Bulletin of the *Transilvania* University of Braşov
Series III: Mathematics and Computer Science, Vol. 3(65), No. 2 - 2023, 1-12
https://doi.org/10.31926/but.mif.2023.3.65.2.1

PĂLTĂNEA'S OPERATORS: OLD AND NEW RESULTS

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Dedicated to Professor Radu Păltănea on the occasion of his 70th anniversary

Abstract

This note concerns the study of an approximation linear positive process introduced by R. Păltănea in 2008. Considering the impact of this class of operators that depends on two parameters, in a distinct section we present a brief radiograph of the main properties highlighted over time in various papers. Our contribution materializes in the definition and study of the approximation properties of King variant of these operators.

2000 *Mathematics Subject Classification:* 41A36, 41A35, 41A25.
Key words: linear approximation process, Durrmeyer type operators, modulus of smoothness, King type operator.

1 Introduction

This note falls under the field of Approximation Theory, more precisely it aims at the study of linear and positive approximation processes. It is known that Szász operators, along with the many discrete and integral generalizations obtained over time, play a significant role in this domain.

Our present study concerns a class of Szász-Durrmeyer type operators introduced by Păltănea in 2008 [9]. They depend on two parameters and are described as follows. For $\alpha > 0$, $\rho > 0$ and $x \in \mathbb{R}_+ = [0, \infty)$,

$$L_\alpha^\rho(f;x) = e^{-\alpha x} f(0) + \sum_{k=1}^{\infty} s_{\alpha,k}(x) \int_0^\infty \Theta_{\alpha,k}^\rho(t) f(t) dt, \qquad (1)$$

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where

$$s_{\alpha,k}(x) = e^{-\alpha x}\,\frac{(\alpha x)^k}{k!},$$
$$\Theta^{\rho}_{\alpha,k}(t) = \frac{\alpha\rho}{\Gamma(k\rho)}\,e^{-\alpha\rho t}(\alpha\rho t)^{k\rho-1} \tag{2}$$

and $f : \mathbb{R}_+ \to \mathbb{R}$ is a locally integrable function for which formula (1) is well defined for all $x \geq 0$.

Păltănea operators preserve affine functions and make a link between the Phillips operators and the classical Szász operators. In the particular case $\rho = 1$ and $\alpha = n \in \mathbb{N}$, the above operators become Phillips operators [11]

$$L_n(f;x) = e^{-nx}f(0) + n\sum_{k=1}^{\infty} s_{n,k}(x)\int_0^{\infty} s_{n,k-1}(t)f(t)dt,\ \ x \geq 0.$$

In the limit case $\rho \to \infty$ (see Theorem 2) one obtains the classical Szász operators defined by

$$S_n(f;x) = \sum_{k=0}^{\infty} s_{n,k}(x)f\left(\frac{k}{n}\right),\ \ x \geq 0. \tag{3}$$

The extension in the Durrmeyer sense of S_n operators was achieved by Mazhar and Totik [7] in 1985. Unlike Păltănea operators, this extension does not reproduce affine functions.

Our goal is twofold: to collect some known properties of Păltănea's operators in a brief synopsis as well as to construct a version of King type by investigating its utility and its approximation properties. Wanting to create a self-contained presentation, all the notions used are described explicitly. As much as possible, we have kept the original notations used in the papers cited.

2 An eclectic collection of known results

We did not propose an exhaustive presentation of the results obtained over time, but scoring of the most significant properties of this approximation process. Păltănea studied this class of operators in the papers [9], [10], the following main properties being proved.

Set W, the space of functions $f : \mathbb{R}_+ \to \mathbb{R}$ which are Riemann integrable on each compact interval of $\mathbb{R}_+$ and for which exist certain numbers $M > 0$, $q > 0$ such that $|f(t)| \leq Me^{qt}$, $t \geq 0$. For $\alpha > 0$ and $\rho > 0$, denote by W^{ρ}_{α} the subspace of W of those functions f which satisfy the above inequality with $q < \alpha\rho$. One has

$$W = \bigcup_{\alpha>0,\,\rho>0} W^{\rho}_{\alpha}.$$

In [9, *Theorem 2.1*] it was showed that $L^{\rho}_{\alpha}f$ exists for any $f \in W^{\rho}_{\alpha}$, $\alpha > 0$, $\rho > 0$.

An approximation property in the particular case $\rho = n \in \mathbb{N}$ will be read as follows.

Theorem 1. ([9, *Theorem 3.4*]). *For any function $f \in W \cap C(\mathbb{R}_+)$ and any number $n \in \mathbb{N}$, there exists $\alpha_0 > 0$ such that $L_\alpha^n f$ exists for $\alpha > \alpha_0$ and for any compact set $K \subset \mathbb{R}_+$ we have*

$$\lim_{\alpha\to\infty} L_\alpha^n f = f, \ \textit{uniformly on } K.$$

The limit of the functions $L_\alpha^\rho f$ has been established when ρ tends to infinity.

Theorem 2. ([10, *Theorem 4*]). *For any $\alpha > 0$, any $f \in W$ and any $b > 0$, there is $\rho_0 > 0$ such that $L_\alpha^\rho f$ exists for all $\rho \geq \rho_0$ and we have*

$$\lim_{\rho\to\infty} L_\alpha^\rho(f;x) = S_\alpha(f;x), \ \textit{uniformly for } x \in [0,b],$$

where the Szász operator $S_\alpha f$ is defined in (3).

Also it was proved that L_α^ρ preserves convexity of higher order and it has the property of simultaneous approximation on the compact sets [10, *Theorems 6, 9*].

Further, we will highlight some papers based on Păltănea operators and which bear his name in the title.

We recall that in [5] the authors gave a generalization of Szász operators based on Appell polynomials. Let $g(z) = \sum_{k=0}^{\infty} a_k z^k$ be an analytic function in the disc $|z| < R$, $R > 1$, and $g(1) \neq 0$. The Appell polynomials p_k, $k \in \mathbb{N}_0 = \{0\} \cup \mathbb{N}$, are defined by the generating function

$$g(u)e^{ux} = \sum_{k=0}^{\infty} p_k(x)u^k. \tag{4}$$

For $f \in C(\mathbb{R}_+)$, Verma and Gupta [14] proposed the Jakimovski-Leviatan-Păltănea operators defined as

$$M_{n,\rho}(f;x) = l_{n,0}(x)f(0) + \sum_{k=1}^{\infty} l_{n,k}(x)\int_0^\infty \Theta_{n,k}^\rho(t)f(t)dt, \tag{5}$$

where $\Theta_{n,k}^\rho$ is defined at (2) and

$$l_{n,k}(x) = \frac{e^{-nx}}{g(1)}p_k(nx), \ k \in \mathbb{N}_0, \tag{6}$$

p_k being described at (4).

To establish the rate of convergence, the authors used the moduli of smoothness of the first and second order which give direct information about the smoothness of f. We recall their definitions.

$$\omega_1(f,\delta) = \sup_{0\leq h\leq\delta}\sup_{x\geq 0}|f(x+h) - f(x)|, \tag{7}$$

$$\omega_2(f,\delta) = \sup_{0\le h\le\delta}\sup_{x\ge 0}|f(x+2h)-2f(x+h)+f(x)|, \tag{8}$$

where $f \in C_B(\mathbb{R}_+)$, the space of continuous and bounded real valued functions defined on $\mathbb{R}_+$. The following result was proved.

Theorem 3. ([14, *Theorem 2*]). *For $f \in C_B(\mathbb{R}_+)$, we have*

$$|M_{n,\rho}(f;x)-f(x)| \le \omega_2(f,\delta)+\omega_1\left(f, \left|\frac{g'(1)}{ng(1)}\right|\right),$$

where $\delta = \left(M_{n,\rho}((\cdot-x)^2;x)+\left(\dfrac{g'(1)}{ng(1)}\right)^2\right)^{1/2}$.

Considering the space

$$B_w(\mathbb{R}_+) = \{f : \mathbb{R}_+ \to \mathbb{R} \mid \ |f(x)| \le M_f(1+x^2),\ x \in \mathbb{R}_+\}$$

M_f being a constant depending on f, a Voronovskaja type asymptotic formula was also obtained.

Theorem 4. ([14, *Theorem 3*]). *For any function $f \in B_w(\mathbb{R}_+)\cap C(\mathbb{R}_+)$ such that f', f'' are continuous and belong to $B_w(\mathbb{R}_+)$, we have*

$$\lim_{n\to\infty} n(M_{n,\rho}(f;x)-f(x)) = \frac{g'(1)}{g(1)}f'(x)+\frac{x}{2}\left(1+\frac{1}{\rho}\right)f''(x),\ x\ge 0.$$

Goyal and Agrawal [3] defined and studied the Bézier variant of the operators $M_{n,\rho}$, $n \in \mathbb{N}$.

Set $C_\gamma(\mathbb{R}_+) = \{f \in C(\mathbb{R}_+) :\ f(t) = \mathcal{O}(e^{\gamma t})$ as $t \to \infty\}$, where $\gamma > 0$ is fixed. For $\theta \ge 1$ and $f \in C_\gamma(\mathbb{R}_+)$, the Jakimovski-Leviatan-Păltănea-Bézier operator is of the form

$$M_{n,\rho}^{\theta}(f;x) = X_{n,0}^{\theta}(x)f(0)+\sum_{k=1}^{\infty}X_{n,k}^{\theta}(x)\int_0^\infty \Theta_{n,k}^{\rho}(t)f(t)dt,$$

where

$$X_{n,k}^{\theta}(x) = (J_{n,k}(x))^\theta-(J_{n,k+1}(x))^\theta,\ J_{n,k}(x) = \sum_{j=k}^{\infty} l_{n,j}(x),$$

see [3, *Eq. (1.2)*].

Clearly, $J_{n,k}(x)-J_{n,k+1}(x) = l_{n,k}(x)$ defined by (6), $k \in \mathbb{N}_0$. For $\theta = 1$, $M_{n,\rho}^1$ turns out to be $M_{n,\rho}$ defined by (5). A substantial result is the establishment of the rate of convergence for functions having a derivative of bounded variation.

Let $DBV_\gamma(\mathbb{R}_+^*)$, $\gamma \ge 0$, be the class of all functions defined on $\mathbb{R}_+^*$ having a derivative of bounded variation on every bounded subinterval of $\mathbb{R}_+^*$ and any function f of this class enjoys the property $|f(t)| \le Mt^\gamma$, $t \in \mathbb{R}_+^*$.

Theorem 5. ([3, *Theorem 5*]). *Let $f \in DBV_{\gamma}(\mathbb{R}_+^*)$, $\theta \geq 1$ and let $V_c^d(f_x')$ be the total variation of f_x' on $[c,d] \subset \mathbb{R}_+^*$. For every $x \in \mathbb{R}_+^*$ and sufficiently large n, we have*

$$\begin{aligned} |M_{n,\rho}^{\theta}(f;x) - f(x)| \leq{}& \frac{\sqrt{\theta}}{\theta+1}\sqrt{\frac{Cx(1+\rho)}{n\rho}}|f'(x+) + \theta f'(x-)| \\ &+ \frac{\theta\sqrt{\theta}}{\theta+1}\sqrt{\frac{Cx(1+\rho)}{n\rho}}|f'(x+) - f'(x-)| \\ &+ \theta\,\frac{C(1+\rho)}{n\rho}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x-(x/k)}^{x}(f_x') + \frac{x}{\sqrt{n}}\bigvee_{x-(x/\sqrt{n})}^{x}(f_x') \\ &+ \theta\,\frac{C(1+\rho)}{n\rho}\sum_{k=1}^{[\sqrt{n}]}\bigvee_{x}^{x+(x/k)}(f_x') + \frac{x}{\sqrt{n}}\bigvee_{x}^{x+(x/\sqrt{n})}(f_x'), \end{aligned}$$

where $C > 1$ and the function f_x' is defined by

$$f_x'(t) = \begin{cases} f'(t) - f'(x-), & 0 \leq t < x, \\ 0, & t = x, \\ f'(t) - f'(x+), & x < t < \infty. \end{cases}$$

Motivated by the above mentioned construction, in [8] the authors introduced the Bézier-Păltănea operators based on Gould-Hopper polynomials. For this new generalization of Păltănea operators, the authors obtained both the quantitative Voronovskaja type theorem in terms of Ditzian-Totik modulus of smoothness and the rate of pointwise convergence for the functions having a derivative of bounded variation.

In the final part we mention a recent result obtained by Gupta and Agrawal [4]. They proposed a hybrid integral type operator containing both Szász as well as Baskakov bases in summation. More precisely, in (1) they replaced $s_{\alpha,k}(x)$, $k \in \mathbb{N}_0$, $x \in \mathbb{R}_+$, with $p_{\alpha,k}(x,c)$, where

$$p_{\alpha,k}(x,c) = \frac{(\alpha/c)_k}{k!}\,\frac{(cx)^k}{(1+cx)^{\alpha/c+k}},$$

c being a constant belonging to the interval $(0,1]$. In the above $(\alpha/c)_k$ stands for rising factorial, also called Pochhammer function. We recall $(\alpha/c)_0$ is taken to be 1. For these new operators, the notation $B_{\alpha}^{\rho}(f;\cdot,c)$ was used. Among the results obtained we mention a Grüss-Voronovskaja type theorem. Setting

$$C_2^*(\mathbb{R}_+) = \left\{ f \in B_w(\mathbb{R}_+) \cap C(\mathbb{R}_+) : \lim_{x\to\infty}\frac{f(x)}{w(x)} \text{ exists and is finite} \right\}, \tag{9}$$

where $w(x) = 1 + x^2$, the following statement was proved.

Theorem 6. ([4, *Theorem 3.2*]). *Let $f, g, f', g', f'', g'', (fg)', (fg)''$ belong to $C_2^*(\mathbb{R}_+)$. For any $x \in \mathbb{R}_+$ we have*

$$\lim_{\alpha\to\infty}\alpha(B_{\alpha}^{\rho}(fg;x,c) - B_{\alpha}^{\rho}(f;x,c)B_{\alpha}^{\rho}(g;x,c)) = \frac{x(1+\rho(1+cx))}{\rho}\,f'(x)g'(x).$$

3 On the King variant of the operators L_α^ρ

Set $e_0(x)=1$, $e_j(x)=x^j$ $(j\in\mathbb{N})$, $x\ge 0$.

Two decades ago, King [6] had the idea to modify the Bernstein operators in order to to reproduce the monomials e_0 and e_2. Consequently, the modified operators enjoy the property of keeping the functions $c_1e_0+c_2e_2$ as fixed points, for any real constants c_1 and c_2.

From approximation theory point of view the construction is useful. In spite of the fact that the new operators have the degree of exactness null, the maximum rate of convergence is smaller. Over time this technique was applied to many linear approximation processes, becoming known as the King method. We propose to apply it to Păltănea operators (1). It is known that

$$L_\alpha^\rho e_0=e_0,\ L_\alpha^\rho e_1=e_1,\ L_\alpha^\rho e_2=e_2+\frac{\rho+1}{\alpha\rho}e_1,\ \alpha>0,\ \rho>0, \tag{10}$$

see [10, *Eq. (2.1)*]. Considering

$$u(x)=\frac{1}{2}\left(\sqrt{\beta^2(\alpha,\rho)+4x^2}-\beta(\alpha,\rho)\right),\ x\ge 0, \tag{11}$$

where $\beta(\alpha,\rho)=(\rho+1)/(\alpha\rho)$, we define the operators

$$L^*_{\alpha,\rho}(f;x)=e^{-\alpha u(x)}f(0)+\sum_{k=1}^{\infty}s_{\alpha,k}(u(x))\int_0^\infty\Theta^\rho_{\alpha,k}(t)f(t)dt,\ x\ge 0, \tag{12}$$

$f\in W$.

Remarks. (i) By using a bivariate kernel, we can write (12) in a more compact form, as follows

$$L^*_{\alpha,\rho}(f;x)=\int_0^\infty H_{\alpha,\rho}(x,t)f(t)dt,\ \alpha>0,$$

where

$$H_{\alpha,\rho}(x,t)=e^{-\alpha u(x)}\delta(0)+\sum_{k=1}^{\infty}s_{\alpha,k}(u(x))\Theta^\rho_{\alpha,k}(t),\ (x,t)\in\mathbb{R}_+\times\mathbb{R}_+.$$

In the above δ represents Dirac delta function for which

$$\int_0^\infty\delta(t)f(t)dt=f(0).$$

(ii) For any $f\in C_B(\mathbb{R}_+)$ we can easily deduce that the operators are non-expansive, this means $\|L^*_{\alpha,\rho}f\|\le\|f\|$. The proof uses the identities

$$\int_0^\infty\Theta^\rho_{\alpha,k}(t)dt=1,\ k\in\mathbb{N}. \tag{13}$$

Relations (10) and (11) involve the identities

$$L^*_{\alpha,\rho}e_0=e_0,\ L^*_{\alpha,\rho}e_1=u,\ L^*_{\alpha,\rho}e_2=e_2,\ \alpha>0,\ \rho>0. \tag{14}$$

Since any compact interval $K\subset\mathbb{R}_+$ is isomorphic to $[0,b]$, $b>0$ arbitrarily fixed, in our approach will use only this interval.

Theorem 7. *For any $b > 0$, any function $f \in W \cap C(\mathbb{R}_+)$ and any number $n \in \mathbb{N}$, there exists $\alpha_0 > 0$ such that for $\alpha > \alpha_0$, $L^*_{\alpha,n}$ is well-defined and we have*

$$\lim_{\alpha\to\infty} L^*_{\alpha,n} f = f, \text{ uniformly on } [0, b].$$

Let $b > 0$ be arbitrarily fixed. The relation $\lim\limits_{\alpha\to\infty} u(x) = x$ uniformly on $[0, b]$ takes place. Based on (14), the proof of the above theorem follows exactly the same line as the proof of Theorem 3.4 from [9], so we omit it.

For a positive linear operator Λ its second central moment defined by

$$\mu_2(\Lambda; x) = (\Lambda\varphi_x^2)(x),$$

where

$$\varphi_x(t) = t - x, \ (t, x) \in \mathbb{R}_+ \times \mathbb{R}_+ \tag{15}$$

plays a crucial role when estimating its local rate of convergence.

The identities (10) and (14) imply

$$\mu_2(L^\rho_\alpha; x) = \beta(\alpha, \rho)x, \ \mu_2(L^*_{\alpha,\rho}; x) = 2x(x - u(x)). \tag{16}$$

It turns out that the second central moment of their King type variant (12) is smaller than the second central moment of the Păltănea operators (1) on the whole interval $(0, +\infty)$.

Lemma 1. *(i) For $x > 0$, there holds*

$$0 < x - u(x) < \beta(\alpha, \rho)/2.$$

(ii) The inequality

$$\mu_2(L^*_{\alpha,\rho}; x) < \mu_2(L^\rho_\alpha; x)$$

is valid, for each $x > 0$.

Proof. (i) Let $x > 0$. The first inequality follows from the observation

$$u(x) = \frac{2x^2}{\sqrt{\beta^2(\alpha, \rho) + 4x^2} + \beta(\alpha, \rho)} < \frac{2x^2}{\sqrt{4x^2}} = x,$$

since $\beta(\alpha, \rho) > 0$. Furthermore, we have

$$2(x - u(x)) = 2x - \sqrt{\beta(\alpha, \rho)^2 + 4x^2} + \beta(\alpha, \rho) < \beta(\alpha, \rho),$$

which proves the second inequality.

(ii) By (16), this statement is a consequence of the previous result. □

By virtue of the classical results regarding the local rate of convergence established by Shisha and Mond [12], the relations (10), (14) and (16) guarantee

$$|L^\rho_\alpha(f; x) - f(x)| \le 2\omega_1\left(f, \sqrt{\beta(\alpha, \rho)x}\right),$$
$$|L^*_{\alpha,\rho}(f; x) - f(x)| \le 2\omega_1\left(f, \sqrt{2x(x - u(x))}\right),$$

for any $f \in C_B(\mathbb{R}_+)$, where ω_1 is defined at (7).

Remark. Since ω_1 associated with a function f is an increasing function, Lemma 1 (ii) demonstrates that the upper bound for the absolute error of $L^*_{\alpha,\rho}$ is smaller than that for L^ρ_α.

The evaluation of the rate of convergence can be carried out in weighted spaces, for example in $C^*_2(\mathbb{R}_+)$ defined at (9) and endowed with the usual norm $\|\cdot\|_{C^*_2(\mathbb{R}_+)}$,

$$\|f\|_{C^*_2(\mathbb{R}_+)} = \sup_{x\ge 0} \frac{|f(x)|}{w(x)},$$

where $w(x) = 1 + x^2$, $x \ge 0$.

Theorem 8. *Let $L^*_{\alpha,\rho}$ be defined by (12). For every $f \in W \cap C^*_2(\mathbb{R}_+)$, $L^*_{\alpha,\rho}$ converges to f in norm, i.e.,*

$$\lim_{\alpha\to\infty} \|L^*_{\alpha,\rho} f - f\|_{C^*_2(\mathbb{R}_+)} = 0. \tag{17}$$

Proof. It is known that $\{e_0, e_1, e_2\}$ is a Korovkin set in $C^*_2(\mathbb{R}_+)$, see, e.g., [1, *Proposition 4.2.5.-(6)*]. Taking in view identities (14), it remains for us to prove (17) only for $f := e_1$. Applying two times Lemma 1 (i), we obtain the estimate

$$\frac{\left|L^*_{\alpha,\rho}(e_1; x) - x\right|}{1+x^2} = \frac{|u(x) - x|}{1+x^2} = \frac{x - u(x)}{1+x^2} \le \frac{x}{1+x^2}\beta(\alpha,\rho) \le \frac{1}{2}\beta(\alpha,\rho),$$

for all $x \ge 0$, from which we deduce

$$\|L^*_{\alpha,\rho} e_1 - e_1\|_{C^*_2(\mathbb{R}_+)} \le \frac{\rho+1}{2\alpha\rho}.$$

Thus, we got what we proposed, consequently (17) takes place. □

To obtain the following new result we need an inequality that we present in what follows. Any discrete or integral linear positive operator Λ of summation type satisfies the classical inequality

$$\Lambda|\varphi_x| \le (\Lambda\varphi_x^2)^{1/2},$$

where φ_x is given at (15). Because the operators $L^*_{\alpha,\rho}$ contain as the first term a quantity not included in the sum, for a self-contained presentation, we prove the relation

$$L^*_{\alpha,\rho}|\varphi_x| \le (L^*_{\alpha,\rho}\varphi_x^2)^{1/2},\ x \ge 0. \tag{18}$$

The proof is based on Cauchy–Schwarz inequality both for integrals and for series, and it runs as follows.

$$\begin{aligned}\int_0^\infty \Theta^\rho_{\alpha,k}(t)|\varphi_x|(t)dt &\le \left(\int_0^\infty \Theta^\rho_{\alpha,k}(t)dt\right)^{1/2}\left(\int_0^\infty \Theta^\rho_{\alpha,k}(t)\varphi_x^2(t)dt\right)^{1/2}\\ &= \left(\int_0^\infty \Theta^\rho_{\alpha,k}(t)\varphi_x^2(t)dt\right)^{1/2},\ k\ge 1,\end{aligned}$$

see (13).

Further, we define $b_0 = \varphi_x^2(0) = x^2$ and $b_k = \int_0^\infty \Theta^\rho_{\alpha,k}(t)\varphi_x^2(t)dt,\ k \geq 1$. We get

$$\begin{aligned}
(L^*_{\alpha,\rho}|\varphi_x|)(x) &\leq \sum_{k=0}^{\infty} s_{\alpha,k}(u(x))b_k^{1/2} = \sum_{k=0}^{\infty} \sqrt{s_{\alpha,k}(u(x))}\sqrt{s_{\alpha,k}(u(x))}\, b_k^{1/2} \\
&\leq \left(\sum_{k=0}^{\infty} s_{\alpha,k}(u(x))\right)^{1/2} \left(\sum_{k=0}^{\infty} s_{\alpha,k}(u(x))b_k\right)^{1/2} \\
&= ((L^*_{\alpha,\rho}\varphi_x^2)(x))^{1/2}
\end{aligned}$$

and the proof of (18) is completed.

A less frequently used tool to approximate signals is the so called Steklov mean. The benefit of special function is that continuous functions can be approximated by smoother functions. For $f \in C_B(\mathbb{R}_+)$, the Stelov mean of second order and step $h/2$ is defined by

$$f_h(x) = \frac{4}{h^2}\int_0^{h/2}\int_0^{h/2}(2f(x+u+v) - f(x+2(u+v)))dudv \tag{19}$$

and verifies the inequalities

$$\|f_h - f\| \leq \omega_2(f,h), \tag{20}$$

and if $f_h', f_h'' \in C_B(\mathbb{R}_+)$ exist,

$$\|f_h'\| \leq \frac{5}{h}\omega_1(f,h),\ \|f_h''\| \leq \frac{9}{h^2}\omega_2(f,h). \tag{21}$$

In the above $\|\cdot\|$ stands for the sup-norm, $\|h\| = \sup\limits_{x\geq 0}|h(x)|,\ h \in C_B(\mathbb{R}_+)$.

The key of the proofs of these relations consists in rewriting the definitions (7) and (8) as follows

$$\begin{aligned}
\omega_1(f,\delta) &= \sup_{\substack{x,u,v\geq 0\\ |u-v|\leq\delta}} |f(x+u) - f(x+v)|, \\
\omega_2(f,\delta) &= \sup_{\substack{x,u,v\geq 0\\ |u-v|\leq\delta}} |f(x+2u) - 2f(x+u+v) + f(x+2v)|,
\end{aligned}$$

where $\delta \geq 0$. The proofs of (20) and (21) can be found in [2, *Eqs. (5.2)-(5.4)*].
Remark. For the full information of the reader, in accordance with *The Great Soviet Encyclopedia*, 3rd Edition (1969-1978), we mention that the initial form of this type of function was introduced in 1907 by Vladimir Steklov (Stekloff) [13] by the equality

$$\Phi(x,h) = \frac{1}{h}\int_x^{x+h} f(t)dt,$$

where $h > 0$ is so small that the interval $(x, x+h)$ lies in the domain of the definition of the locally integrable function f.

Theorem 9. *Let $L^*_{\alpha,\rho}$ be defined by (11). For every $f \in C_B(\mathbb{R}_+)$ and $x \geq 0$, the following inequality*

$$|L^*_{\alpha,\rho}(f;x) - f(x)| \leq 5\omega_1\left(f, \sqrt{2x(x-u(x))}\right) + \frac{13}{2}\omega_2\left(f, \sqrt{2x(x-u(x))}\right)$$

holds.

Proof. Let $f \in C_B(\mathbb{R}_+)$ be arbitrarily fixed. For $x = 0$, our relation is obvious. Let $x > 0$. Applying the Steklov mean f_h given at (19), we can write

$$\begin{aligned} |L^*_{\alpha,\rho}(f;x) - f(x)| &\leq L^*_{\alpha,\rho}(|f - f_h|;x) + |L^*_{\alpha,\rho}(f_h - f_h(x);x)| \\ &\quad + |f_h(x) - f(x)|. \end{aligned} \tag{22}$$

Using the fact that the operators are non-expansive and taking in view (20), we obtain

$$L^*_{\alpha,\rho}(|f - f_h|;x) \leq \|f - f_h\| \leq \omega_2(f,h).$$

Further, using successively Taylor's expansion, the identity $L^*_{\alpha,\rho}e_0 = e_0$ and relations (18), (21) we get

$$\begin{aligned} |L^*_{\alpha,\rho}(f_h - f_h(x);x)| &\leq \|f'_h\|\sqrt{\mu_2(L^*_{\alpha,\rho};x)} + \frac{1}{2}\|f''_h\|\mu_2(L^*_{\alpha,\rho};x) \\ &\leq \frac{5}{h}\omega_1(f,h)\sqrt{2x(x-u(x))} \\ &\quad + \frac{9}{2h^2}\omega_2(f,h)(2x)(x-u(x)). \end{aligned}$$

At this point we choose $h := \sqrt{2x(x-u(x))} > 0$ and returning at (22) we assemble the established increases. Our statement is fully proven. □

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