Bulletin of the *Transilvania* University of Braşov
Series III: Mathematics and Computer Science, Vol. 3(65), No. 2 - 2023, 21-30
https://doi.org/10.31926/but.mif.2023.3.65.2.3

(p, q)-POST-WIDDER OPERATORS OF SEMI EXPONENTIAL TYPE

Gunjan AGRAWAL[*,1] **and Vijay GUPTA**[2]

Dedicated to Professor Radu Păltănea on the occasion of his 70th anniversary

Abstract

We deal here with the (p, q)-variant of the Post-Widder operators of semi-exponential type. By using the basic properties of post-quantum properties, we calculate the moments of these operators and obtain some direct findings for such (p, q)-semi-exponential Post-Widder operators.

2000 *Mathematics Subject Classification:* 41A25, 41A30 .
Key words: (p, q)-Semi-exponential type Post-Widder operators, moments, direct results, Lipschitz condition

1 (p, q)-Variant of Post-Widder operators

Professor Radu Păltănea is a renowned researcher in the area of approximation theory concerning linear positive operators and has produced excellent work on a variety of operators. The papers [13], [14] and [15] contain examples of his work. In [13], the author has dealt with the approximation properties of a modified family of Szász-Mirakjan operators. In the study [14], the author has generated a generic weighted modulus from a class of "admissible" functions and obtained an estimate relevant to general positive linear operators. Also discussed in [15] are the shape-preserving property and the simultaneous approximation by a sequences of Durrmeyer type modifications of Szász-Mirakjan operators with a parameter.

[1*] *Corresponding author*, Department of Mathematics, Netaji Subhas University of Technology, Sector 3 Dwarka, New Delhi 110078, India, e-mail: gunjan.guptaa88@gmail.com
[2]Department of Mathematics, Netaji Subhas University of Technology, Sector 3 Dwarka, New Delhi 110078, India, e-mail: vijaygupta2001@hotmail.com

Tyliba and Wachnicki [16] first generalized the exponential-type operators given in [11], and they captured the semi-exponential Szász-Mirakyan and Gauss-Weierstrass operators. Very recently, Monika Herzog [10] captured one more semi-exponential type operators namely Post-Widder operators, which for $x \in \mathbf{N} \cup \{0\}$ and $\varsigma \in \mathbf{N}$ is defined as follows:

$$(P_\varsigma^\beta f)(x) = \frac{\varsigma^\varsigma}{x^\varsigma e^{\beta x}} \sum_{l=0}^{\infty} \frac{(\varsigma\beta)^l}{l!\Gamma(l+\varsigma)} \int_0^\infty e^{-\varsigma\tau/x} \tau^{l+\varsigma-1} f(\tau) d\tau \tag{1}$$
$$=: \int_0^\infty k_\varsigma^\beta(x,\tau) f(\tau) d\tau.$$

The below mentioned partial differential equation is satisfied by the kernel:

$$\frac{\partial}{\partial x} k_\varsigma^\beta(x,\tau) = \left[\frac{-\varsigma(x-\tau)}{x^2} - \beta\right] k_\varsigma^\beta(x,\tau). \tag{2}$$

This prerequisite must be met for an operator to be of the semi-exponential type. Further, for $\beta = 0$, the PDE (2) for kernel reduces to the condition of exponential type operators and (1) becomes the Post-Widder operators [11, (3.9)] defined by

$$(P_\varsigma f)(x) = \frac{\varsigma^\varsigma}{x^\varsigma} \frac{1}{\Gamma(\varsigma)} \int_0^\infty e^{-\varsigma\tau/x} \tau^{\varsigma-1} f(\tau) d\tau. \tag{3}$$

Its not obvious to introduce the semi-exponential type operators from existing exponential-type operators. Recently Abel et al [1] captured and provided all possible semi-exponential type operators.

In order to investigate the extension of quantum calculus, Acar [2] introduced (p,q)-Szász-Mirakyan operators. Recently, (p,q)-Szász-Durrmeyer operators were introduced by Aral and Gupta [3], who also established some direct results. In [8] and [12], authors also looked into Durrmeyer-style alterations to the Bernstein operators. The (p,q)- Szász-Baskakov operators were recently introduced by Gupta [7], who also produced some direct findings. [6] defined the Kantorovich variation of the (p,q)-Baskakov operators. Certain basic properties of (p,q) calculus is discussed in [9]. The following list includes some fundamental (p,q)-calculus notations, where $0 < q < p \leq 1$:
The (p,q)-numbers are expressed as

$$[\varsigma]_{p,q} = \frac{p^\varsigma - q^\varsigma}{p-q}.$$

It is obvious to observe that $[\varsigma]_{p,q} = p^{\varsigma-1} [\varsigma]_{q/p}$. The definition of (p,q)-factorial is

$$[\varsigma]_{p,q}! = \prod_{l=1}^{\varsigma} [l]_{p,q}, \varsigma \geq 1, \quad [0]_{p,q}! = 1.$$

$$\left[\begin{array}{c}\varsigma\\ l\end{array}\right]_{p,q}=\frac{[\varsigma]_{p,q}!}{[\varsigma-l]_{p,q}!\,[l]_{p,q}!},\quad 0\leq l\leq\varsigma$$

represents the (p,q)-binomial coefficient. The (p,q)-variants of exponential functions, i.e. $e_{p,q}(x)$ and $E_{p,q}(x)$ are given as

$$e_{p,q}\left(x\right)=\sum_{i=0}^{\infty}\frac{p^{i(i-1)/2}}{[i]_{p,q}!}x^{i}\ \text{ and }\ E_{p,q}\left(x\right)=\sum_{j=0}^{\infty}\frac{q^{j(j-1)/2}}{[j]_{p,q}!}x^{j}.$$

It is evident that functions

$$e_{p,q}\left(x\right)E_{p,q}\left(-x\right)=1.$$

For any $\varsigma\in\mathbf{N}$, we suggest (p,q)-Gamma function as

$$\Gamma_{p,q}\left(\varsigma\right)=\int_{0}^{\infty}p^{(\varsigma-1)(\varsigma-2)/2}x^{\varsigma-1}E_{p,q}\left(-qx\right)d_{p,q}x. \tag{4}$$

It can be seen that $\Gamma_{p,q}\left(\varsigma+1\right)=[\varsigma]_{p,q}!,\quad\varsigma\in\mathbf{N}$.

Let $0<q<p\leq1$, then (p,q)-Semi exponential Post-Widder operators can be defined as:

$$(P_{\varsigma}^{\beta,p,q}f)(x)=\frac{[\varsigma]_{p,q}}{E_{p,q}(\beta x)}\sum_{l=0}^{\infty}\frac{q^{l(l-1)/2}\left(\beta x\right)^{l}}{[l]_{p,q}!\Gamma_{p,q}(\varsigma+l)}\times$$
$$\times\int_{0}^{\infty}p^{(\varsigma+l-1)(l+\varsigma-2)/2}E_{p,q}\left(-q[\varsigma]_{p,q}u\right)\left([\varsigma]_{p,q}u\right)^{l+\varsigma-1}f(xuq^{1-\varsigma-l}p^{\varsigma+l})d_{p,q}u$$

For $p=q=1$, we immediately get the semi-exponential Post-Widder operators (1).

2 Moment estimation

Lemma 1. *For $x\in[0,\infty)$, $0<q<p\leq1$, the following holds:*

$$\begin{aligned}(P_{\varsigma}^{\beta,p,q}e_{0})(x)&=1\\(P_{\varsigma}^{\beta,p,q}e_{1})(x)&=q^{1-\varsigma}px+\frac{q^{-\varsigma}\beta x^{2}p^{\varsigma+1}}{[\varsigma]_{p,q}}\\(P_{\varsigma}^{\beta,p,q}e_{2})(x)&=q^{(3-2\varsigma)}px^{2}+\frac{1}{[\varsigma]_{p,q}}\left(q^{2(1-\varsigma)}x^{2}p^{\varsigma+1}+\beta x^{3}p^{\varsigma+1}q^{-2\varsigma+1}(p+q)\right)\\&\quad+\frac{1}{[\varsigma]_{p,q}^{2}}\left(\beta x^{3}p^{\varsigma+1}q^{-2\varsigma}(p+q)+\beta^{2}x^{4}q^{-2\varsigma-1}p^{2\varsigma+3}\right).\end{aligned}$$

Proof. By (p,q)-Gamma function (4), we have

$$\begin{aligned}(P_{\varsigma}^{\beta,p,q}e_0)(x) &= \frac{[\varsigma]_{p,q}}{E_{p,q}(\beta x)}\sum_{l=0}^{\infty}\frac{q^{l(l-1)/2}\,(\beta x)^l}{[l]_{p,q}!\Gamma_{p,q}(\varsigma+l)} \\ &\quad \int_0^{\infty} p^{(l+\varsigma-1)(l+\varsigma-2)/2}E_{p,q}\left(-q[\varsigma]_{p,q}u\right)\left([\varsigma]_{p,q}u\right)^{l+\varsigma-1}d_{p,q}u \\ &= \frac{1}{E_{p,q}(\beta x)}E_{p,q}(\beta x)=1\end{aligned}$$

and by $[\varsigma+l]_{p,q}=q^l[\varsigma]_{p,q}+p^{\varsigma}[l]_{p,q}$, we get

$$\begin{aligned}(P_{\varsigma}^{\beta,p,q}e_1)(x) &= \frac{[\varsigma]_{p,q}}{E_{p,q}(\beta x)}\sum_{l=0}^{\infty}\frac{q^{l(l-1)/2}\,(\beta x)^l}{[l]_{p,q}!\Gamma_{p,q}(\varsigma+l)}\times \\ &\times\int_0^{\infty} p^{(l+\varsigma-1)(l+\varsigma-2)/2}E_{p,q}\left(-q[\varsigma]_{p,q}u\right)\left([\varsigma]_{p,q}u\right)^{l+\varsigma-1}xuq^{1-\varsigma-l}p^{\varsigma+l}d_{p,q}u \\ &= \frac{x[\varsigma]_{p,q}}{[\varsigma]_{p,q}E_{p,q}(\beta x)}\sum_{l=0}^{\infty}q^{1-\varsigma-l}p\frac{q^{l(l-1)/2}\,(\beta x)^l}{[l]_{p,q}!\Gamma_{p,q}(\varsigma+l)} \\ &\quad\int_0^{\infty} p^{(\varsigma+l)(l+\varsigma-1)/2}E_{p,q}\left(-q[\varsigma]_{p,q}u\right)\left([\varsigma]_{p,q}u\right)^{\varsigma+l}d_{p,q}u \\ &= \frac{x}{[\varsigma]_{p,q}E_{p,q}(\beta x)}\sum_{l=0}^{\infty}q^{1-\varsigma-l}p\frac{q^{l(l-1)/2}\,(\beta x)^l}{[l]_{p,q}!\Gamma_{p,q}(\varsigma+l)}\Gamma_{p,q}(l+\varsigma+1) \\ &= \frac{x}{[\varsigma]_{p,q}E_{p,q}(\beta x)}\sum_{l=0}^{\infty}q^{1-\varsigma-l}p\frac{q^{l(l-1)/2}\,(\beta x)^l}{[l]_{p,q}!}[\varsigma+l]_{p,q} \\ &= \frac{x}{[\varsigma]_{p,q}E_{p,q}(\beta x)}\sum_{l=0}^{\infty}q^{1-\varsigma-l}p\frac{q^{l(l-1)/2}\,(\beta x)^l}{[l]_{p,q}!}\left(q^l[\varsigma]_{p,q}+p^{\varsigma}[l]_{p,q}\right) \\ &= q^{1-\varsigma}px+\frac{\beta x^2p^{\varsigma+1}}{[\varsigma]_{p,q}E_{p,q}(\beta x)}\sum_{l=0}^{\infty}q^{-\varsigma-l}\frac{q^{l(l+1)/2}\,(\beta x)^l}{[l]_{p,q}!} \\ &= q^{1-\varsigma}px+\frac{q^{-\varsigma}\beta x^2p^{\varsigma+1}}{[\varsigma]_{p,q}}\end{aligned}$$

and

$$\begin{aligned}(P_{\varsigma}^{\beta,p,q}e_2)(x) &= \frac{[\varsigma]_{p,q}}{E_{p,q}(\beta x)}\sum_{l=0}^{\infty}\frac{q^{l(l-1)/2}\,(\beta x)^l}{[l]_{p,q}!\Gamma_{p,q}(\varsigma+l)}\times \\ &\times\int_0^{\infty} p^{(\varsigma+l-1)(l+\varsigma-2)/2}E_{p,q}\left(-q[\varsigma]_{p,q}u\right)\left([\varsigma]_{p,q}u\right)^{l+\varsigma-1}x^2u^2q^{2(1-\varsigma-l)}p^{2(\varsigma+l)}d_{p,q}u \\ &= \frac{x^2[\varsigma]_{p,q}}{[\varsigma]_{p,q}^2E_{p,q}(\beta x)}\sum_{l=0}^{\infty}q^{2(1-\varsigma-l)}p\frac{q^{l(l-1)/2}\,(\beta x)^l}{[l]_{p,q}!\Gamma_{p,q}(\varsigma+l)} \\ &\quad\int_0^{\infty} p^{(l+\varsigma+1)(\varsigma+l)/2}E_{p,q}\left(-q[\varsigma]_{p,q}u\right)\left([\varsigma]_{p,q}u\right)^{l+\varsigma+1}d_{p,q}u\end{aligned}$$

$$
\begin{aligned}
&= \frac{x^2}{[\varsigma]_{p,q}^2 E_{p,q}(\beta x)} \sum_{l=0}^{\infty} q^{2(1-\varsigma-l)} p \frac{q^{l(l-1)/2}(\beta x)^l}{[l]_{p,q}!\Gamma_{p,q}(\varsigma+l)} \Gamma_{p,q}(l+\varsigma+2) \\
&= \frac{x^2}{[\varsigma]_{p,q}^2 E_{p,q}(\beta x)} \sum_{l=0}^{\infty} q^{2(1-\varsigma-l)} p \frac{q^{l(l-1)/2}(\beta x)^l}{[l]_{p,q}!} [l+\varsigma+1]_{p,q}[\varsigma+l]_{p,q} \\
&= \frac{x^2}{[\varsigma]_{p,q}^2 E_{p,q}(\beta x)} \sum_{l=0}^{\infty} q^{2(1-\varsigma-l)} p \frac{q^{l(l-1)/2}(\beta x)^l}{[l]_{p,q}!} \\
&\quad \left[q^{2l}[\varsigma+1]_{p,q}[\varsigma]_{p,q} + p^{\varsigma} q^l [l]_{p,q} \left(p[\varsigma]_{p,q} + [\varsigma+1]_{p,q} + \frac{p^{\varsigma+1}}{q} \right) \right. \\
&\quad \left. + p^{2\varsigma+2}[l]_{p,q}[l-1]_{p,q} \right] \\
&= \frac{[\varsigma+1]_{p,q}[\varsigma]_{p,q} x^2}{[\varsigma]_{p,q}^2} q^{2(1-\varsigma)} p + \frac{\beta x^3 p^{\varsigma+1} q^{-2\varsigma}}{[\varsigma]_{p,q}^2} \left(pq[\varsigma]_{p,q} + q[\varsigma+1]_{p,q} + p^{\varsigma+1} \right) \\
&\quad + \frac{\beta^2 x^4 q^{-2\varsigma-1} p^{2\varsigma+3}}{[\varsigma]_{p,q}^2} \\
&= q^{(3-2\varsigma)} p x^2 + \frac{1}{[\varsigma]_{p,q}} \left(q^{2(1-\varsigma)} x^2 p^{\varsigma+1} + \beta x^3 p^{\varsigma+1} q^{-2\varsigma+1}(p+q) \right) \\
&\quad + \frac{1}{[\varsigma]_{p,q}^2} \left(\beta x^3 p^{\varsigma+1} q^{-2\varsigma}(p+q) + \beta^2 x^4 q^{-2\varsigma-1} p^{2\varsigma+3} \right).
\end{aligned}
$$

□

Lemma 2. *If the m-th order central moment is denoted by* $\mu_{\varsigma,m}^{\beta,p,q}(x) = (P_{\varsigma}^{\beta,p,q}(e_1 - xe_0)^m)(x)$, *then*

$$
\begin{aligned}
\mu_{\varsigma,0}^{\beta,p,q}(x) &= 1 \\
\mu_{\varsigma,1}^{\beta,p,q}(x) &= (q^{1-\varsigma} p - 1)x + \frac{q^{-\varsigma}\beta x^2 p^{\varsigma+1}}{[\varsigma]_{p,q}} \\
\mu_{\varsigma,2}^{\beta,p,q}(x) &= x^2 \left(q^{3-2\varsigma} p - 2q^{1-\varsigma} + 3 \right) + \frac{x^2 p^{\varsigma+1} q^{-\varsigma}}{[\varsigma]_{p,q}} \left(q^{2-\varsigma} + \beta x q^{-\varsigma+1}(p+q) - 2x\beta \right) \\
&\quad + \frac{\beta x^3 p^{2\varsigma+1} q^{-2\varsigma}}{[\varsigma]_{p,q}^2} \left(q + p + \beta x q^{-1} p^2 \right).
\end{aligned}
$$

3 Direct estimates

Let's use the symbol $H_{x^4}[0, \infty)$ to represent the collection of all functions f defined on the positive real axis that meet the statement $|f(x)| \leq K_{g\eta}\left(1 + x^4\right)$, where $K_{g\eta}$ is an absolute constant dependent on f. The subspace of continuous functions that are a part of $H_{x^4}[0, \infty)$ is referred to as $C_{x^4}[0, \infty)$. Additionally, let $C_{x^4}^*[0, \infty)$ be the subspace of all functions $f \in C_{x^4}[0, \infty)$, for which $\lim_{|x| \to \infty} \frac{f(x)}{1+x^4}$

has finite value. The norm of the class $C^*_{x^4}\left[0,\ \infty\right)$ is

$$\|f\|_{x^4} = \sup_{x\in[0,\ \infty)} \frac{|f(x)|}{1+x^4}.$$

The weighted approximation theorem, which holds that the the approximation formula is accurate along the positive real axis, is covered below (refer [5]).

Theorem 1. *Assume $p = p_\varsigma$ and $q = q_\varsigma$ satisfy the conditions $0 < q_\varsigma < p_\varsigma \leq 1$ and for sufficiently large ς, $q_\varsigma^\varsigma \to a$, $p_\varsigma^\varsigma \to b$ and $p_\varsigma \to 1$, $q_\varsigma \to 1$. For each $f \in C^*_{x^4}\left[0,\ \infty\right)$, we have*

$$\lim_{\varsigma\to\infty} \left\|(P_\varsigma^{\beta,p_\varsigma,q_\varsigma} f) - f\right\|_{x^4} = 0.$$

Proof. It is enough to confirm the following three requirements using Korovkin's theorem:

$$\lim_{\varsigma\to\infty} \left\|(P_\varsigma^{\beta,p_\varsigma,q_\varsigma} e_\nu) - e_\nu\right\|_{x^4} = 0, \nu = 0, 1, 2. \tag{5}$$

For $\nu = 0$, the first condition of the above equality is satisfied since $(P_\varsigma^{\beta,p,q} e_0)(x) = 1$.

Now, for $\varsigma \in \mathbf{N}$, we have

$$\begin{aligned} \left\|(P_\varsigma^{\beta,p_\varsigma,q_\varsigma} e_1) - e_1\right\|_{x^4} &\leq (q_\varsigma^{1-\varsigma} p_\varsigma - 1) \sup_{x\in[0,\ \infty)} \frac{x}{1+x^4} \\ &\quad + \frac{q_\varsigma^{-\varsigma} \beta p_\varsigma^{\varsigma+1}}{[\varsigma]_{p_\varsigma,q_\varsigma}} \sup_{x\in[0,\ \infty)} \frac{x^2}{1+x^4} \end{aligned}$$

and

$$\begin{aligned} \left\|(P_\varsigma^{\beta,p_\varsigma,q_\varsigma} e_2) - e_2\right\|_{x^4} &\leq \left(\frac{[\varsigma+1]_{p_\varsigma,q_\varsigma}[\varsigma]_{p_\varsigma,q_\varsigma}}{[\varsigma]^2_{p_\varsigma,q_\varsigma}} q_\varsigma^{2(1-\varsigma)} p_\varsigma - 1\right) \sup_{x\in[0,\ \infty)} \frac{x^2}{1+x^4} \\ &\quad + \frac{\beta p_\varsigma^{\varsigma+1} q_\varsigma^{-2\varsigma}}{[\varsigma]^2_{p_\varsigma,q_\varsigma}} \left(q_\varsigma[\varsigma+1]_{p_\varsigma,q_\varsigma} + p_\varsigma q_\varsigma [\varsigma]_{p_\varsigma,q_\varsigma} + p_\varsigma^{\varsigma+1}\right) \sup_{x\in[0,\ \infty)} \frac{x^3}{1+x^4} \\ &\quad + \frac{\beta^2 q_\varsigma^{-2\varsigma-1} p_\varsigma^{2\varsigma+3}}{[\varsigma]^2_{p_\varsigma,q_\varsigma}} \sup_{x\in[0,\ \infty)} \frac{x^4}{1+x^4}, \end{aligned}$$

which implies that for $v = 1, 2$

$$\lim_{\varsigma\to\infty} \left\|(P_\varsigma^{\beta,p_\varsigma,q_\varsigma} e_\nu) - e_\nu\right\|_{x^4} = 0.$$

Thus the proof is completed. □

Let a function $g \in C[0,\ \infty)$, then g is considered to meet Lipschitz condition Lip_η in I, $\eta \in (0,\ 1]$, $I \subset [0,\ \infty)$ provided

$$|g(t) - g(x)| \leq K_{g\eta} |t - x|^\eta, \quad x \in I, \quad t \in [0,\ \infty),$$

where $K_{g\eta}$ is a constant that depends on η and g.

Theorem 2. *Let $g \in Lip_\eta$ on $I \subset [0,\ \infty)$ and $\eta \in (0,\ 1]$, then*

$$\left|(P_\varsigma^{\beta,p,q} g)(x) - g(x)\right| \leq K_{g\eta}\left(\left(\mu_{\varsigma,2}^{\beta,p,q}(x)\right)^{\frac{\eta}{2}} + 2\,(d(x,I))^\eta\right)$$

where the distance between x and I is shown by the expression $d(x, I)$.

Proof. Let $\overline{I}$ represent the set $I's$ closure. Then, for $x_0 \in \overline{I}$, where x_0 is the closest point of $\overline{I}$ from x and $x \in [0,\ \infty)$, we have

$$|g(t) - g(x)| \leq |g(x_0) - g(x)| + |g(t) - g(x_0)|, \quad t \in [0,\ \infty).$$

By definition of Lipschitz class, we get

$$\begin{aligned} \left|(P_\varsigma^{\beta,p,q} g)(x) - g(x)\right| &\leq |g(x_0) - g(x)| + (P_\varsigma^{\beta,p,q} |g(t)) - g(x_0)|)(x) \\ &\leq K_{g\eta} |x_0 - x|^\eta + K_{g\eta}(P_\varsigma^{\beta,p,q} |t - x_0|^\eta)(x). \end{aligned} \tag{6}$$

As $(P_\varsigma^{\beta,p,q})$ is monotone

$$(P_\varsigma^{\beta,p,q} |t - x_0|^\eta)(x) \leq |x_0 - x|^\eta + \left((P_\varsigma^{\beta,p,q} |t - x|^2)(x)\right)^{\frac{\eta}{2}}.$$

Then, using Hölder's inequality with $p := \frac{2}{\eta}$ and $\frac{1}{r} := 1 - \frac{1}{p}$, we are led to

$$(P_\varsigma^{\beta,p,q} |t - x|^\eta)(x) \leq \left((P_\varsigma^{\beta,p,q} e_0)(x)\right)^{1-\frac{\eta}{2}} \left((P_\varsigma^{\beta,p,q} |t - x|^2)(x)\right)^{\frac{\eta}{2}}. \tag{7}$$

Using (6), (7) and Lemma 2, we immediately get the desired result. □

Theorem 3. *Let $f \in C_B[0, \infty)$, the class of continuous and bounded functions defined on $R^+ \cup \{0\}$ and $0 < q < p \leq 1$. Then for all natural numbers ς and $x \in R^+ \cup \{0\}$, the absolute constant $C > 0$ exists in such a way that*

$$|\left(P_\varsigma^{\beta,p,q} f\right)(x) - f(x)| \leq \omega\left(f, |\mu_{\varsigma,1}^{\beta,p,q}(x)|\right) + C\omega_2\left(f, \sqrt{\mu_{\varsigma,2}^{\beta,p,q}(x) + \left(\mu_{\varsigma,1}^{\beta,p,q}(x)\right)^2}\right).$$

Proof. Let $h \in W_\infty^2 = \{h \in C_B[0, \infty) : h', h'' \in C_B[0, \infty)\}$ Using Taylor's formula, we get

$$h(t) = h(x) + h'(x)(t - x) + \int_x^t (t - u)h''(u)du, \tag{8}$$

where $x,t\in[0,\infty)$. Consider the following operator:

$$\left(\check{P}_{\varsigma}^{\beta,p,q}f\right)(x)=\left(P_{\varsigma}^{\beta,p,q}f\right)(x)-f\left(q^{1-\varsigma}px+\frac{q^{-\varsigma}\beta x^2p^{\varsigma+1}}{[\varsigma]_{p,q}}\right)+f(x). \qquad (9)$$

Applying the operators $\check{P}_{\varsigma}^{\beta,p,q}$ on (8), we get

$$\begin{aligned}\left(\check{P}_{\varsigma}^{\beta,p,q}h\right)(x)-h(x) &= h'(x)\left(\check{P}_{\varsigma}^{\beta,p,q}(t-x)\right)(x)+\\ &+\left(\check{P}_{\varsigma}^{\beta,p,q}\left(\int_x^t(t-u)h''(u)du\right)\right)(x).\end{aligned}$$

Therefore

$$\begin{aligned}\left|\left(\check{P}_{\varsigma}^{\beta,p,q}h\right)(x)-h(x)\right| &\leq \left(P_{\varsigma}^{\beta,p,q}\left|\int_x^t|t-u||h''(u)|du\right|\right)(x)+\\ &+\left|\int_x^{q^{1-\varsigma}px+\frac{q^{-\varsigma}\beta x^2p^{\varsigma+1}}{[\varsigma]_{p,q}}}\left(q^{1-\varsigma}px+\frac{q^{-\varsigma}\beta x^2p^{\varsigma+1}}{[\varsigma]_{p,q}}-u\right)h''(u)du\right|\\ &\leq\left\{\mu_{\beta,\varsigma,2}^{p,q}(x)+\left((q^{1-\varsigma}p-1)x+\frac{q^{-\varsigma}\beta x^2p^{\varsigma+1}}{[\varsigma]_{p,q}}\right)^2\right\}||h''||,\end{aligned}$$

where norm-$||.||$ is the supremum norm. Therefore

$$\begin{aligned}&|\left(P_{\varsigma}^{\beta,p,q}f\right)(x)-f(x)|\leq|\left(\check{P}_{\varsigma}^{\beta,p,q}(f-h)\right)(x)-(f-h)(x)|+\\ &\quad+|\left(\check{P}_{\varsigma}^{\beta,p,q}h\right)(x)-h(x)|+\left|f\left(q^{1-\varsigma}px+\frac{q^{-\varsigma}\beta x^2p^{\varsigma+1}}{[\varsigma]_{p,q}}\right)-f(x)\right|\\ &\leq 2||f-h||+\left\{\mu_{\varsigma,2}^{\beta,p,q}(x)+\left((q^{1-\varsigma}p-1)x+\frac{q^{-\varsigma}\beta x^2p^{\varsigma+1}}{[\varsigma]_{p,q}}\right)^2\right\}||h''||+\\ &+\omega\left(f,\left|(q^{1-\varsigma}p-1)x+\frac{q^{-\varsigma}\beta x^2p^{\varsigma+1}}{[\varsigma]_{p,q}}\right|\right).\end{aligned}$$

Finally taking the infimum over all $h\in W_\infty^2$ and using the Peetre's K-functional:

$$K_2(f,\delta)=\inf[\{||f-h||+\delta||h"||:h\in W_\infty^2\}$$

Then apply the inequality $K_2(f,\delta)\leq C\omega_2(f,\delta^{1/2}),\delta>0$ (see [4, pp. 177]), the required result follows.

□

Acknowledgements

The authors are extremely thankful to the reviewers for their valuable suggestions, leading to overall improvements in the manuscript.

References

[1] Abel, U., Gupta, V. and Sisodia, M., *Some new semi-exponential type operators*, Rev. R. Acad. Cienc. Exactas F ́ıs. Nat. Ser. A Math. RACSAM **116**, 87 (2022).

[2] Acar, T., (p, q)-*Generalization of Szász–Mirakyan operators*, Math. Methods Appl. Sci. **39** (2016), no. 10, 2685-2695.

[3] Aral, A. and Gupta, V., *Applications of* (p, q)-*Gamma function to Szász Durrmeyer operators*, Publ. Inst. Math. (Beograd) **102** (116) (2017), 211-220.

[4] DeVore, R. A. and Lorentz, G. G., *Constructive Approximation*, Springer, Berlin, 1993.

[5] Gadzhiev, A. D., *Theorems of the type of P. P. Korovkin type theorems*, Math. Zametki **20** (1976), no. 5, 781–786.

[6] Gupta, V., (p, q)-*Baskakov–Kantorovich operators*, Appl. Math. Inform. Sci. **10** (2016), no. 4, 1551–1556.

[7] Gupta, V., (p, q)-*Szász–Mirakyan–Baskakov operators*, Complex Anal. Oper. Theory **12** (2018), 17–25 .

[8] Gupta, V. and Aral, A., *Bernstein Durrmeyer operators based on two parameters*, Facta Univ Ser. Math. Infor. **31** (2016), no.1, 79–95.

[9] Gupta, V., Rassias, T. M., Agrawal, P. N. and Acu, A. M., *Basics of Post-quantum Calculus*, In: *Recent advances in constructive approximation theory*, Springer, Optimization and Its Applications, **138**, Springer, Berlin, Germany, 73-89.

[10] Herzog, M., *Semi-exponential operators*, Symmetry, **13**, 637 (2021).

[11] Ismail, M. and May, C. P., *On a family of approximation operators*, J. Math. Anal. Appl. **63** (1978), no. 1, 446–462.

[12] Milovanović, G. V., Gupta, V. and Malik, N., (p, q)-*Beta functions and applications in approximation*, Boletín de la Sociedad Matemática Mexicana **24** (2018), no. 1, 219–237.

[13] Păltănea, R., *Modified Szász-Mirakjan operators of integral form*, Carpathian J. Math. **24** (2008), no. 3, 378-385.

[14] Păltănea, R., *Estimates of approximation in terms of a weighted modulus of continuity*, Bull. Transilvania Univ. Braşov, **54** (2011), no. 4, 67–74.

[15] Păltănea, R., *Simultaneous approximation by a class of Szász Mirakjan operators*, J. Appl. Funct. Anal. **9** (2014), no. 3-4, 356-368.

[16] Tyliba, A. and Wachnicki, E., *On some class of exponential type operators*, Comment. Math. **45** (2005), 59-73.