3)\left(\frac{1-\sqrt{2}}{2} i\right)^{M+2} \neq 0.$$

So, the matrix A is invertible.
Case 2. $\mu = 0$ and $\lambda < \lambda_0$. Let $\delta^2 = 2(\lambda_0 - \lambda)(2\lambda + \sqrt{2} + 1)$. Standard computations also yield that

$$Det_{M+2}(A) = -\frac{1}{2^{M+3}\delta}\left[(\delta + 1 - 2\lambda)^{M+3} + (-\delta + 1 - 2\lambda)^{M+3}\right] i^{M+3} \neq 0.$$

So, here-also the matrix A is invertible.
Case 3. $\mu = 0$ and $\lambda > \lambda_0$. Let $\delta^2 = 2(\lambda - \lambda_0)(2\lambda + \sqrt{2} + 1)$. We have

$$Det_{M+2}(A) = \frac{1}{2^{M+3}\delta}\left[(\delta + (1-2\lambda)i)^{M+3} - (-\delta + (1-2\lambda)i)^{M+3}\right] \neq 0.$$

So, A is invertible.
Case 4. $\mu \neq 0$. Let $\delta = \sqrt{a_2^2 - 8a_1^2} \in \mathbb{C}$. It holds that

$$Det_{M+2}(A) = \frac{1}{\delta}\left(\zeta^{M+3} - \xi^{M+3}\right) \neq 0$$

where

$$\zeta = \frac{a_2 + \delta}{2} \quad \text{and} \quad \xi = \frac{a_2 - \delta}{2}.$$

So, A is invertible.

4 The discrete scheme convergence investigation

In this part, the discrete scheme convergence will be investigated and proved to be precisely unconditionally convergent. We will use the same approach developed by the authors in [7] and [6].

Let us write for this goal $U_m^k = e^{im\theta}e^{ik\psi}$ where $\theta \in \mathbb{R}$ and $\psi = \psi_1 + i\psi_2 \in \mathbb{C}$. Let also $X = e^{i\psi}$. It results from equations (5)-(6) that

$$AX^2 + BX + C = 0, \tag{8}$$

where

$$A = \alpha_1 + i\beta_1, \quad B = \alpha_2 + \nu_1\beta_2,$$

$$C = \alpha_1 + (1-\nu_1)\beta_2 + 2l\varepsilon\varphi_{k-1}(\psi_2) - i\beta_1,$$

$$\alpha_1 = -8\mu\sigma\sin^2\left(\frac{\theta}{2}\right), \quad \alpha_2 = 8(2\mu-1)\sigma\sin^2\left(\frac{\theta}{2}\right),$$

$$\beta_1 = 1 - 4\lambda\sin^2\left(\frac{\theta}{2}\right), \quad \beta_2 = 2\widetilde{g}_p l,$$

and finally

$$\varphi_k(x) = e^{-k(q-1)x}.$$

It is immediate from equation (8) that U_m^k is bounded, whatever $\psi \in \mathbb{R}$. Otherwise,

$$|e^{i\psi}| \leq \frac{|B|}{|A|}.$$

We now investigate the term on the right. We split the proof according to λ and μ into four cases. We will develop the computations for $\lambda = 0$. The remaining case $\lambda \neq 0$ can be checked by similar techniques as in [7] where we have used some analogous techniques and is therefore left to the reader. However, we will develop all the cases for μ because it is quite more technical.

Case 1. $0 \leq \mu \leq \frac{1}{3}$. Observing that $h = o(l^2)$, standard computations show that

$$|B|^2 - |A|^2 \leq 4\nu_1^2\widetilde{g}_p^2 l^2 - \frac{32}{3}\nu_1\widetilde{g}_p\sigma l - \frac{1}{2},$$

which is negative whenever $l \leq \dfrac{1}{4\nu_1\widetilde{g}_p}$, which is always possible.

Case 2. $\frac{1}{3} \leq \mu \leq \frac{1}{2}$. The same hypothesis and computations show that

$$|B|^2 - |A|^2 \leq 4\nu_1^2\widetilde{g}_p^2 l^2 - 1,$$

which is negative for the same condition as above.

Case 3. $\frac{1}{2} \leq \mu \leq \frac{2}{3}$. In this case, we obtain

$$|B|^2 - |A|^2 \leq -16\sigma^2 + 4\nu_1^2\widetilde{g}_p^2 l^2 + \frac{32}{3}\nu_1\widetilde{g}_p\sigma l - 1$$

which is negative for the same conditions.
Case 4. $\frac{2}{3} \leq \mu \leq 1$. It holds that

$$|B|^2 - |A|^2 \leq 4\nu_1^2 \widetilde{g}_p^2 l^2 + \frac{32}{3}\nu_1 \widetilde{g}_p \sigma l - 1.$$

So, it is negative as previously.
From all these cases one observe that

$$|B|^2 - |A|^2 < 0 \Leftrightarrow |e^{i\psi}| \leq \frac{|B|}{|A|} < 1$$

which completes the proof.

Theorem 2. *Let u be a sufficiently regular solution of (1) and U^n be the solutions of (3,4). and denote u^n its value at the time t_n. Then, for l small enough, there exist a constant C independent of h and l such that,*

$$\|U^n - u^n\|_h \leq C\left(h^2 + l^2\right)$$

Proof. The proof of this theorem is not complicated and will not be exposed here. It follows quite similar techniques developpend in [5] and [2]. □

5 Stability and consistency of the discrete scheme

Following the scheme described previously, we get the following principal local truncation error

$$\mathcal{L}(t,x) = i\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} + i\frac{l^2}{6}\frac{\partial^3 u}{\partial t^3} + \frac{h^2}{24}\frac{\partial^4 u}{\partial x^4} + \mu l^2 \frac{\partial^4 u}{\partial t^2 \partial x^2} + \mu\frac{h^2}{24} l^2 \frac{\partial^6 u}{\partial t^2 \partial x^4} + o\left(l^2 + h^2\right)$$

Then, the local truncation error tends to 0 as the step size h and time step l approaches 0. Consequently, our discrete scheme is surely consistent.

To investigate the stability, the problem (5) is reformulated in the following way

$$\begin{aligned} & (2\mu\sigma + i\lambda)\Big(U_{m+1}^{k+1} - 2U_m^{k+1} + U_{m-1}^{k+1}\Big) + iU_m^{k+1} \\ = \;& -2(1-2\mu)\sigma\Big(U_{m+1}^{k} - 2U_m^{k} + U_{m-1}^{k}\Big) \\ + \;& (-2\mu\sigma + i\lambda)\Big(U_{m+1}^{k-1} - 2U_m^{k-1} + U_{m-1}^{k-1}\Big) + iU_m^{k-1} \\ - \;& 2\widetilde{g}_p l\Big(\nu_1 U_m^k + (1-\nu_1)U_m^{k-1}\Big) - 2l g_q(U_m^{k-1}). \end{aligned} \tag{9}$$

Next, we apply the Von Neumann criterion by studying the Fourier decomposition of the numerical error. To do this, the numerical value of $U_{j,m}^n$ (subject of round-off computer errors) will be denoted by $\widetilde{U}_{j,m}^n$. We next define a small error as

$$S_m^k = U_m^k - \widetilde{U}_m^k.$$

Equation (9) implies that

$$\begin{aligned}
& (2\mu\sigma + i\lambda)\Big(S_{m+1}^{k+1} - 2S_m^{k+1} + S_{m-1}^{k+1}\Big) + iS_m^{k+1} \\
= \;& -2(1-2\mu)\sigma\Big(S_{m+1}^{k} - 2S_m^{k} + S_{m-1}^{k}\Big) \\
+ \;& (-2\mu\sigma + i\lambda)\Big(S_{m+1}^{k-1} - 2S_m^{k-1} + S_{m-1}^{k-1}\Big) + iS_m^{k-1} \\
- \;& 2\widetilde{g}_p l\Big(\nu_1 S_m^k + (1-\nu_1)S_m^{k-1}\Big) \\
- \;& 2l\left(g_q(U_m^{k-1}) - g_q(\widetilde{U}_m^{k-1})\right).
\end{aligned} \tag{10}$$

Let nest

$$S_m^k = e^{\gamma kl}e^{im\beta h},\ \gamma = \gamma_1 + i\gamma_2 \in \mathbb{C},\ \beta \in \mathbb{R},\ \rho = e^{\gamma l},\ \chi = e^{i\beta h}.$$

It holds that

$$A(\chi)\rho^2 + B(\chi)\rho + C(\chi) = -D(\chi),$$

where

$$A(\chi) = (2\mu\sigma + i\lambda)(\chi-1)^2 + i\chi,$$

$$B(\chi) = 2\sigma(1-2\mu)(\chi-1)^2 + 2\widetilde{g}_p\nu_1 l\chi,$$

$$C(\chi) = (2\mu\sigma - i\lambda)(\chi-1)^2 + (2\widetilde{g}_p(1-\nu_1)l - i)\chi$$

and

$$D(\chi) = 2l\left(g_q(U_m^{k-1}) - g_q(\widetilde{U}_m^{k-1})\right).$$

From Euler's formula, and the fact that g_q is q-Hölder continuous, ρ is solution of the quadratic equation

$$A_0\rho^2 + B_0\rho + C_0 = C\left[\nu_2 e^{q\gamma_1 kl + i\varphi_m^k} + (1-\nu_2)e^{q\gamma_1(k-1)l + i\varphi_m^{k-1}}\right]$$

where

$$\varphi_m^k = \gamma_2 lk + m\beta h,$$

$$A_0 = -8\mu\sigma\sin^2\left(\frac{\beta h}{2}\right) + i(1 - 4\lambda\sin^2\left(\frac{\beta h}{2}\right)),$$

$$B_0 = -8(1-2\mu)\sigma\sin^2\frac{\beta h}{2} + 2\widetilde{g}_p\nu_1 l$$

and

$$C_0 = -8\mu\sigma\sin^2\left(\frac{\beta h}{2}\right) + 2\widetilde{g}_p(1-\nu_1)l + i(4\lambda\sigma\sin^2\left(\frac{\beta h}{2}\right) - 1).$$

Now, the application of Von Neumann requires the amplification factor to be no larger than one, which means that

$$\left|\frac{B(\chi)}{A(\chi)}\right| = \left|\frac{B_0}{A_0}\right| \leq 1.$$

By the fact that $l = o(h^2)$ small enough, and by applying preliminary calculus, we obtain

$$l \leq \min\Big(\frac{4|\lambda - 1|}{2\widetilde{g}_p\nu_1}, \frac{\sqrt{(4|\lambda| - 1)^2 - 64(\mu - 1)(3\mu - 1)}}{2\widetilde{g}_p\nu_1}\Big).$$

The famous Lax-Wendro-Richtmayer theorm permit to obtain the notions of stability and convergence at least one from others (See [23], [26]).

Theorem 3. *[Lax Theorem] For a well posed problem and a consisten numerical scheme, stability is equivalent to convergence.*

Corollary 1. *As the numerical scheme is consistent and stable, it is then convergent.*

6 Numerical experimentation

In this experimental part, we provide a numerical implementation of a concrete example. We firstly have to use the assumption

$$u(L_1, t) = u(L_2, t) = 0, \ \ \forall\, t.$$

This assumption is in fact artificial as our solutions are not necessary compactly supported. However, it may be justified clearly from soliton solutions, for example, which are rapidly decaying at infinity, which makes sense to the assumption. We fix as in section 4, the parameters

$$\mu = \frac{3}{4} \quad \text{and} \quad \lambda = \nu_2 = 0.$$

We study the asymptotic limits on $q \to 1$ and $\varepsilon \to 0$. Numerical estimations of the error of closeness between the limits and the solutions of the limit problems for $q = 1$ and $\varepsilon = 0$ are provided. We consider the exact solutions of

$$iu_t + u_{xx} + |u|^{p-1}u = 0, \tag{11}$$

where our initial problem (1) appears as a perturbed version of this one. There are in fact collection of exact solutions of (11) given by

$$u(x,t) = \frac{C\exp\left(i(at+b)\right)}{[\cosh(\alpha x + \beta)]^{2/(p-1)}}, \tag{12}$$

and

$$u(x,t) = \frac{C\exp\left(i(ax+bt+\Phi_0)\right)}{[\cosh(\alpha x + \beta t + \varphi_0)]^{2/(p-1)}}, \tag{13}$$

where C, a, b, α, β, Φ_0 and φ_0 are fixed constants.

Notice that by fixing the time variable t, we obtain for both cases above exponentially decaying functions as $|x| \to \infty$. Next, we consider the space domain

$[L_1, L_2] = [-80, 100]$, equipped with step $h = 1$, and the time interval $[0, T] = [0, 10]$, with a step $l = 0,01$. In the sequel, we focus on a soliton-type solution following the model (12) above. We then have the following parameter relations

$$\alpha^2 = \frac{a(p-1)^2}{4} = \frac{C^{p-1}(p-1)^2}{2p}.$$

We fix $p = 1,5$ and $\alpha = 0,1$, the phase parameters $\Phi_0 = \varphi_0 = 0$ and finally $q = 0.5$. The figures 1 to 5 show the asymptotic limit as ε tends to 0. We investigate the limit $u_\varepsilon \longrightarrow u_{\varepsilon=0}$, where $u_{\varepsilon=0}$ is the solution of the limit problem (11).

For figures 6 to 10, we fix $p = 3,5$ and $\alpha = 0,1$, the phase parameters $\Phi_0 = \varphi_0 = 0$ and finally $\varepsilon = 1$.

It is noticeable easily from Figures 1 to 6, that the effect of the sublinear term $\varepsilon|u|^{q-1}u$, which is clearly illustrated in Figures 1 and 2 starts to be reduced as the parameter ε approaches 0, it somehow disappear for $\varepsilon = 0.0001$. This is also confirmed in Table 1 illustrating the variation of the error

$$E_\infty = \|u - U\|_\infty = f(\varepsilon),$$

where we put

$$\|u - U\|_\infty = \max_{k,m} |u(t^k, x_m) - U_m^k|.$$

Besides, we notice easily that the solution tends to be more and more regular (smooth curve) as $\varepsilon \to 0$.

ε	0.0001	0.001	0.01	0.1	1
E_∞	5.14e-05	2.31e-04	0.0054	0.0490	0.1270

Table 1: Error estimates relatively to ε.

Next, Figures 7 to 12 show the asymptotic limit as q tends to 1, illustrating the fact that $u_q \longrightarrow u_1$ as $q \to 1$, where u_1 is the solution of the problem (1) with $q = 1$.

It is noticeable here-also from Figures 7 to 12 the influence of the sublinear term $\varepsilon|u|^{q-1}u$, which is clearly illustrated in Figures 7, 8 and also 9, where the curves present some irregularity. Such irregular behavior starts to be reduced as the parameter q approaches 1, and disappears completely for $q = 1$, which is reasonable as in this case the nonlinear term is f_ε becomes $f_\varepsilon(u) = |u|^{p-1}u + \varepsilon u$, which is regular enough as $p > 1$. This is also confirmed in Table 2 illustrating the variation of the error $E_\infty = \|u - U\|_\infty = f(q)$, although the convergence is somehow slowly.

q	0.70	0.75	0.80	0.85	0.90	1
E_∞	0.0498	0.0483	0.0470	0.0455	0.0081	0.0072

Table 2: Error estimates relative to q.

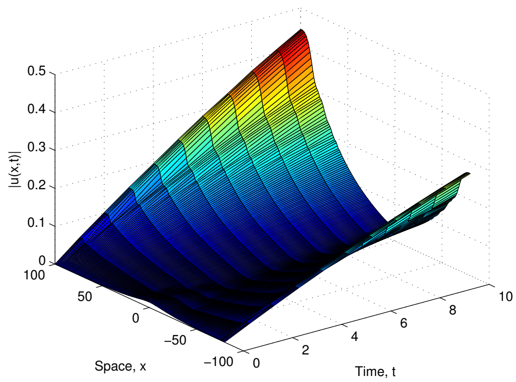

Figure 1: $\varepsilon = 10$.

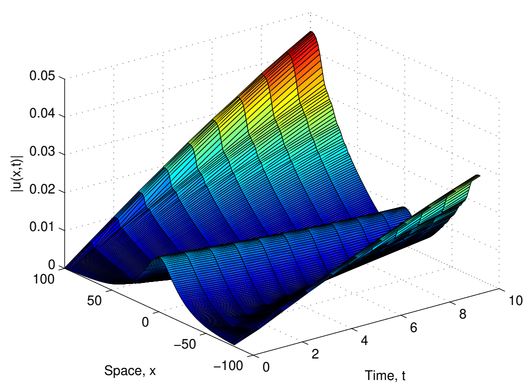

Figure 2: $\varepsilon = 1$.

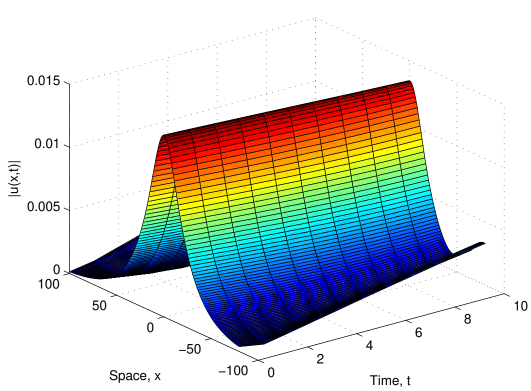

Figure 3: $\varepsilon = 0,01$.

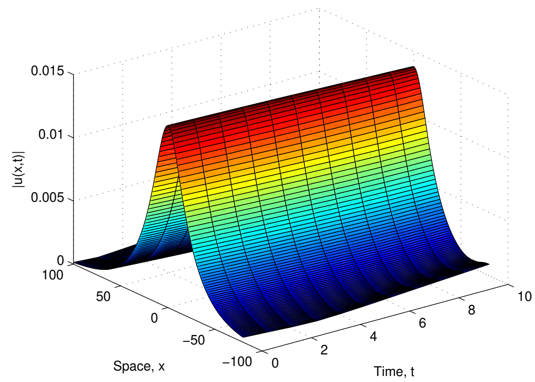

Figure 4: $\varepsilon = 0.001$.

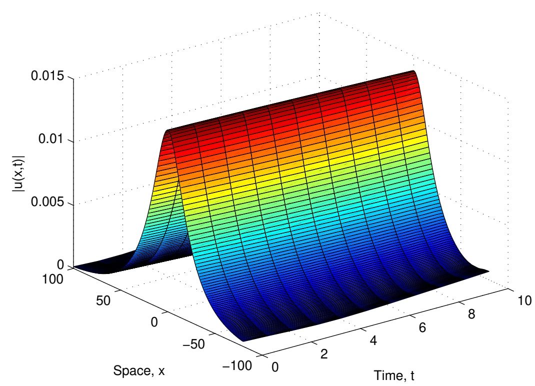

Figure 5: $\varepsilon = 0.0001$.

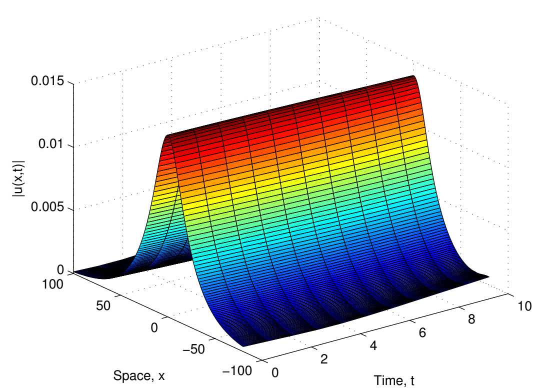

Figure 6: $\varepsilon = 0$.

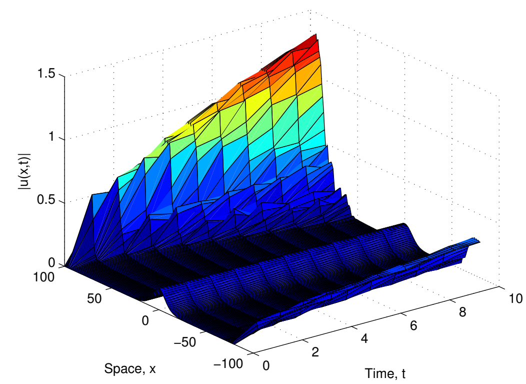

Figure 7: $q = 0,70$.

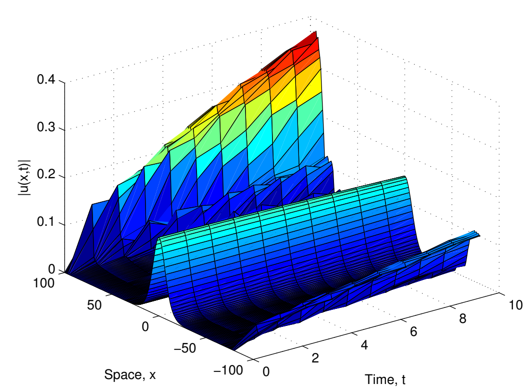

Figure 8: $q = 0,75$.

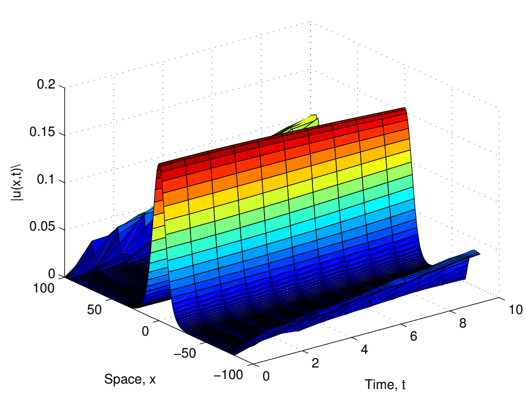

Figure 9: $q = 0,80$.

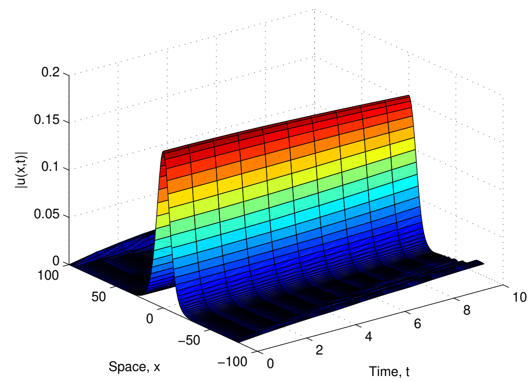

Figure 10: $q = 0,85$.

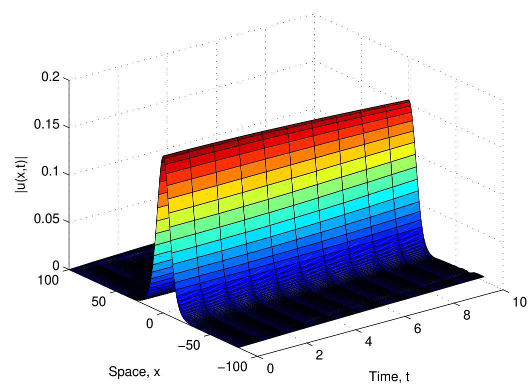

Figure 11: $q = 0,90$.

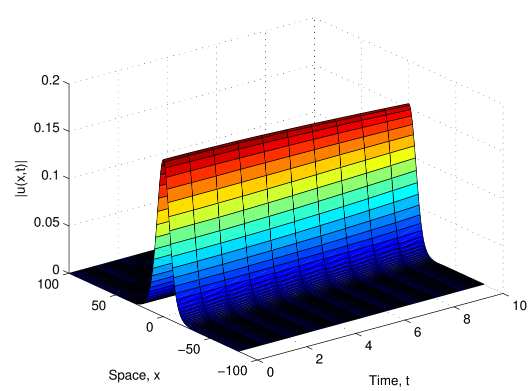

Figure 12: $q = 1$.

7 Conclusion

The present paper is concerned to the development of a numerical approach for the estimation of the solution of a mixed nonlinear Schrödinger equation in one-dimensional case in one side, and to test the possible convergence of the solution of the original problem to the solution of the limit problem according to two parameters due to the sublinear perturbation term. Firstly, the asymptotic limit according to the focusing-defocusing parameter is investigated numerically at 0. Besides, the second limit is due to the sublinear power exponent when it approaches the linear power $q = 1$.

We noticed however from the numerical tests indicate a convergence of the solution of the original problem (1) to the solution of the limit problem that such convergence is sometimes slowly. Besides, it did not guarantee that no surprises occur. This leads us to think about extending ideas based on the theoretical study of the dependence on the limit parameters.

One extending idea to emphasize more the behavior of $u(.,\varepsilon)$ according to ε may be based on the Taylor-expansion by considering the solution of problem (1) as a function $u_\varepsilon = u(t,x,\varepsilon)$ of the variable ϵ on a small neighborhood of 0, and write therefore

$$u_0(t,x) = u(t,x,0), \quad u_1(t,x) = \frac{\partial u_\varepsilon}{\partial \varepsilon}(t,x,0) \ \ , u_2(t,x) = \frac{1}{2}\frac{\partial^2 u_\varepsilon}{\partial \varepsilon^2}(t,x,0).$$

At the order 2, for example, we obtain

$$u_\varepsilon = u_0 + u_1\varepsilon + u_2\varepsilon^2 + \varepsilon^2 o(\varepsilon),$$

where $o(\varepsilon) \to 0$ as $\varepsilon \to 0$. By substituting in problem (1) we get the following system of nonlinear PDEs,

$$\left\{\begin{array}{l} iu_{0,t} + u_{0,xx} + |u_0|^{p-1}u_0 = 0, \\ iu_{1,t} + u_{1,xx} + |u_0|^{p-1}(A_p u_0 + u_1) + |u_0|^{q-1}u_0 = 0, \\ iu_{2,t} + u_{2,xx} + |u_0|^{p-1}(B_p u_0 + A_p u_1 + u_2) + |u_0|^{q-1}(A_q u_0 + u_1) = 0, \end{array}\right. \tag{14}$$

where for $r \in \mathbb{R}$, we denoted

$$A_r = (r-1)\frac{u_1}{u_0} \text{ and } B_r = (r-1)\frac{u_2}{u_0} + \frac{(r-1)(r-2)}{2}\frac{u_1^2}{u_0^2}.$$

Notice for example that u_0 is a solution of the non perturbed problem (11). Therefore, it may take the form of models (12) and (13). It is consequently of interest to investigate the behavior of such a system. The properties of (12) and (13) may serve to investigate the behaviors of u_1 and u_2. Proving for example that u_0, u_1 and u_2 are uniformly bounded (at least ob bounded domains, which means that there are no blow-up surprises especially in time) will permit to conclude about the convergence of the solution u_ε to u_0 is some sense to be fixed.

Similarly, to emphasize the influence of the parameter q, we denote $u_q = u(t,x,q)$ the solution of problem (1) as a function of the variable q on a small neighborhood of 1, and we write

$$u_0(t,x) = u(t,x,1), \quad u_1(t,x) = \frac{\partial u_q}{\partial q}(t,x,1) \ \ , u_2(t,x) = \frac{1}{2}\frac{\partial^2 u_q}{\partial q^2}(t,x,1).$$

At the order 2, we get similarly to above

$$u_q = u_0 + u_1(q-1) + u_2(q-1)^2 + (q-1)^2 o(q-1),$$

where $o(q-1) \to 0$ as $q \to 1$. By substituting in problem (1) we get here also a system of nonlinear PDEs,

$$\left\{ \begin{array}{l} iu_{0,t} + u_{0,xx} + |u_0|^{p-1}u_0 + \varepsilon u_0 = 0, \\ iu_{1,t} + u_{1,xx} + |u_0|^{p-1}(A_p u_0 + u_1) + \varepsilon C(u_0,u_1) = 0, \\ iu_{2,t} + u_{2,xx} + |u_0|^{p-1}(B_p u_0 + A_p u_1 + u_2) + \varepsilon D(u_0,u_1,u_2) = 0, \end{array} \right. \tag{15}$$

where

$$C(u_0,u_1) = u_0 \log|u_0| + u_1 \text{ and } D(u_0,u_1,u_2) = \frac{u_0 \log^2|u_0|}{2} + u_1 \log|u_0| + u_1 + u_2.$$

Notice here also that we get u_0 a solution of the limit problem $q = 1$ due to (1), which is also a linear perturbation of (11). Furthermore, it resembles to an extension of the famous Brezis-Nirenberg problem, where the coefficient ε in the linear perturbation may be compared to the eigenvalues of the Laplacian operator and thus gives again a framwork for the limit on ε.

The readers may for the moment refer to [1, 3, 4, 5, 7, 10, 11, 12, 13, 20, 25] for full theoretical study of similar problems, as well as systems of analogue NLS equations.

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