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BERNSTEIN POLYNOMIALS AND DUAL FUNCTIONALS

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Dedicated to Professor Radu Păltănea on the occasion of his 70th anniversary

Abstract

The divided differences of Bernstein polynomials were investigated by Alexandru Lupaş in 1995. We extend the results of that investigation. Moreover, we establish new relations between them and the theory of dual functionals.

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1 Introduction

Among other interesting results, A. Lupaş presented in [6] a method for computing the divided differences of the Bernstein polynomials B_nf in terms of divided differences of the function f. In this context he introduced an array of numbers α_{nsj}. This paper is devoted to extending these results. In Section 2 we present the pertinent definition and compute explicitly some values α_{nsj}. Section 3 is devoted to the operators F_n introduced in [2] and later investigated in [3], [4], [9]. We give a new expression of F_nf and then present new properties of α_{nsj}, in particular a method for computing them recursively. In Section 4 we recall some definitions and properties of the dual functionals associated with Bernstein operators. Then we establish new relations between them and the divided differences, using the numbers α_{nsj}. These relations are used in order to describe new methods for computing divided differences of B_nf.

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2 Bernstein polynomials and divided diferences

The Bernstein operators are given by

$$B_n : C[0,1] \to C[0,1], \quad B_n(f;x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right) b_{n,k}(x), \tag{2.1}$$

where

$$b_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k},\ x \in [0,1].$$

The divided difference of a function f at the distinct points $x_0, x_1, \dots, x_m$ is denoted by $[x_0, x_1, \dots, x_m; f]$.

From the theory of divided differences it is well known that

$$[x_0, x_1, \dots, x_m; e_m] = 1, \tag{2.2}$$

$$[x_0, x_1, \dots, x_{m-1}; e_m] = x_0 + \dots + x_{m-1}, \tag{2.3}$$

$$[x_0, x_1, \dots, x_m; f] = \frac{[x_1, \dots, x_m; f] - [x_0, \dots, x_{m-1}; f]}{x_m - x_0}, \tag{2.4}$$

where $e_k(x) := x^k,\ k = 0, 1, \dots$

Let n, s, j be integers, $1 \le s \le n$, $0 \le j \le n - s$. Define

$$\Psi_{nsj}(x) := s\binom{n}{s} \int_0^x (x-y)^{s-1} b_{n-s,j}(y)dy,$$

$$\alpha_{nsj} := \left[0, -\frac{1}{n}, \dots, -\frac{s}{n}; \Psi_{nsj}\right] = \frac{n^s}{s!} \sum_{k=0}^{s} (-1)^k \binom{s}{k} \Psi_{nsj}\left(-\frac{k}{n}\right).$$

Moreover, set

$$\alpha_{n0j} := 0,\ j = 1, \dots, n;\ \alpha_{n00} = 1.$$

Example 2.1. *By a direct calculation we find*

1) $\Psi_{nn0}(x) = x^n,\ \ \alpha_{nn0} = 1,$

2) $\Psi_{n,n-1,0}(x) = nx^{n-1} - x^n,\ \alpha_{n,n-1,0} = \frac{3n-1}{2},$

3) $\Psi_{n,n-1,1}(x) = x^n,\ \ \alpha_{n,n-1,1} = -\frac{n-1}{2}.$

In Tables 1 and 2 we present the explicit expressions of the functions Ψ_{nsj}, respectively the numbers α_{nsj}, for $n = 5$.

Table 1. *Functions* Ψ_{nsj}, $n = 5$

$j \setminus s$	1	2	3	4	5
0	$x^5-5x^4+10x^3-10x^2+5x$	$-x^5+5x^4-10x^3+10x^2$	$x^5-5x^4+10x^3$	$-x^5+5x^4$	x^5
1	$-4x^5+15x^4-20x^3+10x^2$	$3x^5-10x^4+10x^3$	$-2x^5+5x^4$	x^5	
2	$6x^5-15x^4+10x^3$	$-3x^5+5x^4$	x^5		
3	$-4x^5+5x^4$	x^5			
4	x^5				

Table 2. *Values of* α_{nsj}, $n = 5$

$j \setminus s$	1	2	3	4	5
0	4651/625	438/25	17	7	1
1	−1829/625	−229/25	−8	−2	
2	331/625	44/25	1		
3	−29/625	−3/25			
4	1/625				

The following result was obtained by Lupaş in [6, pp.206-207].

Theorem 2.1. *With the above notation,*

$$\frac{1}{s!}\frac{d^s}{dx^s}\Psi_{nsj}(x) = \binom{n}{s} b_{n-s,j}(x),\ 1 \le s \le n,\ 0 \le j \le n-s, \tag{2.5}$$

$$\left[0, -\frac{1}{n}, \ldots, -\frac{s}{n}; B_n f\right] = \frac{s!}{n^s}\sum_{j=0}^{n-s}\alpha_{nsj}\left[\frac{j}{n}, \frac{j+1}{n}, \ldots, \frac{j+s}{n}; f\right],\ 0 \le s \le n, \tag{2.6}$$

$$\frac{s!}{n^s}\sum_{j=0}^{n-s}\alpha_{nsj} = \left[0, -\frac{1}{n}, \cdots, -\frac{s}{n}; B_n e_s\right] = \frac{s!}{n^s}\binom{n}{s},\ 0 \le s \le n. \tag{2.7}$$

Remark 2.1. *We will need (2.6) in the next section. A more general result, expressing* $[x_0, x_1, \ldots, x_s; B_n f]$ *for arbitrary knots, can be found in [6, p.207].*

3 The operator F_n

The Beta operators introduced by Mühlbach [7, 8] and Lupaş [5] are defined as

$$\overline{\mathbb{B}}_n(f;x) = \begin{cases} f(0), & x = 0, \\ \dfrac{1}{B(nx, n(1-x))}\displaystyle\int_0^1 t^{nx-1}(1-t)^{n(1-x)-1}f(t)dt, & x \in (0,1), \\ f(1), & x = 1, \end{cases} \tag{3.1}$$

where $B(\cdot,\cdot)$ is the Beta function, $f \in C[0,1]$ and $x \in [0,1]$.

Let Π_n be the linear space of all real polynomials of degree $\le n$. Using the inverse of Beta operator, by composing the operators

$$B_n : C[0,1] \to \Pi_n \quad \text{and} \quad \overline{\mathbb{B}}_n^{-1} : \Pi_n \to \Pi_n,$$

one obtains the operators $F_n := \mathbb{B}_n^{-1} \circ B_n$, $F_n : C[0,1] \to \Pi_n$, $n \ge 1$. They were introduced in [2] and later investigated in [3], [4], [9].

Theorem 3.1. *For $f \in C[0,1]$ we have*

$$F_n f = \sum_{s=0}^{n} \frac{(n)_s}{n^s} \frac{s!}{n^s} \sum_{j=0}^{n-s} \alpha_{nsj} \left[\frac{j}{n}, \frac{j+1}{n}, \dots, \frac{j+s}{n}; f \right] e_s, \tag{3.2}$$

where $(n)_s := n(n+1)\dots(n+s-1)$, $(n)_0 := 1$.

Proof. It was proved in [9] that

$$F_n f = \sum_{j=0}^{n} \frac{(n)_s}{n^s} \left[0, -\frac{1}{n}, \dots, -\frac{s}{n}; B_n f \right] e_s. \tag{3.3}$$

Now (3.2) is a consequence of (3.3) and (2.6). □

The next result presents some properties of the numbers α_{nsj}.

Corollary 3.1. *a) For $0 \le s \le n$ one has*

$$\sum_{j=0}^{n-s} \alpha_{nsj} = \binom{n}{s}. \tag{3.4}$$

b) Let $1 \le s \le n$, $0 \le j \le n-s$. Then, $\alpha_{nsj} > 0$ for j even, and $\alpha_{nsj} < 0$ for j odd.

Proof. a) For $s = 0$, (3.4) can be verified directly. For $1 \le s \le n$, it follows from (2.7).

b) According to the mean value theorem for divided differences, there exists $\xi_{ns} \in \left(-\frac{s}{n}, 0\right)$ such that

$$\alpha_{nsj} = \left[0, -\frac{1}{n}, \dots, -\frac{s}{n}; \Psi_{nsj} \right] = \frac{1}{s!} \frac{d^s}{dx^s} \Psi_{nsj}(\xi_{ns}).$$

Now (2.5) shows that

$$\alpha_{nsj} = \binom{n}{s} b_{n-s,j}(\psi_{ns}) = \binom{n}{s} \binom{n-s}{j} \xi_{ns}^j (1-\xi_{ns})^{n-s-j},$$

and this entails the conclusion b). □

Let $S(j,s)$ be the Stirling numbers of second kind. Denote

$$c_{nsj} := \frac{(n+s-1)!(-1)^j S(s+j,s)}{(n-s-j)!n^{2j-1}s!(n)_s},\ 0 \le s \le n,\ 0 \le j \le n-s.$$

Theorem 3.2. *For $f \in C[0,1]$ we have*

$$\sum_{j=0}^{n-s} \left[\frac{j}{n}, \frac{j+1}{n}, \dots, \frac{j+s}{n}; f\right] \alpha_{nsj} = \sum_{j=0}^{n-s} \left[0, \frac{1}{n}, \dots, \frac{j+s}{n}; f\right] c_{nsj}. \tag{3.5}$$

Proof. It was proved in [2, (19)] that

$$F_n f = \sum_{j=0}^{n} \left[0, \frac{1}{n}, \dots, \frac{j}{n}; f\right] \frac{1}{(n-j)!n^{2j-1}} \sum_{s=0}^{j} (-1)^{j-s}(n+s-1)!S(j,s)e_s.$$

This can be written as

$$F_n f = \sum_{s=0}^{n} \left(\sum_{j=s}^{n} \frac{(n+s-1)!}{(n-j)!n^{2j-1}} (-1)^{j-s} S(j,s) \left[0, \frac{1}{n}, \dots, \frac{j}{n}; f\right] \right) e_s,$$

and also

$$F_n f = \sum_{s=0}^{n} \left(\sum_{i=0}^{n-s} \frac{(n+s-1)!(-1)^i S(s+i,s)}{(n-s-i)!n^{2s+2i-1}} \left[0, \frac{1}{n}, \dots, \frac{i+s}{n}; f\right] \right) e_s. \tag{3.6}$$

From (3.2) and (3.6) we get

$$\frac{s!(n)_s}{n^{2s}} \sum_{j=0}^{n-s} \alpha_{nsj} \left[\frac{j}{n}, \frac{j+1}{n}, \dots, \frac{j+s}{n}; f\right]$$

$$= \sum_{j=0}^{n-s} \frac{(n+s-1)!(-1)^j S(s+j,s)}{(n-s-j)!n^{2s+2j-1}} \left[0, \frac{1}{n}, \dots, \frac{j+s}{n}; f\right],$$

and this implies (3.5). □

Remark 3.1. *The relation (3.5) can be used in order to compute successively the numbers $\alpha_{ns0}, \alpha_{ns1}, \dots, \alpha_{ns,n-s}$. Indeed, let $i \in \{0,1,\dots,n\}$ and $a_{ni} \in C[0,1]$ piecewise linear, $a_{ni}\left(\frac{i}{n}\right) = 1$, $a_{ni}\left(\frac{k}{n}\right) = 0$, $k \ne i$. Taking in (3.5) $f = a_{n0}$ we compute a_{ns0}; then $f = a_{n1}$ leads to α_{ns1}, and so on.*

4 Dual functionals and divided differences

In this section we use results from [1]. Let $\lambda_k^{(n)}$ be the eigenvalues of B_n, $p_k^{(n)}$ the associated monic eigenpolynomials and $\mu_k^{(n)}$ the corresponding dual functionals, $k = 0, 1, \dots, n$. It is known that

$$\lambda_k^{(n)} = \frac{n!}{(n-k)!n^k}, \tag{4.1}$$

$$p_k^{(n)}(x) = x^k - \frac{k}{2}x^{k-1} + \text{terms of lower degree}, \tag{4.2}$$

$$B_n f = \sum_{k=0}^{n} \lambda_k^{(n)} p_k^{(n)} \mu_k^{(n)}(f), \ \ f \in C[0,1]. \tag{4.3}$$

From(2.6) and (4.3) we get the following result.

Theorem 4.1. *For $s = 0, 1, \dots, n$ one has*

$$\sum_{k=s}^{n} \lambda_k^{(n)} \left[0, -\frac{1}{n}, \dots, -\frac{s}{n}; p_k^{(n)}\right] \mu_k^{(n)} = \sum_{j=0}^{n-s} \frac{s!}{n^s} \alpha_{nsj} \left[\frac{j}{n}, \frac{j+1}{n}, \dots, \frac{j+s}{n}; \cdot\right]. \tag{4.4}$$

Example 4.1. *Let $s = n$. In this case (4.4) entails*

$$\lambda_n^{(n)} \left[0, -\frac{1}{n}, \dots, -\frac{n}{n}; p_n^{(n)}\right] \mu_n^{(n)} = \frac{n!}{n^n} \alpha_{nn0} \left[0, \frac{1}{n}, \dots, \frac{n}{n}; \cdot\right].$$

Since $\lambda_n^{(n)} = \dfrac{n!}{n^n}$ (see (4.1)) and $\alpha_{nn0} = 1$ (see Example 2.1), we obtain

$$\mu_n^{(n)} = \left[0, \frac{1}{n}, \dots, \frac{n}{n}; \cdot\right]. \tag{4.5}$$

So, we recover a result from [1, p.164].

Example 4.2. *Let $s = n-1$. Now (4.4) yields*

$$\lambda_{n-1}^{(n)} \left[0, -\frac{1}{n}, \dots, -\frac{n-1}{n}; p_{n-1}^{(n)}\right] \mu_{n-1}^{(n)} + \lambda_n^{(n)} \left[0, -\frac{1}{n}, \dots, -\frac{n-1}{n}; p_n^{(n)}\right] \mu_n^{(n)}$$
$$= \frac{(n-1)!}{n^{n-1}} \left\{ \alpha_{n,n-1,0} \left[0, \frac{1}{n}, \dots, \frac{n-1}{n}; \cdot\right] + \alpha_{n,n-1,1} \left[\frac{1}{n}, \frac{2}{n} \dots, \frac{n}{n}; \cdot\right] \right\}.$$

Using (4.1), (4.2), (4.5), (2.2) and (2.3), we get

$$n\mu_{n-1}^{(n)} = \frac{2n-1}{2} \left[0, \frac{1}{n}, \dots, \frac{n}{n}; \cdot\right]$$
$$+ \alpha_{n,n-1,0} \left[0, \frac{1}{n}, \dots, \frac{n-1}{n}; \cdot\right] + \alpha_{n,n-1,1} \left[\frac{1}{n}, \frac{2}{n} \dots, \frac{n}{n}; \cdot\right].$$

Combined with (2.4) and Example 2.1, this implies after some calculation

$$\mu_{n-1}^{(n)} = \frac{1}{2}\left[0, \frac{1}{n}, \dots, \frac{n-1}{n}; \cdot\right] + \frac{1}{2}\left[\frac{1}{n}, \frac{2}{n} \dots, \frac{n}{n}; \cdot\right].$$

So, we recover another result from [1, p.164]. With the above arguments one can determine the other dual functionals.

Theorem 4.2. *For $f \in C[0,1]$ we have*

$$[x_0, x_1, \dots, x_n; B_n f] = \frac{n!}{n^n}\left[0, \frac{1}{n}, \dots, \frac{n}{n}; f\right], \tag{4.6}$$

$$\begin{aligned}[x_0, x_1, \dots, x_{n-1}; B_n f] = \frac{(n-1)!}{n^{n-1}} \Bigg\{ &\left(n - \sum_{i=0}^{n-1} x_i\right)\left[0, \frac{1}{n}, \dots, \frac{n-1}{n}; f\right] \\ &+ \left(\sum_{i=0}^{n-1} x_i\right)\left[\frac{1}{n}, \frac{2}{n} \dots, \frac{n}{n}; f\right]\Bigg\}.\end{aligned} \tag{4.7}$$

Proof. Using (4.3) we see that

$$[x_0, x_1, \dots, x_s; B_n f] = \sum_{k=s}^{n} \lambda_k^{(n)} [x_0, x_1, \dots, x_s; p_k^{(n)}] \mu_k^{(n)}(f) \tag{4.8}$$

Therefore, (4.8), (4.1), (4.2), (4.5) and (2.2) show that

$$\begin{aligned}[x_0, x_1, \dots, x_n; B_n f] &= \lambda_n^{(n)} [x_0, x_1, \dots, x_n; p_n^{(n)}] \mu_n^{(n)}(f) \\ &= \frac{n!}{n^n}\left[0, \frac{1}{n}, \dots, \frac{n}{n}; f\right],\end{aligned}$$

and (4.6) is proved. Now

$$\begin{aligned}&[x_0, x_1, \dots, x_{n-1}; B_n f] \\ &= \lambda_{n-1}^{(n)} [x_0, x_1, \dots, x_{n-1}; p_{n-1}^{(n)}] \mu_{n-1}^{(n)}(f) + \lambda_n^{(n)} [x_0, x_1, \dots, x_{n-1}; p_n^{(n)}] \mu_n^{(n)}(f) \\ &= \frac{n!}{n^{n-1}} [x_0, x_1, \dots, x_{n-1}; e_{n-1}] \mu_{n-1}^{(n)}(f) + \frac{n!}{n^n}\left[x_0, x_1, \dots, x_{n-1}; e_n - \frac{n}{2} e_{n-1}\right] \mu_n^{(n)}(f) \\ &= \frac{n!}{n^{n-1}} \mu_{n-1}^{(n)}(f) + \frac{n!}{n^n}\left(\sum_{i=0}^{n-1} x_i - \frac{n}{2}\right) \mu_n^{(n)}(f) \\ &= \frac{n!}{n^{n-1}} \Bigg\{ \frac{1}{2}\left[0, \frac{1}{n}, \dots, \frac{n-1}{n}; f\right] + \frac{1}{2}\left[\frac{1}{n}, \dots, \frac{n}{n}; f\right] \\ &+ \left(\frac{1}{n}\sum_{i=0}^{n-1} x_i - \frac{1}{2}\right)\left[0, \frac{1}{n}, \dots, \frac{n}{n}; f\right]\Bigg\} \\ &= \frac{(n-1)!}{n^{n-1}} \left\{ \left(n - \sum_{i=0}^{n-1} x_i\right)\left[0, \frac{1}{n}, \dots, \frac{n-1}{n}; f\right] + \left(\sum_{i=0}^{n-1} x_i\right)\left[\frac{1}{n}, \frac{2}{n}, \dots, \frac{n}{n}; f\right]\right\}.\end{aligned}$$

This concludes the proof. □

Remark 4.1. *The divided differences of $B_n f$ on other knots can be calculated similarly.*

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References

[1] Cooper, S. and Waldron, S., *The Eigenstructure of the Bernstein Operator*, J. Approx. Theory **105** (2000), no. 1, 133-165.

[2] Gonska, H., Heilmann, M., Lupaş, A. and Raşa, I., *On the composition and decomposition of positive linear operators III: A non-trivial decomposition of the Bernstein operator*, arXiv:1204.2723v1 [math.CA], (2012).

[3] Heilmann, M., Nasaireh, F. and Raşa, I., *Beta and related operators revisited*, Proceedings of Constructive Theory of Functions, Sozopol 2016 (K. Ivanov, G. Nikolov and R. Uluchev, Eds.), 175-185. Prof. Marin Drinov Academic Publishing House, Sofia, 2018, 175-185.

[4] Heilmann, M. and Raşa, I., *On the decomposition of Bernstein operators*, Numer. Funct. Anal. Optimiz. **36** (2015), 72-85.

[5] Lupaş, A., *Die Folge der Betaoperatoren, Dissertation*, Universität Stuttgart, 1972.

[6] Lupaş, A., *The approximation by means of some linear positive operators*, Approximation Theory, Proc. IDoMAT 95, Edited by M.W. Müller, M. Felten, D.H. Mache, Mathematical Research, 86, 201-229, Academic Verlag, Berlin, 1995.

[7] Mühlbach, G. *Verallgemeinerungen der Bernstein- und der Lagrangepolynome*, Bemerkungen zu einer Klasse linearer Polynomoperatoren von D.D. Stancu, Rev. Roumaine Math. Pure Appl. **15** (1970), 1235-1252.

[8] Mühlbach, G., *Rekursionsformeln für die zentralen Momente der Polya- und der Beta-Verteilung*, Metrika **19** (1972), 171-177.

[9] Nasaireh, F. and Raşa, I., *Another Look at Voronovskaja Type Formulas*, J. Math. Ineq. **12** (2018), no.1, 95-105.