Bulletin of the *Transilvania* University of Braşov
Series III: Mathematics and Computer Science, Vol. 3(65), No. 2 - 2023, 31-52
https://doi.org/10.31926/but.mif.2023.3.65.2.4

q-DEFORMED AND λ-PARAMETRIZED A-GENERALIZED LOGISTIC FUNCTION BASED BANACH SPACE VALUED ORDINARY AND FRACTIONAL NEURAL NETWORK APPROXIMATION

George A. ANASTASSIOU[1]

Dedicated to Professor Radu Păltănea on the occasion of his 70th anniversary

Abstract

Here we research the univariate quantitative approximation, ordinary and fractional, of Banach space valued continuous functions on a compact interval or all the real line by quasi-interpolation Banach space valued neural network operators. These approximations are derived by establishing Jackson type inequalities involving the modulus of continuity of the engaged function or its Banach space valued high order derivative of fractional derivatives. Our operators are defined by using a density function generated by a q-deformed and λ-parametrized A-generalized logistic function, which is a sigmoid function. The approximations are pointwise and of the uniform norm. The related Banach space valued feed-forward neural networks are with one hidden layer.

2000 *Mathematics Subject Classification:* 26A33, 41A17, 41A25, 41A30, 46B25.

Key words: q-deformed and λ-parametrized A-generalized logistic function, Banach space valued neural network approximation, Banach space valued quasi-interpolation operator, modulus of continuity, Banach space valued Caputo fractional derivative, Banach space valued fractional approximation.

1 Introduction

The author in [1] and [2], see Chapters 2-5, was the first to establish neural network approximation to continuous functions with rates by very specifically defined neural network operators of Cardaliaguet-Euvrard and "Squashing" types, by employing the modulus of continuity of the engaged function or its high order derivative, and producing very tight Jackson type inequalities. He treats there

[1]Department of Mathematical Sciences, University of Memphis, Memphis, TN 38152, U.S.A., e-mail: ganastss@memphis.edu

both the univariate and multivariate cases. The defining these operators "bell-shaped" and "squashing" functions are assumed to be of compact support. Also in [2] he gives the Nth order asymptotic expansion for the error of weak approximation of these two operators to a special natural class of smooth functions, see Chapters 4-5 there.

Again the author inspired by [17], continued his studies on neural networks approximation by introducing and using the proper quasi-interpolation operators of sigmoidal and hyperbolic tangent type which resulted into [3]-[7], by treating both the univariate and multivariate cases. He did also the corresponding fractional cases [8], [9], [13].

Let h be a general sigmoid function with $h(0) = 0$, and $y = \pm 1$ the horizontal asymptotes. Of course h is strictly increasing over $\mathbb{R}$. Let the parameter $0 < r < 1$ and $x > 0$. Then clearly $-x < x$ and $-x < -rx < rx < x$, furthermore it holds $h(-x) < h(-rx) < h(rx) < h(x)$. Consequently the sigmoid $y = h(rx)$ has a graph inside the graph of $y = h(x)$, of course with the same asymptotes $y = \pm 1$. Therefore $h(rx)$ has derivatives (gradients) at more points x than $h(x)$ has different than zero or not as close to zero, thus killing less number of neurons! And of course $h(rx)$ is more distant from $y = \pm 1$, than $h(x)$ it is. A highly desired fact in Neural Networks theory.

Different activation functions allow for different non-linearities which might work better for solving a specific function. So the need to use neural networks with various activation functions is vivid. Thus, performing neural network approximations using different activation functions is not only necessary but fully justified.

Also brain asymmetry has been observed in animals and humans in terms of structure, function and behaviour. This lateralization is thought to reflect evolutionary, hereditary, developmental, experiential and pathological factors. Therefore it is natural to consider for our study deformed neural network activation functions and operators. So this paper is a specific study under this philosophy of approaching reality as close as possible.

Consequently the author here performs q-deformed and λ-parametrized A-generalized logistic function activated neural network approximations to continuous functions over compact intervals of the real line or over the whole $\mathbb{R}$ with valued to an arbitrary Banach space $(X, \|\cdot\|)$. Finally he treats completely the related X-valued fractional approximation. All convergences here are with rates expressed via the modulus of continuity of the involved function or its X-valued high order derivative, or X-valued fractional derivatives and given by very tight Jackson type inequalities.

Our compact intervals are not necessarily symmetric to the origin. Some of our upper bounds to error quantity are very flexible and general. In preparation to prove our results we describe important properties of the basic density function defining our operators which is induced by a q-deformed and λ-parametrized A-generalized logistic function, which is a sigmoid function.

Feed-forward X-valued neural networks (FNNs) with one hidden layer, the only type of networks we deal with in this article, are mathematically expressed as

$$N_n(x) = \sum_{j=0}^{n} c_j \sigma(\langle a_j \cdot x \rangle + b_j), \quad x \in \mathbb{R}^s, \ s \in \mathbb{N},$$

where for $0 \leq j \leq n$, $b_j \in \mathbb{R}$ are the thresholds, $a_j \in \mathbb{R}^s$ are the connection weights, $c_j \in X$ are the coefficients, $\langle a_j \cdot x \rangle$ is the inner product of a_j and x, and σ is the activation function of the network. About neural networks in general read [19], [21], [23].

2 Preliminaries

The following background comes from [16].

We consider here the q-deformed and λ-parametrized function acting as an activation function

$$\varphi_{q,\lambda}(x) = \frac{1}{1 + qA^{-\lambda x}}, \quad x \in \mathbb{R}, \text{ where } q, \lambda > 0, \ A > 1. \tag{1}$$

This is an A-generalized logistic type function.

We easily observe that

$$\varphi_{q,\lambda}(+\infty) = 1, \ \varphi_{q,\lambda}(-\infty) = 0. \tag{2}$$

Furthermore we have

$$1 - \varphi_{\frac{1}{q},\lambda}(-x) = 1 - \frac{1}{1 + \frac{1}{q}A^{\lambda x}} = \frac{1 + \frac{1}{q}A^{\lambda x} - 1}{1 + \frac{1}{q}A^{\lambda x}} =$$

$$\frac{\frac{1}{q}A^{\lambda x}}{1 + \frac{1}{q}A^{\lambda x}} = \frac{1}{\frac{1}{\frac{1}{q}A^{\lambda x}} + 1} = \frac{1}{1 + qA^{-\lambda x}} = \varphi_{q,\lambda}(x),$$

proving

$$\varphi_{q,\lambda}(x) = 1 - \varphi_{\frac{1}{q},\lambda}(-x). \tag{3}$$

We also have that

$$\varphi_{q,\lambda}(0) = \frac{1}{1+q}. \tag{4}$$

Consider the function

$$G_{q,\lambda}(x) := \frac{1}{2}\left(\varphi_{q,\lambda}(x+1) - \varphi_{q,\lambda}(x-1)\right), \quad x \in \mathbb{R}. \tag{5}$$

Then

$$G_{q,\lambda}(-x) = \frac{1}{2}\left(\varphi_{q,\lambda}(-x+1) - \varphi_{q,\lambda}(-x-1)\right) =$$

$$\frac{1}{2}\left(1 - \varphi_{\frac{1}{q},\lambda}(x-1) - 1 + \varphi_{\frac{1}{q},\lambda}(x+1)\right) =$$

$$\frac{1}{2}\left(\varphi_{\frac{1}{q},\lambda}(x+1) - \varphi_{\frac{1}{q},\lambda}(x-1)\right) = G_{\frac{1}{q},\lambda}(x). \tag{6}$$

That is

$$G_{q,\lambda}(-x) = G_{\frac{1}{q},\lambda}(x), \quad \forall\, x \in \mathbb{R}. \tag{7}$$

We have

$$\varphi'_{q,\lambda}(x) = \left(\left(1 + qA^{-\lambda x}\right)^{-1}\right)' =$$

$$-1\left(1 + qA^{-\lambda x}\right)^{-2} q(\ln A) A^{-\lambda x}(-\lambda) = q\lambda(\ln A)\left(1 + qA^{-\lambda x}\right)^{-2} A^{-\lambda x} > 0. \tag{8}$$

So that $\varphi_{q,\lambda}$ is a striclty increasing function over $\mathbb{R}$.

Hence it holds

$$\varphi'_{q,\lambda}(x) = \frac{q\lambda(\ln A)}{\left(1 + qA^{-\lambda x}\right)^2 A^{\lambda x}} =$$

$$\frac{q\lambda(\ln A)}{\left(1 + q^2 A^{-2\lambda x} + 2qA^{-\lambda x}\right) A^{\lambda x}} = \frac{q\lambda(\ln A)}{\left(A^{\lambda x} + q^2 A^{-\lambda x} + 2q\right)}. \tag{9}$$

That is

$$\varphi'_{q,\lambda}(x) = q\lambda(\ln A)\left(A^{\lambda x} + q^2 A^{-\lambda x} + 2q\right)^{-1}. \tag{10}$$

Therefore it holds

$$\varphi''_{q,\lambda}(x)$$

$$= q\lambda(\ln A)(-1)\left(A^{\lambda x} + q^2 A^{-\lambda x} + 2q\right)^{-2}\left((\ln A) A^{\lambda x}\lambda + q^2(\ln A) A^{-\lambda x}(-\lambda)\right)$$

$$= q\lambda^2(\ln A)^2\left(A^{\lambda x} + q^2 A^{-\lambda x} + 2q\right)^{-2}\left(q^2 A^{-\lambda x} - A^{\lambda x}\right). \tag{11}$$

That is

$$\varphi''_{q,\lambda}(x) = q\lambda^2(\ln A)^2\left(A^{\lambda x} + q^2 A^{-\lambda x} + 2q\right)^{-2}\left(q^2 A^{-\lambda x} - A^{\lambda x}\right) \in C(\mathbb{R}). \tag{12}$$

We have

$$q^2 A^{-\lambda x} - A^{\lambda x} > 0, \text{ iff } q^2 A^{-\lambda x} > A^{\lambda x}, \text{ iff } q^2 > A^{2\lambda x}, \text{ iff } q > A^{\lambda x},$$

$$\text{iff } \log_A q > \lambda x, \text{ iff } x < \frac{\log_A q}{\lambda}.$$

So, $\varphi''_{q,\lambda}(x) > 0$, for $x < \frac{\log_A q}{\lambda}$ and there $\varphi_{q,\lambda}$ is concave up.

When $x > \frac{\log_A q}{\lambda}$, we have $\varphi''_{q,\lambda}(x) < 0$ and $\varphi_{q,\lambda}$ is concave down.

Of course

$$\varphi''_{q,\lambda}\left(\frac{\log_A q}{\lambda}\right) = 0.$$

So, $\varphi_{q,\lambda}$ is a sigmoid function, see [14].

We have that

$$G'_{q,\lambda}(x) = \frac{1}{2}\left(\varphi'_{q,\lambda}(x+1) - \varphi'_{q,\lambda}(x-1)\right).$$

We got that $\varphi'_{q,\lambda}$ is strictly increasing for $x < \frac{\log_A q}{\lambda}$. Let $x < \frac{\log_A q}{\lambda} - 1$, then $x - 1 < x + 1 < \frac{\log_A q}{\lambda}$. Hence $\varphi'_{q,\lambda}(x+1) > \varphi'_{q,\lambda}(x-1)$. Thus $G'_{q,\lambda} > 0$, i.e. $G_{q,\lambda}$ is striclty increasing over $\left(-\infty, \frac{\log_A q}{\lambda} - 1\right)$.

Let now $x > \frac{\log_A q}{\lambda} + 1$, then $x + 1 > x - 1 > \frac{\log_A q}{\lambda}$, and $\varphi'_{q,\lambda}(x+1) < \varphi'_{q,\lambda}(x-1)$,by $\varphi'_{q,\lambda}$ being strictly decreasing over $\left(\frac{\log_A q}{\lambda}, +\infty\right)$. Hence $G'_{q,\lambda} < 0$, and $G_{q,\lambda}$ is strictly decreasing over $\left(\frac{\log_A q}{\lambda} + 1, \infty\right)$.

Let now $\frac{\log_A q}{\lambda} - 1 \leq x \leq \frac{\log_A q}{\lambda} + 1$. We have that

$$G''_{q,\lambda}(x) = \frac{1}{2}\left(\varphi''_{q,\lambda}(x+1) - \varphi''_{q,\lambda}(x-1)\right) =$$

$$\frac{q\lambda^2(\ln A)^2}{2}\left[\frac{\left(q^2A^{-\lambda(x+1)} - A^{\lambda(x+1)}\right)}{\left(A^{\lambda(x+1)} + q^2A^{-\lambda(x+1)} + 2q\right)^2}\frac{\left(q^2A^{-\lambda(x-1)} - A^{\lambda(x-1)}\right)}{\left(A^{\lambda(x-1)} + q^2A^{-\lambda(x-1)} + 2q\right)^2}\right] =$$

$$\frac{q\lambda^2(\ln A)^2}{2}\Big[\frac{\left(q^2 - A^{2\lambda(x+1)}\right)}{\left(A^{\lambda(x+1)} + q^2A^{-\lambda(x+1)} + 2q\right)^2 A^{\lambda(x+1)}} -$$

$$-\frac{\left(q^2 - A^{2\lambda(x-1)}\right)}{\left(A^{\lambda(x-1)} + q^2A^{-\lambda(x-1)} + 2q\right)^2 A^{\lambda(x-1)}}\Big] =$$

$$\frac{q\lambda^2(\ln A)^2}{2}\Big[\frac{\left(q - A^{\lambda(x+1)}\right)\left(q + A^{\lambda(x+1)}\right)}{\left(A^{\lambda(x+1)} + q^2A^{-\lambda(x+1)} + 2q\right)^2 A^{\lambda(x+1)}} -$$

$$-\frac{\left(q - A^{\lambda(x-1)}\right)\left(q + A^{\lambda(x-1)}\right)}{\left(A^{\lambda(x-1)} + q^2A^{-\lambda(x-1)} + 2q\right)^2 A^{\lambda(x-1)}}\Big]. \tag{13}$$

By $\frac{\log_A q}{\lambda} \leq x + 1 \Leftrightarrow \log_A q \leq \lambda(x+1) \Leftrightarrow q \leq A^{\lambda(x+1)} \Leftrightarrow q - A^{\lambda(x+1)} \leq 0$.

By $x \leq \frac{\log_A q}{\lambda} + 1 \Leftrightarrow x - 1 \leq \frac{\log_A q}{\lambda} \Leftrightarrow \lambda(x-1) \leq \log_A q \Leftrightarrow A^{\lambda(x-1)} \leq q \Leftrightarrow q - A^{\lambda(x-1)} \geq 0$.

Clearly, when $\frac{\log_A q}{\lambda} - 1 \leq x \leq \frac{\log_A q}{\lambda} + 1$ by the above we get that $G''_{q,\lambda}(x) \leq 0$, that is $G''_{q,\lambda}$ is concave down there.

Clearly $G_{q,\lambda}$ is strictly concave down over $\left(\frac{\log_A q}{\lambda} - 1, \frac{\log_A q}{\lambda} + 1\right)$.

Overall $G_{q,\lambda}$ is a bell-shaped function over $\mathbb{R}$.

Of course it holds $G''_{q,\lambda}\left(\frac{\log_A q}{\lambda}\right) < 0$.

We have that

$$G'_{q,\lambda}\left(\frac{\log_A q}{\lambda}\right) = \frac{1}{2}\left(\varphi'_{q,\lambda}\left(\frac{\log_A q}{\lambda} + 1\right) - \varphi'_{q,\lambda}\left(\frac{\log_A q}{\lambda} - 1\right)\right) =$$

$$\frac{q\lambda(\ln A)}{2}\Big[\frac{1}{A^{\lambda\left(\frac{\log_A q}{\lambda}+1\right)} + q^2A^{-\lambda\left(\frac{\log_A q}{\lambda}+1\right)} + 2q} -$$

$$-\frac{1}{A^{\lambda\left(\frac{\log_A q}{\lambda}-1\right)} + q^2A^{-\lambda\left(\frac{\log_A q}{\lambda}-1\right)} + 2q}\Big] = \tag{14}$$

$$= \frac{q\lambda (\ln A)}{2} T \tag{15}$$

where

$$T = \frac{A^{\lambda\left(\frac{\log_A q}{\lambda}-1\right)} + q^2 A^{-\lambda\left(\frac{\log_A q}{\lambda}-1\right)} - A^{\lambda\left(\frac{\log_A q}{\lambda}+1\right)} - q^2 A^{-\lambda\left(\frac{\log_A q}{\lambda}+1\right)}}{\left(A^{\lambda\left(\frac{\log_A q}{\lambda}+1\right)} + q^2 A^{-\lambda\left(\frac{\log_A q}{\lambda}+1\right)} + 2q\right)\left(A^{\lambda\left(\frac{\log_A q}{\lambda}-1\right)} + q^2 A^{-\lambda\left(\frac{\log_A q}{\lambda}-1\right)} + 2q\right)}.$$

Then

$$\frac{q\lambda (\ln A)}{2} T = \frac{q\lambda (\ln A)}{2} \left[\frac{qA^{-\lambda} + q^2 q^{-1} A^{\lambda} - qA^{\lambda} - q^2 q^{-1} A^{-\lambda}}{\left(qA^{-\lambda} + q^2 q^{-1} A^{\lambda} + 2q\right)\left(qA^{\lambda} + q^2 q^{-1} A^{-\lambda} + 2q\right)}\right] =$$

$$\frac{q\lambda (\ln A)}{2} \left[\frac{qA^{-\lambda} + qA^{\lambda} - qA^{\lambda} - qA^{-\lambda}}{\left(qA^{-\lambda} + qA^{\lambda} + 2q\right)\left(qA^{\lambda} + qA^{-\lambda} + 2q\right)}\right] = 0. \tag{16}$$

So $\frac{\log_A q}{\lambda}$ is the only critical number of $G_{q,\lambda}$ over $\mathbb{R}$. Therefore $G_{q,\lambda}\left(\frac{\log_A q}{\lambda}\right)$ is the maximum of $G_{q,\lambda}$.

We calculate it:

We have that

$$G_{q,\lambda}\left(\frac{\log_A q}{\lambda}\right) = \frac{1}{2}\left(\varphi_{q,\lambda}\left(\frac{\log_A q}{\lambda}+1\right) - \varphi_{q,\lambda}\left(\frac{\log_A q}{\lambda}-1\right)\right) =$$

$$\frac{1}{2}\left(\frac{1}{1+qA^{-\lambda\left(\frac{\log_A q}{\lambda}+1\right)}} - \frac{1}{1+qA^{-\lambda\left(\frac{\log_A q}{\lambda}-1\right)}}\right) = \tag{17}$$

$$\frac{1}{2}\left(\frac{1}{1+qq^{-1}A^{-\lambda}} - \frac{1}{1+qq^{-1}A^{\lambda}}\right) = \frac{1}{2}\left(\frac{1}{1+A^{-\lambda}} - \frac{1}{1+A^{\lambda}}\right) =$$

$$\frac{1}{2}\left(\frac{A^{\lambda}-A^{-\lambda}}{\left(1+A^{-\lambda}\right)\left(1+A^{\lambda}\right)}\right) = \frac{A^{\lambda}-1}{2\left(A^{\lambda}+1\right)}.$$

Yhe global maximul of $G_{q,\lambda}$ is

$$G_{q,\lambda}\left(\frac{\log_A q}{\lambda}\right) = \frac{A^{\lambda}-1}{2\left(A^{\lambda}+1\right)}. \tag{18}$$

Finally we have that

$$\lim_{x\to+\infty} G_{q,\lambda}(x) = \frac{1}{2}\left(\varphi_{q,\lambda}(+\infty) - \varphi_{q,\lambda}(+\infty)\right) = 0, \tag{19}$$

and

$$\lim_{x\to-\infty} G_{q,\lambda}(x) = \frac{1}{2}\left(\varphi_{q,\lambda}(-\infty) - \varphi_{q,\lambda}(-\infty)\right) = 0. \tag{20}$$

Consequently the x-axis is the horizontal asymptote of $G_{q,\lambda}$. Of course $G_{q,\lambda}(x) > 0$, $\forall\ x \in \mathbb{R}$.

We need

Theorem 1. *([16]) It holds*

$$\sum_{i=-\infty}^{\infty} G_{q,\lambda}(x-i) = 1, \ \ \forall\, x \in \mathbb{R}, \forall\, q, \lambda > 0, \ A > 1. \tag{21}$$

Thus

$$\sum_{i=-\infty}^{\infty} G_{q,\lambda}(nx-i) = 1, \forall\, n \in \mathbb{N}, \ \forall\, x \in \mathbb{R}. \tag{22}$$

Similarly, it holds

$$\sum_{i=-\infty}^{\infty} G_{\frac{1}{q},\lambda}(x-i) = 1, \forall\, x \in \mathbb{R}. \tag{23}$$

But $G_{\frac{1}{q},\lambda}(x-i) \overset{(7)}{=} G_{q,\lambda}(i-x), \forall\, x \in \mathbb{R}$.

Hence

$$\sum_{i=-\infty}^{\infty} G_{q,\lambda}(i-x) = 1, \forall\, x \in \mathbb{R}, \tag{24}$$

and

$$\sum_{i=-\infty}^{\infty} G_{q,\lambda}(i+x) = 1, \forall\, x \in \mathbb{R}. \tag{25}$$

It follows

Theorem 2. *([16]) It holds*

$$\int_{-\infty}^{\infty} G_{q,\lambda}(x)\, dx = 1, \quad \lambda, q > 0, \ A > 1. \tag{26}$$

So that $G_{q,\lambda}$ is a density function on $\mathbb{R}$; $\lambda, q > 0, \ A > 1$.

We need the following result

Theorem 3. *([16]) Let $0 < \alpha < 1$, and $n \in \mathbb{N}$ with $n^{1-\alpha} > 2$. Then*

$$\sum_{\substack{k=-\infty \\ |nx-k| \geq n^{1-\alpha}}}^{\infty} G_{q,\lambda}(nx-k) < \max\left\{q, \frac{1}{q}\right\} \frac{1}{A^{\lambda(n^{1-\alpha}-2)}} = \gamma A^{-\lambda\left(n^{1-\alpha}-2\right)}, \tag{27}$$

where $q, \lambda > 0, \ A > 1$; $\gamma := \max\left\{q, \frac{1}{q}\right\}$.

Let $\lceil \cdot \rceil$ the ceiling of the number, and $\lfloor \cdot \rfloor$ the integral part of the number.

We also need

Theorem 4. *([16]) Let $x \in [a,b] \subset \mathbb{R}$ and $n \in \mathbb{N}$ so that $\lceil na \rceil \leq \lfloor nb \rfloor$. For $q > 0$, $\lambda > 0, \ A > 1$, we consider the number $\lambda_q > z_0 > 0$ with $G_{q,\lambda}(z_0) = G_{q,\lambda}(0)$ and $\lambda_q > 1$. Then*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} G_{q,\lambda}(nx-k)} < max\left\{\frac{1}{G_{q,\lambda}(\lambda_q)}, \frac{1}{G_{\frac{1}{q},\lambda}\left(\lambda_{\frac{1}{q}}\right)}\right\} =: K(q). \tag{28}$$

We finally mention

Remark 1. *([16]) (i) We have that*

$$\lim_{n\to+\infty}\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} G_{q,\lambda}(nx-k)\neq 1,\quad \text{for at least some } x\in[a,b], \tag{29}$$

where $\lambda, q>0$.

(ii) Let $[a,b]\subset\mathbb{R}$. *For large* n *we always have* $\lceil na\rceil\leq\lfloor nb\rfloor$. *Also* $a\leq\frac{k}{n}\leq b$, *iff* $\lceil na\rceil\leq k\leq\lfloor nb\rfloor$. *In general it holds*

$$\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} G_{q,\lambda}(nx-k)\leq 1. \tag{30}$$

Let $(X,\|\cdot\|)$ be a Banach space.

Definition 1. *Let* $f\in C([a,b],X)$ *and* $n\in\mathbb{N}:\lceil na\rceil\leq\lfloor nb\rfloor$. *We introduce and define the* X*-valued linear neural network operators*

$$L_n(f,x):=\frac{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f\left(\frac{k}{n}\right)G_{q,\lambda}(nx-k)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} G_{q,\lambda}(nx-k)},\quad x\in[a,b]. \tag{31}$$

Clearly here $L_n(f,x)\in C([a,b],X)$.

For convenience we use the same L_n for real valued functions when needed. We study here the pointwise and uniform convergence of $L_n(f,x)$ to $f(x)$ with rates.

For convenience, also we call

$$L_n^*(f,x):=\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f\left(\frac{k}{n}\right)G_{q,\lambda}(nx-k), \tag{32}$$

(similarly, L_n^* can be defined for real valued functions) that is

$$L_n(f,x):=\frac{L_n^*(f,x)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} G_{q,\lambda}(nx-k)}. \tag{33}$$

So that

$$L_n(f,x)-f(x)=\frac{L_n^*(f,x)}{\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} G_{q,\lambda}(nx-k)}-f(x)= \tag{34}$$

$$\frac{L_n^*(f,x) - f(x)\left(\sum\limits_{k=\lceil na\rceil}^{\lfloor nb\rfloor} G_{q,\lambda}(nx-k)\right)}{\sum\limits_{k=\lceil na\rceil}^{\lfloor nb\rfloor} G_{q,\lambda}(nx-k)}.$$

Consequently, we derive that

$$\|L_n(f,x) - f(x)\| \leq K(q)\left\|L_n^*(f,x) - f(x)\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} G_{q,\lambda}(nx-k)\right)\right\| =$$

$$K(q)\left\|\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left(f\left(\frac{k}{n}\right) - f(x)\right)G_{q,\lambda}(nx-k)\right\|. \tag{35}$$

We will estimate the right hand side of the last quantity.

For that we need, for $f \in C([a,b],X)$ the first modulus of continuity

$$\omega_1(f,\delta) := \sup_{\substack{x,y\in[a,b]\\ |x-y|\leq\delta}} \|f(x) - f(y)\|, \quad \delta > 0. \tag{36}$$

Similarly, it is defined ω_1 for $f \in C_{uB}(\mathbb{R},X)$ (uniformly continuous and bounded functions from $\mathbb{R}$ into X), for $f \in C_B(\mathbb{R},X)$ (continuous and bounded X-valued), and for $f \in C_u(\mathbb{R},X)$ (uniformly continuous).

The fact $f \in C([a,b],X)$ or $f \in C_u(\mathbb{R},X)$, is equivalent to $\lim\limits_{\delta\to 0}\omega_1(f,\delta) = 0$, see [11].

We make

Definition 2. *When $f \in C_{uB}(\mathbb{R},X)$, or $f \in C_B(\mathbb{R},X)$, we define*

$$\overline{L_n}(f,x) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) G_{q,\lambda}(nx-k), \tag{37}$$

$n \in \mathbb{N}$, $x \in \mathbb{R}$, the X-valued quasi-interpolation neural network operator.

We make

Remark 2. *We have that*

$$\left\|f\left(\frac{k}{n}\right)\right\| \leq \|f\|_{\infty,\mathbb{R}} < +\infty,$$

and

$$\left\|f\left(\frac{k}{n}\right)\right\| G_{q,\lambda}(nx-k) \leq \|f\|_{\infty,\mathbb{R}}\, G_{q,\lambda}(nx-k) \tag{38}$$

and

$$\sum_{k=-\lambda}^{\lambda}\left\|f\left(\frac{k}{n}\right)\right\| G_{q,\lambda}(nx-k) \leq \|f\|_{\infty,\mathbb{R}}\left(\sum_{k=-\lambda}^{\lambda} G_{q,\lambda}(nx-k)\right),$$

and finally

$$\sum_{k=-\infty}^{\infty} \left\| f\left(\frac{k}{n}\right) \right\| G_{q,\lambda}(nx-k) \leq \|f\|_{\infty,\mathbb{R}}, \tag{39}$$

a convergent series in $\mathbb{R}$.

So, the series $\sum_{k=-\infty}^{\infty} \left\| f\left(\frac{k}{n}\right) \right\| G_{q,\lambda}(nx-k)$ *is absolutely convergent in* X*, hence it is convergent in* X *and* $\overline{L_n}(f,x) \in X$*. We denote by* $\|f\|_\infty := \sup_{x\in[a,b]} \|f(x)\|$*, for* $f \in C([a,b],X)$*, similarly it is defined for* $f \in C_B(\mathbb{R},X)$.

3 Main results

We present a set of X-valued neural network approximations to a function given with rates.

Theorem 5. *Let* $f \in C([a,b],X)$*,* $0 < \alpha < 1$*,* $n \in \mathbb{N} : n^{1-\alpha} > 2$*,* $x \in [a,b]$*. Then*

i)

$$\|L_n(f,x) - f(x)\| \leq K(q)\left[\omega_1\left(f, \frac{1}{n^\alpha}\right) + 2\|f\|_\infty \gamma A^{-\lambda\left(n^{1-\alpha}-2\right)}\right] =: \rho, \tag{40}$$

and

ii)

$$\|L_n(f) - f\|_\infty \leq \rho. \tag{41}$$

We get that $\lim_{n\to\infty} L_n(f) = f$*, pointwise and uniformly.*

Proof. We see that

$$\left\| \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left(f\left(\frac{k}{n}\right) - f(x) \right) G_{q,\lambda}(nx-k) \right\| \leq$$

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left\| f\left(\frac{k}{n}\right) - f(x) \right\| G_{q,\lambda}(nx-k) =$$

$$\sum_{\substack{k=\lceil na \rceil \\ \left|\frac{k}{n}-x\right| \leq \frac{1}{n^\alpha}}}^{\lfloor nb \rfloor} \left\| f\left(\frac{k}{n}\right) - f(x) \right\| G_{q,\lambda}(nx-k) +$$

$$\sum_{\substack{k=\lceil na \rceil \\ \left|\frac{k}{n}-x\right| > \frac{1}{n^\alpha}}}^{\lfloor nb \rfloor} \left\| f\left(\frac{k}{n}\right) - f(x) \right\| G_{q,\lambda}(nx-k) \leq \tag{42}$$

$$\sum_{\substack{k=\lceil na \rceil \\ \left|\frac{k}{n}-x\right| \leq \frac{1}{n^\alpha}}}^{\lfloor nb \rfloor} \omega_1\left(f, \left|\frac{k}{n}-x\right|\right) G_{q,\lambda}(nx-k) +$$

$$2\|f\|_\infty \sum_{\substack{k=\lceil na\rceil \\ |k-nx|>n^{1-\alpha}}}^{\lfloor nb\rfloor} G_{q,\lambda}(nx-k) \leq$$

$$\omega_1\left(f, \frac{1}{n^\alpha}\right) \sum_{\substack{k=-\infty \\ \left|\frac{k}{n}-x\right|\leq\frac{1}{n^\alpha}}}^{\infty} G_{q,\lambda}(nx-k) +$$

$$2\|f\|_\infty \sum_{\substack{k=-\infty \\ |k-nx|>n^{1-\alpha}}}^{\infty} G_{q,\lambda}(nx-k) \underset{\text{(by Theorem 3)}}{\leq}$$

$$\omega_1\left(f, \frac{1}{n^\alpha}\right) + 2\|f\|_\infty \gamma A^{-\lambda\left(n^{1-\alpha}-2\right)}. \tag{43}$$

That is

$$\left\|\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left(f\left(\frac{k}{n}\right)-f(x)\right)G_{q,\lambda}(nx-k)\right\| \leq$$

$$\omega_1\left(f, \frac{1}{n^\alpha}\right) + 2\|f\|_\infty \gamma A^{-\lambda\left(n^{1-\alpha}-2\right)}. \tag{44}$$

Using the last equality we derive (40). □

Next we give

Theorem 6. *Let $f \in C_B(\mathbb{R}, X)$, $0<\alpha<1$, $n\in\mathbb{N} : n^{1-\alpha}>2$, $x\in\mathbb{R}$. Then*

i)

$$\left\|\overline{L}_n(f,x)-f(x)\right\| \leq \omega_1\left(f, \frac{1}{n^\alpha}\right) + 2\|f\|_\infty \gamma A^{-\lambda\left(n^{1-\alpha}-2\right)} =: \gamma^*, \tag{45}$$

and

ii)

$$\left\|\overline{L}_n(f)-f\right\|_\infty \leq \gamma^*. \tag{46}$$

For $f \in C_{uB}(\mathbb{R}, X)$ we get $\lim_{n\to\infty}\overline{L}_n(f)=f$, pointwise and uniformly.

Proof. We observe that

$$\left\|\overline{L}_n(f,x)-f(x)\right\| = \left\|\sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right)G_{q,\lambda}(nx-k) - f(x)\sum_{k=-\infty}^{\infty}G_{q,\lambda}(nx-k)\right\| =$$

$$\left\|\sum_{k=-\infty}^{\infty}\left(f\left(\frac{k}{n}\right)-f(x)\right)G_{q,\lambda}(nx-k)\right\| \leq$$

$$\sum_{k=-\infty}^{\infty}\left\|f\left(\frac{k}{n}\right)-f(x)\right\|G_{q,\lambda}(nx-k) =$$

$$\sum_{\substack{k=-\infty \\ \left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\alpha}}}}^{\infty}\left\|f\left(\frac{k}{n}\right)-f(x)\right\|G_{q,\lambda}(nx-k)+$$

$$\sum_{\substack{k=-\infty \\ \left|\frac{k}{n}-x\right|>\frac{1}{n^{\alpha}}}}^{\infty}\left\|f\left(\frac{k}{n}\right)-f(x)\right\|G_{q,\lambda}(nx-k)\leq \tag{47}$$

$$\sum_{\substack{k=-\infty \\ \left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\alpha}}}}^{\infty}\omega_1\left(f,\left|\frac{k}{n}-x\right|\right)G_{q,\lambda}(nx-k)+$$

$$2\|f\|_{\infty}\sum_{\substack{k=-\infty \\ \left|\frac{k}{n}-x\right|>\frac{1}{n^{\alpha}}}}^{\infty}G_{q,\lambda}(nx-k)\leq$$

$$\omega_1\left(f,\frac{1}{n^{\alpha}}\right)\sum_{\substack{k=-\infty \\ \left|\frac{k}{n}-x\right|\leq\frac{1}{n^{\alpha}}}}^{\infty}G_{q,\lambda}(nx-k)+2\|f\|_{\infty}\gamma A^{-\lambda\left(n^{1-\alpha}-2\right)}\leq$$

$$\omega_1\left(f,\frac{1}{n^{\alpha}}\right)+2\|f\|_{\infty}\gamma A^{-\lambda\left(n^{1-\alpha}-2\right)}, \tag{48}$$

proving the claim. □

We need the X-valued Taylor's formula in an appropiate form:

Theorem 7. *([10], [12]) Let $N\in\mathbb{N}$, and $f\in C^N([a,b],X)$, where $[a,b]\subset\mathbb{R}$ and X is a Banach space. Let any $x,y\in[a,b]$. Then*

$$f(x)=\sum_{i=0}^{N}\frac{(x-y)^i}{i!}f^{(i)}(y)+\frac{1}{(N-1)!}\int_y^x(x-t)^{N-1}\left(f^{(N)}(t)-f^{(N)}(y)\right)dt. \tag{49}$$

The derivatives $f^{(i)}$, $i\in\mathbb{N}$, are defined like the numerical ones, see [24], p. 83. The integral $\int_y^x$ in (49) is of Bochner type, see [22].

By [12], [20] we have that: if $f\in C([a,b],X)$, then $f\in L_{\infty}([a,b],X)$ and $f\in L_1([a,b],X)$.

In the next we discuss high order neural network X-valued approximation by using the smoothness of f.

Theorem 8. *Let $f\in C^N([a,b],X)$, $n,N\in\mathbb{N}$, $0<\alpha<1$, $x\in[a,b]$ and $n^{1-\alpha}>2$. Then*

i)

$$\|L_n(f,x)-f(x)\|\leq K(q)\left\{\sum_{j=1}^{N}\frac{\left\|f^{(j)}(x)\right\|}{j!}\left[\frac{1}{n^{\alpha j}}+(b-a)^j\gamma A^{-\lambda\left(n^{1-\alpha}-2\right)}\right]+\right. \tag{50}$$

$$\left[\omega_1\left(f^{(N)},\frac{1}{n^{\alpha}}\right)\frac{1}{n^{\alpha N}N!}+\frac{2\left\|f^{(N)}\right\|_{\infty}(b-a)^N\gamma A^{-\lambda\left(n^{1-\alpha}-2\right)}}{N!}\right]\right\},$$

ii) assume further $f^{(j)}(x_0)=0$, $j=1,...,N$, *for some* $x_0\in[a,b]$, *it holds*

$$\|L_n(f,x_0)-f(x_0)\|\le K(q)\cdot$$

$$\left\{\omega_1\left(f^{(N)},\frac{1}{n^{\alpha}}\right)\frac{1}{n^{\alpha N}N!}+\frac{2\left\|f^{(N)}\right\|_{\infty}(b-a)^N\gamma A^{-\lambda\left(n^{1-\alpha}-2\right)}}{N!}\right\}, \quad (51)$$

and

iii)

$$\|L_n(f)-f\|_{\infty}\le K(q)\left\{\sum_{j=1}^{N}\frac{\left\|f^{(j)}\right\|_{\infty}}{j!}\left[\frac{1}{n^{\alpha j}}+(b-a)^j\gamma A^{-\lambda\left(n^{1-\alpha}-2\right)}\right]+\right.$$

$$\left.\left[\omega_1\left(f^{(N)},\frac{1}{n^{\alpha}}\right)\frac{1}{n^{\alpha N}N!}+\frac{2\left\|f^{(N)}\right\|_{\infty}(b-a)^N\gamma A^{-\lambda\left(n^{1-\alpha}-2\right)}}{N!}\right]\right\}. \quad (52)$$

Again we obtain $\lim_{n\to\infty}L_n(f)=f$, *pointwise and uniformly.*

Proof. It is lengthy and similar to [15]. As such it is omitted. □

All integrals from now on are of Bochner type [22].

Definition 3. *([12]) Let* $[a,b]\subset\mathbb{R}$, *X be a Banach space,* $\alpha>0$; $m=\lceil\alpha\rceil\in\mathbb{N}$, *(*$\lceil\cdot\rceil$ *is the ceiling of the number),* $f:[a,b]\to X$. *We assume that* $f^{(m)}\in L_1([a,b],X)$. *We call the Caputo-Bochner left fractional derivative of order* α*:*

$$\left(D_{*a}^{\alpha}f\right)(x):=\frac{1}{\Gamma(m-\alpha)}\int_a^x(x-t)^{m-\alpha-1}f^{(m)}(t)\,dt,\quad\forall\ x\in[a,b]. \quad (53)$$

If $\alpha\in\mathbb{N}$, *we set* $D_{*a}^{\alpha}f:=f^{(m)}$ *the ordinary X-valued derivative (defined similar to numerical one, see [24], p. 83), and also set* $D_{*a}^{0}f:=f$.

By [12], $\left(D_{*a}^{\alpha}f\right)(x)$ exists almost everywhere in $x\in[a,b]$ and $D_{*a}^{\alpha}f\in L_1([a,b],X)$.

If $\left\|f^{(m)}\right\|_{L_{\infty}([a,b],X)}<\infty$, then by [12], $D_{*a}^{\alpha}f\in C([a,b],X)$, hence $\left\|D_{*a}^{\alpha}f\right\|\in C([a,b])$.

We mention

Lemma 1. *([11]) Let* $\alpha>0$, $\alpha\notin\mathbb{N}$, $m=\lceil\alpha\rceil$, $f\in C^{m-1}([a,b],X)$ *and* $f^{(m)}\in L_{\infty}([a,b],X)$. *Then* $D_{*a}^{\alpha}f(a)=0$.

We mention

Definition 4. *([10]) Let $[a,b] \subset \mathbb{R}$, X be a Banach space, $\alpha > 0$, $m := \lceil \alpha \rceil$. We assume that $f^{(m)} \in L_1([a,b], X)$, where $f : [a,b] \to X$. We call the Caputo-Bochner right fractional derivative of order α:*

$$\left(D_{b-}^{\alpha} f\right)(x) := \frac{(-1)^m}{\Gamma(m-\alpha)} \int_x^b (z-x)^{m-\alpha-1} f^{(m)}(z)\, dz, \quad \forall\, x \in [a,b]. \tag{54}$$

We observe that $\left(D_{b-}^{m} f\right)(x) = (-1)^m f^{(m)}(x)$, for $m \in \mathbb{N}$, and $\left(D_{b-}^{0} f\right)(x) = f(x)$.

By [10], $\left(D_{b-}^{\alpha} f\right)(x)$ exists almost everywhere on $[a,b]$ and $\left(D_{b-}^{\alpha} f\right) \in L_1([a,b], X)$.

If $\left\| f^{(m)} \right\|_{L_\infty([a,b],X)} < \infty$, and $\alpha \notin \mathbb{N}$, by [10], $D_{b-}^{\alpha} f \in C([a,b], X)$, hence $\left\| D_{b-}^{\alpha} f \right\| \in C([a,b])$.

We need

Lemma 2. *([11]) Let $f \in C^{m-1}([a,b], X)$, $f^{(m)} \in L_\infty([a,b], X)$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $\alpha \notin \mathbb{N}$. Then $D_{b-}^{\alpha} f(b) = 0$.*

We mention the left fractional Taylor formula

Theorem 9. *([12]) Let $m \in \mathbb{N}$ and $f \in C^m([a,b], X)$, where $[a,b] \subset \mathbb{R}$ and X is a Banach space, and let $\alpha > 0 : m = \lceil \alpha \rceil$. Then*

$$f(x) = \sum_{i=0}^{m-1} \frac{(x-a)^i}{i!} f^{(i)}(a) + \frac{1}{\Gamma(\alpha)} \int_a^x (x-z)^{\alpha-1} \left(D_{*a}^{\alpha} f\right)(z)\, dz, \tag{55}$$

$\forall\, x \in [a,b]$.

We also mention the right fractional Taylor formula

Theorem 10. *([10]) Let $[a,b] \subset \mathbb{R}$, X be a Banach space, $\alpha > 0$, $m = \lceil \alpha \rceil$, $f \in C^m([a,b], X)$. Then*

$$f(x) = \sum_{i=0}^{m-1} \frac{(x-b)^i}{i!} f^{(i)}(b) + \frac{1}{\Gamma(\alpha)} \int_x^b (z-x)^{\alpha-1} \left(D_{b-}^{\alpha} f\right)(z)\, dz, \tag{56}$$

$\forall\, x \in [a,b]$.

Convention 1. *We assume that*

$$D_{*x_0}^{\alpha} f(x) = 0, \text{ for } x < x_0, \tag{57}$$

and

$$D_{x_0-}^{\alpha} f(x) = 0, \text{ for } x > x_0, \tag{58}$$

for all $x, x_0 \in [a,b]$.

We mention

Proposition 1. *([11]) Let $f \in C^n([a,b],X)$, $n = \lceil \nu \rceil$, $\nu > 0$. Then $D_{*a}^{\nu} f(x)$ is continuous in $x \in [a,b]$.*

Proposition 2. *([11]) Let $f \in C^m([a,b],X)$, $m = \lceil \alpha \rceil$, $\alpha > 0$. Then $D_{b-}^{\alpha} f(x)$ is continuous in $x \in [a,b]$.*

We also mention

Proposition 3. *([11]) Let $f \in C^{m-1}([a,b],X)$, $f^{(m)} \in L_\infty([a,b],X)$, $m = \lceil \alpha \rceil$, $\alpha > 0$ and*

$$D_{*x_0}^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_{x_0}^{x} (x-t)^{m-\alpha-1} f^{(m)}(t)\, dt, \tag{59}$$

for all $x, x_0 \in [a,b] : x \geq x_0$.

Then $D_{*x_0}^{\alpha} f(x)$ is continuous in x_0.

Proposition 4. *([11]) Let $f \in C^{m-1}([a,b],X)$, $f^{(m)} \in L_\infty([a,b],X)$, $m = \lceil \alpha \rceil$, $\alpha > 0$ and*

$$D_{x_0-}^{\alpha} f(x) = \frac{(-1)^m}{\Gamma(m-\alpha)} \int_{x}^{x_0} (\zeta-x)^{m-\alpha-1} f^{(m)}(\zeta)\, d\zeta, \tag{60}$$

for all $x, x_0 \in [a,b] : x_0 \geq x$.

Then $D_{x_0-}^{\alpha} f(x)$ is continuous in x_0.

Corollary 1. *([11]) Let $f \in C^m([a,b],X)$, $m = \lceil \alpha \rceil$, $\alpha > 0$, $x, x_0 \in [a,b]$. Then $D_{*x_0}^{\alpha} f(x)$, $D_{x_0-}^{\alpha} f(x)$ are jointly continuous functions in (x, x_0) from $[a,b]^2$ into X, X is a Banach space.*

We need

Theorem 11. *([11]) Let $f : [a,b]^2 \to X$ be jointly continuous, X is a Banach space. Consider*

$$G(x) = \omega_1(f(\cdot, x), \delta, [x,b]), \tag{61}$$

$\delta > 0$, $x \in [a,b]$.

Then G is continuous on $[a,b]$.

Theorem 12. *([11]) Let $f : [a,b]^2 \to X$ be jointly continuous, X is a Banach space. Then*

$$H(x) = \omega_1(f(\cdot, x), \delta, [a,x]), \tag{62}$$

$x \in [a,b]$, is continuous in $x \in [a,b]$, $\delta > 0$.

We make

Remark 3. *([11]) Let $f \in C^{n-1}([a,b])$, $f^{(n)} \in L_\infty([a,b])$, $n = \lceil \nu \rceil$, $\nu > 0$, $\nu \notin \mathbb{N}$. Then*

$$\left\| D_{*a}^{\nu} f(x) \right\| \leq \frac{\left\| f^{(n)} \right\|_{L_\infty([a,b],X)}}{\Gamma(n-\nu+1)} (x-a)^{n-\nu}, \quad \forall\, x \in [a,b]. \tag{63}$$

Thus we observe

$$\omega_1\left(D^{\nu}_{*a}f,\delta\right)=\sup_{\substack{x,y\in[a,b]\\|x-y|\leq\delta}}\left\|D^{\nu}_{*a}f(x)-D^{\nu}_{*a}f(y)\right\|\leq \tag{64}$$

$$\sup_{\substack{x,y\in[a,b]\\|x-y|\leq\delta}}\left(\frac{\left\|f^{(n)}\right\|_{L_\infty([a,b],X)}}{\Gamma(n-\nu+1)}(x-a)^{n-\nu}+\frac{\left\|f^{(n)}\right\|_{L_\infty([a,b],X)}}{\Gamma(n-\nu+1)}(y-a)^{n-\nu}\right)$$

$$\leq\frac{2\left\|f^{(n)}\right\|_{L_\infty([a,b],X)}}{\Gamma(n-\nu+1)}(b-a)^{n-\nu}.$$

Consequently

$$\omega_1\left(D^{\nu}_{*a}f,\delta\right)\leq\frac{2\left\|f^{(n)}\right\|_{L_\infty([a,b],X)}}{\Gamma(n-\nu+1)}(b-a)^{n-\nu}. \tag{65}$$

Similarly, let $f\in C^{m-1}([a,b])$, $f^{(m)}\in L_\infty([a,b])$, $m=\lceil\alpha\rceil$, $\alpha>0$, $\alpha\notin\mathbb{N}$, *then*

$$\omega_1\left(D^{\alpha}_{b-}f,\delta\right)\leq\frac{2\left\|f^{(m)}\right\|_{L_\infty([a,b],X)}}{\Gamma(m-\alpha+1)}(b-a)^{m-\alpha}. \tag{66}$$

So for $f\in C^{m-1}([a,b])$, $f^{(m)}\in L_\infty([a,b])$, $m=\lceil\alpha\rceil$, $\alpha>0$, $\alpha\notin\mathbb{N}$, *we find*

$$\sup_{x_0\in[a,b]}\omega_1\left(D^{\alpha}_{*x_0}f,\delta\right)_{[x_0,b]}\leq\frac{2\left\|f^{(m)}\right\|_{L_\infty([a,b],X)}}{\Gamma(m-\alpha+1)}(b-a)^{m-\alpha}, \tag{67}$$

and

$$\sup_{x_0\in[a,b]}\omega_1\left(D^{\alpha}_{x_0-}f,\delta\right)_{[a,x_0]}\leq\frac{2\left\|f^{(m)}\right\|_{L_\infty([a,b],X)}}{\Gamma(m-\alpha+1)}(b-a)^{m-\alpha}. \tag{68}$$

By [12] we get that $D^{\alpha}_{*x_0}f\in C([x_0,b],X)$, and by [10] we obtain that $D^{\alpha}_{x_0-}f\in C([a,x_0],X)$.

We present the following X-valued fractional approximation result by neural networks.

Theorem 13. *Let* $\alpha>0$, $N=\lceil\alpha\rceil$, $\alpha\notin\mathbb{N}$, $f\in C^N([a,b],X)$, $0<\beta<1$, $x\in[a,b]$, $n\in\mathbb{N}:n^{1-\beta}>2$. *Then*

i)

$$\left\|L(f,x)-\sum_{j=1}^{N-1}\frac{f^{(j)}(x)}{j!}L_n\left((\cdot-x)^j\right)(x)-f(x)\right\|\leq$$

$$K(q)\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left(\omega_1\left(D^{\alpha}_{x-}f,\frac{1}{n^{\beta}}\right)_{[a,x]}+\omega_1\left(D^{\alpha}_{*x}f,\frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}}+\right.$$

$$\left.\gamma A^{-\lambda\left(n^{1-\beta}-2\right)}\left(\left\|D^{\alpha}_{x-}f\right\|_{\infty,[a,x]}(x-a)^{\alpha}+\left\|D^{\alpha}_{*x}f\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\right\}, \tag{69}$$

ii) if $f^{(j)}(x) = 0$, *for* $j = 1, ..., N-1$, *we have*

$$\|L_n(f, x) - f(x)\| \leq K(q) \frac{1}{\Gamma(\alpha+1)}$$

$$\left\{ \frac{\left(\omega_1 \left(D_{x-}^{\alpha} f, \frac{1}{n^{\beta}} \right)_{[a,x]} + \omega_1 \left(D_{*x}^{\alpha} f, \frac{1}{n^{\beta}} \right)_{[x,b]} \right)}{n^{\alpha\beta}} + \right.$$

$$\left. \gamma A^{-\lambda\left(n^{1-\beta}-2\right)} \left(\left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]} (x-a)^{\alpha} + \left\| D_{*x}^{\alpha} f \right\|_{\infty,[x,b]} (b-x)^{\alpha} \right) \right\}, \tag{70}$$

iii)

$$\|L_n(f, x) - f(x)\| \leq K(q) \cdot$$

$$\left\{ \sum_{j=1}^{N-1} \frac{\left\| f^{(j)}(x) \right\|}{j!} \left\{ \frac{1}{n^{\beta j}} + (b-a)^j \gamma A^{-\lambda\left(n^{1-\beta}-2\right)} \right\} + \right.$$

$$\frac{1}{\Gamma(\alpha+1)} \left\{ \frac{\left(\omega_1 \left(D_{x-}^{\alpha} f, \frac{1}{n^{\beta}} \right)_{[a,x]} + \omega_1 \left(D_{*x}^{\alpha} f, \frac{1}{n^{\beta}} \right)_{[x,b]} \right)}{n^{\alpha\beta}} + \right.$$

$$\left. \left. \gamma A^{-\lambda\left(n^{1-\beta}-2\right)} \left(\left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]} (x-a)^{\alpha} + \left\| D_{*x}^{\alpha} f \right\|_{\infty,[x,b]} (b-x)^{\alpha} \right) \right\} \right\}, \tag{71}$$

$\forall\, x \in [a, b]$,

and

iv)

$$\|L_n f - f\|_{\infty} \leq K(q) \cdot$$

$$\left\{ \sum_{j=1}^{N-1} \frac{\left\| f^{(j)} \right\|_{\infty}}{j!} \left\{ \frac{1}{n^{\beta j}} + (b-a)^j \gamma A^{-\lambda\left(n^{1-\beta}-2\right)} \right\} + \right.$$

$$\frac{1}{\Gamma(\alpha+1)} \left\{ \frac{\left(\sup_{x \in [a,b]} \omega_1 \left(D_{x-}^{\alpha} f, \frac{1}{n^{\beta}} \right)_{[a,x]} + \sup_{x \in [a,b]} \omega_1 \left(D_{*x}^{\alpha} f, \frac{1}{n^{\beta}} \right)_{[x,b]} \right)}{n^{\alpha\beta}} + \right.$$

$$\left. \left. \gamma A^{-\lambda\left(n^{1-\beta}-2\right)} (b-a)^{\alpha} \left(\sup_{x \in [a,b]} \left\| D_{x-}^{\alpha} f \right\|_{\infty,[a,x]} + \sup_{x \in [a,b]} \left\| D_{*x}^{\alpha} f \right\|_{\infty,[x,b]} \right) \right\} \right\}. \tag{72}$$

Above, when $N = 1$ *the sum* $\sum_{j=1}^{N-1} \cdot = 0$.

As we see here we obtain X*-valued fractionally type pointwise and uniform convergence with rates of* $L_n \to I$ *the unit operator, as* $n \to \infty$.

Proof. It is very lengthy and very similar to [15]. As such it is omitted. □

Next we apply Theorem 13 for $N = 1$.

Theorem 14. *Let* $0 < \alpha, \beta < 1$, $f \in C^1([a,b], X)$, $x \in [a,b]$, $n \in \mathbb{N} : n^{1-\beta} > 2$. *Then*

i)

$$\|L_n(f,x) - f(x)\| \leq$$

$$K(q)\frac{1}{\Gamma(\alpha+1)}\left\{\frac{\left(\omega_1\left(D_{x-}^{\alpha}f, \frac{1}{n^{\beta}}\right)_{[a,x]} + \omega_1\left(D_{*x}^{\alpha}f, \frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}} + \right.$$

$$\left.\gamma A^{-\lambda\left(n^{1-\beta}-2\right)}\left(\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]}(x-a)^{\alpha} + \left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}(b-x)^{\alpha}\right)\right\}, \quad (73)$$

and

ii)

$$\|L_nf - f\|_{\infty} \leq K(q)\frac{1}{\Gamma(\alpha+1)} \cdot$$

$$\left\{\frac{\left(\sup_{x\in[a,b]}\omega_1\left(D_{x-}^{\alpha}f, \frac{1}{n^{\beta}}\right)_{[a,x]} + \sup_{x\in[a,b]}\omega_1\left(D_{*x}^{\alpha}f, \frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\alpha\beta}} + \right.$$

$$\left.(b-a)^{\alpha}\gamma A^{-\lambda\left(n^{1-\beta}-2\right)}\left(\sup_{x\in[a,b]}\left\|D_{x-}^{\alpha}f\right\|_{\infty,[a,x]} + \sup_{x\in[a,b]}\left\|D_{*x}^{\alpha}f\right\|_{\infty,[x,b]}\right)\right\}. \quad (74)$$

When $\alpha = \frac{1}{2}$ we derive

Corollary 2. *Let* $0 < \beta < 1$, $f \in C^1([a,b], X)$, $x \in [a,b]$, $n \in \mathbb{N} : n^{1-\beta} > 2$. *Then*

i)

$$\|L_n(f,x) - f(x)\| \leq$$

$$\frac{2K(q)}{\sqrt{\pi}}\left\{\frac{\left(\omega_1\left(D_{x-}^{\frac{1}{2}}f, \frac{1}{n^{\beta}}\right)_{[a,x]} + \omega_1\left(D_{*x}^{\frac{1}{2}}f, \frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\frac{\beta}{2}}} + \right.$$

$$\left.\gamma A^{-\lambda\left(n^{1-\beta}-2\right)}\left(\left\|D_{x-}^{\frac{1}{2}}f\right\|_{\infty,[a,x]}\sqrt{(x-a)} + \left\|D_{*x}^{\frac{1}{2}}f\right\|_{\infty,[x,b]}\sqrt{(b-x)}\right)\right\}, \quad (75)$$

and

ii)

$$\|L_nf - f\|_{\infty} \leq \frac{2K(q)}{\sqrt{\pi}} \cdot$$

$$\left\{\frac{\left(\sup_{x\in[a,b]}\omega_1\left(D_{x-}^{\frac{1}{2}}f, \frac{1}{n^{\beta}}\right)_{[a,x]} + \sup_{x\in[a,b]}\omega_1\left(D_{*x}^{\frac{1}{2}}f, \frac{1}{n^{\beta}}\right)_{[x,b]}\right)}{n^{\frac{\beta}{2}}} + \right.$$

$$\sqrt{(b-a)}\gamma A^{-\lambda\left(n^{1-\beta}-2\right)}\left(\sup_{x\in[a,b]}\left\|D_{x-}^{\frac{1}{2}}f\right\|_{\infty,[a,x]}+\sup_{x\in[a,b]}\left\|D_{*x}^{\frac{1}{2}}f\right\|_{\infty,[x,b]}\right)\Bigg\}<\infty. \tag{76}$$

We make

Remark 4. *Some convergence analysis follows based on Corollary 2.*

Let $0<\beta<1$, $f\in C^1([a,b],X)$, $x\in[a,b]$, $n\in\mathbb{N}:n^{1-\beta}>2$. *We elaborate on (76). Assume that*

$$\omega_1\left(D_{x-}^{\frac{1}{2}}f,\frac{1}{n^\beta}\right)_{[a,x]}\leq\frac{R_1}{n^\beta}, \tag{77}$$

and

$$\omega_1\left(D_{*x}^{\frac{1}{2}}f,\frac{1}{n^\beta}\right)_{[x,b]}\leq\frac{R_2}{n^\beta}, \tag{78}$$

$\forall\ x\in[a,b]$, $\forall\ n\in\mathbb{N}$, *where* $R_1,R_2>0$.

Then it holds

$$\frac{\left[\sup_{x\in[a,b]}\omega_1\left(D_{x-}^{\frac{1}{2}}f,\frac{1}{n^\beta}\right)_{[a,x]}+\sup_{x\in[a,b]}\omega_1\left(D_{*x}^{\frac{1}{2}}f,\frac{1}{n^\beta}\right)_{[x,b]}\right]}{n^{\frac{\beta}{2}}}\leq$$

$$\frac{\frac{(R_1+R_2)}{n^\beta}}{n^{\frac{\beta}{2}}}=\frac{(R_1+R_2)}{n^{\frac{3\beta}{2}}}=\frac{R}{n^{\frac{3\beta}{2}}}, \tag{79}$$

where $R:=R_1+R_2>0$.

The other summand of the right hand side of (76), for large enough n, *converges to zero at the speed* $A^{-\lambda\left(n^{1-\beta}-2\right)}$, *so it is about* $LA^{-\lambda\left(n^{1-\beta}-2\right)}$, *where* $L>0$ *is a constant.*

Then, for large enough $n\in\mathbb{N}$, *by (76), (79) and the above comment, we obtain that*

$$\|L_nf-f\|_\infty\leq\frac{M}{n^{\frac{3\beta}{2}}}, \tag{80}$$

where $M>0$, *converging to zero at the high speed of* $\frac{1}{n^{\frac{3\beta}{2}}}$.

In Theorem 5, for $f\in C([a,b],X)$ *and for large enough* $n\in\mathbb{N}$, *the speed is* $\frac{1}{n^\beta}$. *So by (80),* $\|L_nf-f\|_\infty$ *converges much faster to zero. The last comes because we assumed differentiability of* f. *Notice that in Corollary 2 no initial condition is assumed.*

References

[1] Anastassiou, G.A., *Rate of convergence of some neural network operators to the unit-univariate case*, J. Math. Anal. Appl, **212** (1997), 237-262.

[2] Anastassiou, G.A., *Quantitative Approximations*, Chapman & Hall / CRC, Boca Raton, New York, 2001.

[3] Anastassiou, G.A., *Univariate hyperbolic tangent neural network approximation*, Mathematics and Computer Modelling, **53** (2011), 1111-1132.

[4] Anastassiou, G.A., *Multivariate hyperbolic tangent neural network approximation*, Computers and Mathematics, **61** (2011), 809-821.

[5] Anastassiou, G.A., *Multivariate sigmoidal neural network approximation*, Neural Networks, **24** (2011), 378-386.

[6] Anastassiou, G.A., *Inteligent Systems: Approximation by Artificial Neural Networks*, Intelligent Systems Reference Library, Vol. 19, Springer, Heidelberg, 2011.

[7] Anastassiou, G.A., *Univariate sigmoidal neural network approximation*, J. Comput. Anal. Appl. **14** (2012), no. 4, 659-690.

[8] Anastassiou, G.A., *Fractional neural network approximation*, Computers and Mathematics with Applications, **64** (2012), 1655-1676.

[9] Anastassiou, G.A., *Intelligent Systems II: Complete Approximation by Neural Network Operators*, Springer, Heidelberg, New York, 2016.

[10] Anastassiou, G.A., *Strong Right Fractional Calculus for Banach space valued functions*, 'Revista Proyecciones, **36** (2017), no. 1, 149-186.

[11] Anastassiou, G.A., *Vector fractional Korovkin type Approximations*, Dynamic Systems and Applications, **26** (2017), 81-104.

[12] Anastassiou, G.A., *A strong Fractional Calculus Theory for Banach space valued functions*, Nonlinear Functional Analysis and Applications (Korea), **22(3)** (2017), 495-524.

[13] Anastassiou, G.A., *Nonlinearity: Ordinary and Fractional Approximations by Sublinear and Max-Product Operators*, Springer, Heidelberg, New York, 2018.

[14] Anastassiou, G.A., *General sigmoid based Banach space valued neural network approximation*, J. Comput. Anal. Appl., **31(4)** (2023), 520-534.

[15] Anastassiou, G.A.; Karateke, S., *Parametrized hyperbolic tangent induced Banach space valued ordinary and fractional neural network approximation*, Progr. Fract. Differ. Appl., accepted, 2022.

[16] Anastassiou, G.A., *q-Deformed and λ-parametrized A-generalized logistic function induced Banach space valued multivariate multi layer neural network approximations*, submitted, 2022.

[17] Chen, Z.; Cao, F., *The approximation operators with sigmoidal functions*, Computers and Mathematics with Applications, **58** (2009), 758-765.

[18] El-Shehawy, S.A.; Abdel-Salam, E.A-B., *The q-deformed hyperbolic Secant family*, Intern. J. of Applied Math. & Stat., **29 (5)** (2012), 51-62.

[19] Haykin, S., *Neural Networks: A Comprehensive Foundation* (2 ed.), Prentice Hall, New York, 1998.

[20] Kreuter, M., *Sobolev Spaces of Vector-valued functions*, Ulm Univ., Master Thesis in Math., Ulm, Germany, 2015.

[21] McCulloch, W.; Pitts, W., *A logical calculus of the ideas immanent in nervous activity*, Bulletin of Mathematical Biophysis, **7** (1943), 115-133.

[22] Mikusinski, J., *The Bochner integral*, Academic Press, New York, 1978.

[23] Mitchell, T.M., *Machine Learning*, WCB-McGraw-Hill, New York, 1997.

[24] Shilov, G.E., *Elementary Functional Analysis*, Dover Publications, Inc., New York, 1996.