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EDGE DETECTION TECHNIQUES USING NONLINEAR DIFFUSION-BASED MODELS

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Abstract

An overview of the edge detection techniques based on partial differential equations (PDE) is presented in this work. Nonlinear anisotropic diffusion-based boundary extraction approaches, like the influential Perona-Malik model and some improved variants of it, are described first. Anisotropic diffusion-based detection schemes using the mean curvature motion and non-linear PDE-based approaches combining anisotropic diffusion to the bilateral filter, are then disscused here. Some nonlinear reaction-diffusion based edge detection methods are described next. Variational edge detection solutions using the total variation (TV) regularization or combining the anisotropic diffusion to the TV-based models are then presented. Directional diffusion-based image edge extraction algorithms, are also disscused. Our own contributions in this computer vision domain are finally described.

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Key words: edge detection, partial differential equation, anisotropic diffusion model, total variation regularization, multi-scale analysis.

1 Introduction

Edge detection has the purpose of detecting sharp changes in image brightness and represents a fundamental tool in image processing and analysis, and computer vision. It includes various mathematical techniques that identify those points in a digital image corresponding to discontinuities in depth or in surface orientation, variations in scene illumination or changes in material properties. The edge detection has important applications in various image processing and computer vision domains, such as the image segmentation, object detection and tracking, face recognition and remote sensing [1].

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Numerous image edge detection techniques have been developed in the last decades. There are two main categories of image boundary detectors. The first-order differential edge detectors include operators like Canny, Roberts, Sobel and Prewitt [2]. Canny edge detector, which also have some variations) still represents the state-of-the-art in this computer vision field, although it was introduced almost 40 years ago [3].

The second-order derivative-based edge detection techniques search for zero crossings in a second-order derivative expression computed from the image in order to find the boundaries. These zero-crossing based approaches include the Marr–Hildreth edge detector [4] and *differential approaches* that detect the zero-crossings of the second-order directional derivative in the gradient direction.

Another important category of image edge extraction techniques is based on the nonlinear partial differential equations (PDE) and variational models. A survey of some of these nonlinear diffusion-based edge detection techniques is described here. Also, our own contributions in this image analysis field are also presented in this work.

2 Anisotropic diffusion-based edge detection

Edge preserving anisotropic diffusion-based filtering techniques represent an effective tool for edge detection. An influential anisotropic diffusion model was introduced by Perona and Malik in 1990 [5]. It is based on the following nonlinear PDE:

$$u_t = \frac{\partial u}{\partial t} = \operatorname{div}(g(\|\nabla u\|^2) \cdot \nabla u) \tag{1}$$

where its edge stopping (diffusivity) function may have the next two forms:

$$g(s^2) = e^{-\frac{s^2}{k^2}}; \quad g(s^2) = \frac{1}{1 + \left(\frac{s}{k}\right)^2} \tag{2}$$

It filters successfully the image and sharpens the edges, since the diffusion process is performed along them and not across them. A scale space was constructed by using this nonlinear PDE model, each scale being given by a t value. The image edges are estimated at each scale by using the magnitude of the gradient of the brightness function, $\|\nabla u\|$ [5]. The edges determined at various scales can be then combined by using a fine-to-coarse tracking approach. An example of image edges obtained at finer and coarser scales using the Perona–Malik detector is described in Fig. 1 [5].

Other nonlinear anisotropic diffusion-based edge detectors inspired by the Perona–Malik scheme were constructed by using other diffusivity functions, g. Other diffusion-based models introduced a spatially adaptive term in the anisotropic scheme. Thus, the one developed by V. Prasath and A. Singh in 2008 has the following form [6]:

$$u_t = \operatorname{div}(\alpha(x) g(|\nabla u|) \nabla u) \tag{3}$$

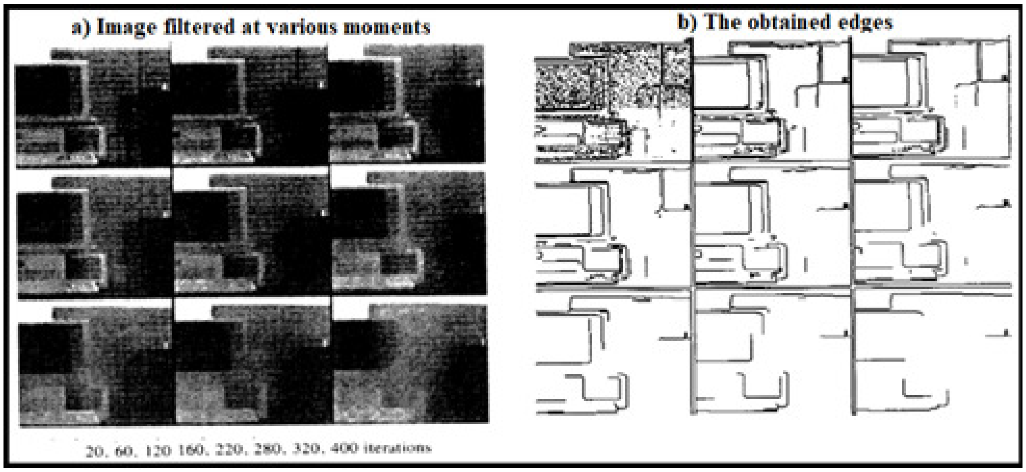

Figure 1: Edge detection performed by Perona–Malik model

where the introduced function $\alpha(x)$ provides a pixel-wise edge characterization. This avoids the stability problems of the gradient magnitude and overcomes the localization error of the smoothed gradient method [6].

Other detection models, such as the one introduced by C. Bazan and P. Blomgren in 2007 [7], combine the nonlinear anisotropic diffusion to bilateral filter:

$$\partial_t u - \nabla \cdot (g(\|\nabla u_{BF}\|^2)\nabla u) = 0 \tag{4}$$

where

$$\begin{aligned} BF(u(\mathbf{x})) &= \frac{1}{W(\mathbf{x})}\int_\Omega G_{\sigma_z}(\xi,\mathbf{x})G_{\sigma_r}(u(\xi),u(\mathbf{x})) * u(\mathbf{x})d\xi \\ W(\mathbf{x}) &= \int_\Omega G_{\sigma_z}(\xi,\mathbf{x})G_{\sigma_r}(u(\xi),u(\mathbf{x}))d\xi \end{aligned} \tag{5}$$

A nonlinear diffusion-based edge detection approach using the mean curvature motion was proposed by Catte et al. in 1992 [8]. Their model has the following form:

$$\begin{aligned} &\frac{\partial u}{\partial t} = g(|G * Du|)|Du|\mathrm{div}\,\frac{Du}{|Du|} \\ &u(0,x,y) = u_0(x,y) \end{aligned} \tag{6}$$

If Du has a large average value in the neighborhood of (x,y), then (x,y) represents an edge point. If Du has a small mean in a neighborhood of such a point, then (x,y) is the interior point of a smooth region [8]. The mean curvature motion-based filter provides an improved edge-preserving smoothing and better edge detection results.

A directional diffusion-based edge detector was introduced by S. Belfiih, P. Montesinos and R. Beuscart in 2003 [9]. They define a PDE model, called Perpendicular Edge Smoothing (PES), having the form:

$$\begin{cases} \dfrac{\partial I}{\partial t} = I_{\eta\eta} \\ I(0,x,y) = I_0(x,y) \end{cases} \tag{7}$$

It uses the second-order diffusion in the direction of the gradient of the image I. This diffusion-based scheme has the fundamental property to smooth every edge point independently of its curvature [9]. Thus, the proposed edge detection algorithm has the next steps:

- Compute the smoothed image at time $t_1 : I_t$
- Compute the smoothed image at time $t_2 : I_{t_2}$ $(t_2 > t_1)$
- Substract I_{t_2} to $I_{t_1} : I_s$
- Compute zero crossings of $I_s : I_z$

In order to achieve a more precise edge localization, one should consider $t_1 = 0$ and $t_2 >> t_1$. A PES-based edge detection example using $t_1 = 0$ and $t_2 = 10$ is displayed in Fig. 2.

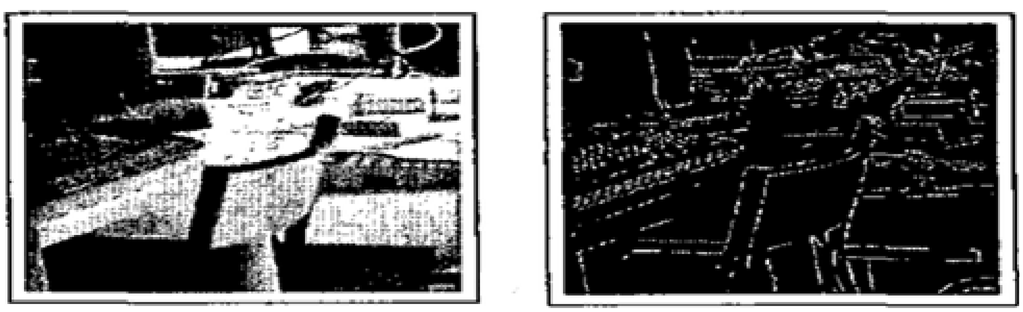

Figure 2: PES-based edge detection example

An edge detection algorithm for the grayscale images which uses a modified version of the FitzHugh–Nagumo reaction-diffusion equation was proposed in 2014 by A. Nomura et al. [10]. While the original FitzHugh–Nagumo reaction-diffusion equations have the function of detecting the boundaries from binary images [11, 12], this approach detects those edges having large spatial gradients in the grayscale images.

The FitzHugh–Nagumo reaction-diffusion equation system has the following form:

$$\begin{aligned} \partial_t u &= D_u \nabla^2 u + [u(u-a)(1-u) - v]/\varepsilon \\ \partial_t v &= D_v \nabla^2 v + u - bv \end{aligned} \tag{8}$$

where the parameter a refers to the threshold value for binarisation. So, they added the next equation to the system:

$$\partial_t a = D_a \nabla^2 a \tag{9}$$

where D_a represents the diffusion coefficient and $a(x, y, t)$ is a spatio-temporal variable [10]. The initial conditions for the three variables are the following:

$$\begin{aligned} u(x, y, 0) &= a(x, y, 0) = a_0 I(x, y) \\ v(x, y, t = 0) &= 0 \end{aligned} \tag{10}$$

The two diffusion coefficients, D_u and D_v, have to satisfy the condition $D_u << D_v$ for static edge patterns. The system is then solved by discretizing the equations by the finite difference method with the Crank–Nicolson scheme [10].

The solutions u and v are next used to detect the edges. Thus, one determines the zero-crossings in $[u- - v]$. These identified zero-crossings correspond to the edge points. The proposed algorithm outperforms the previous FitzHugh–Nagumo equation-based edge detectors and the DoG-based model [4]. A method comparison example is provided in Fig. 3. One can see that the original FitzHugh–Nagumo obtains much fewer edge points.

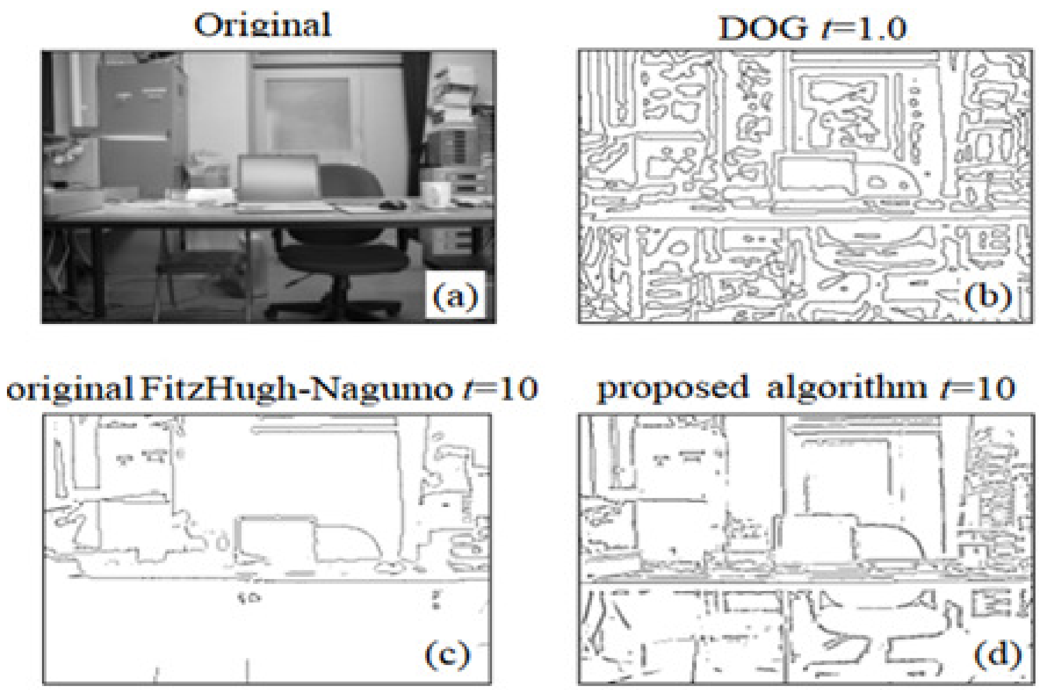

Figure 3: Edge detection method comparison

3 Total variation-based edge detectors

Some edge detection solutions combine the anisotropic diffusion and total variation filtering models. Thus, such a variational edge detector was proposed in 2019 by A. Yahya et al. [13]. It combines the TV Denoising model (ROF) to a Perona–Malik inspired anisotropic diffusion scheme transformed into a variational model.

The combined energy functional is the following one:

$$E(u) = \int\int_{\Omega} \left[\lambda|\nabla u| + (1-\lambda)k^2 \ln\left(\frac{1}{g(|\nabla u|)}\right)\right] dxdy \tag{11}$$

where the adaptive weight function used to detect the edges is

$$\lambda = \begin{cases} \delta M(i,j)_p & \text{if } P \in \text{edges areas} \\ \delta S(i,j)_p & \text{otherwise} \end{cases} \qquad M = \exp\left(\frac{-1}{|\nabla u|}\right),\ S = 1 - M \tag{12}$$

By applying the gradient descent method, one obtains the nonlinear parabolic PDE model:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \lambda \nabla \cdot \left(\dfrac{\nabla u}{|\nabla u|}\right) + (1-\lambda)\nabla \cdot (g(|\nabla u|)\nabla u) \\ u(x,y,t)|_{p=0} = u_0(x,y) \end{cases} \tag{13}$$

which is then solved numerically by applying the finite difference method [13].

A zero-crossing detector is used to distinguish the edges areas from the flat areas. Applying TV filter on the double edges areas will remove most of the false edges, thus obtaining much sharper edges. Applying anisotropic diffusion filter on the discontinuous edges areas will lead to obtaining robust and continuous edges. The proposed scheme has succeeded in removing most of the false edges and achieved a high edge localization performance. It achieves high values of the performance measures, such as the edge detection PSNR. See some edge detection results achieved by this approach in Fig. 4.

A variational regularization-based edge detection scheme was proposesd by K.N. Nordström in 1990 [14]. It is based on the minimization of the following energy cost functional:

$$C_{\zeta}(w,z) \doteq \int_B [\lambda f \circ w + (z-\zeta)^2 + w\|\nabla z^T\|^2] dx \tag{14}$$

where $\zeta : B \to R$ is the original image function, $w : B \to R$, represents the proposed edge function, the estimated image function is $z : B \to R$, and $f : R_+ \to R$ represents the edge cost density function.

By applying the Euler–Lagrange equations, one obtains:

$$\begin{aligned} z(x) = \zeta(x) + \nabla \cdot (w\nabla z)(x) \qquad & \forall x \in B \\ w(x) = g(\|\nabla z(x)^T\|) \qquad & \forall x \in B \\ \frac{\partial z}{\partial e_n}(x) = 0 \qquad & \forall x \in \partial B \end{aligned} \tag{15}$$

where $g(\gamma) \doteq (f'|]0,1])^{-1}\left(-\frac{\gamma^2}{\lambda}\right) \gamma \geq 0$ [14].

The image edges are obtained by thresholding the magnitude of the gradient of the estimated image function z. This variational edge detection provides effective results, as illustrated by the example in Fig. 5.

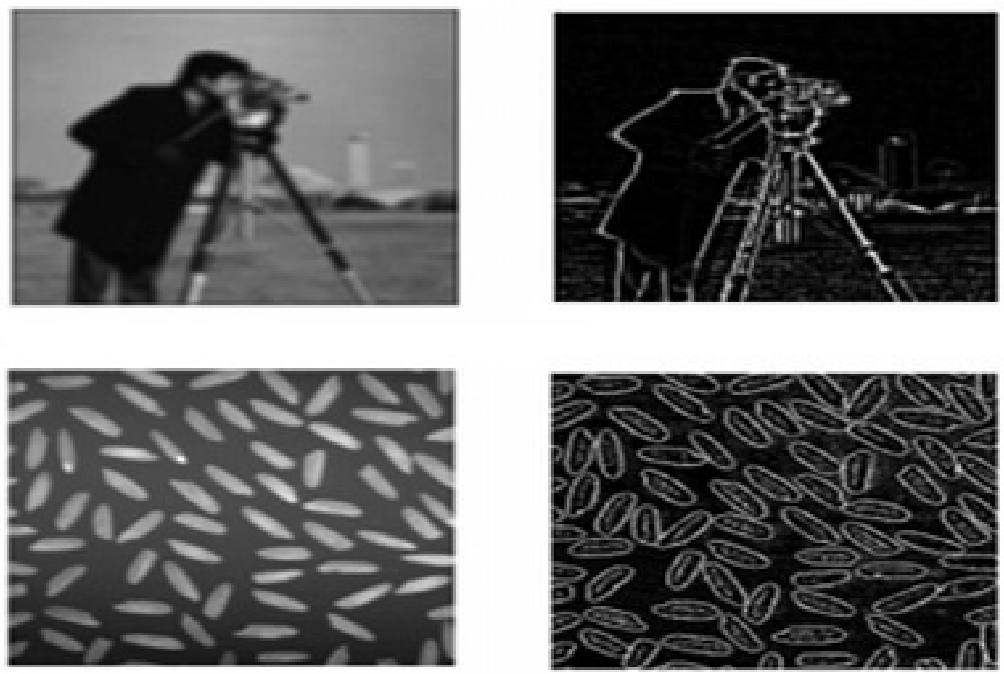

Figure 4: Edge detection results produced by this model

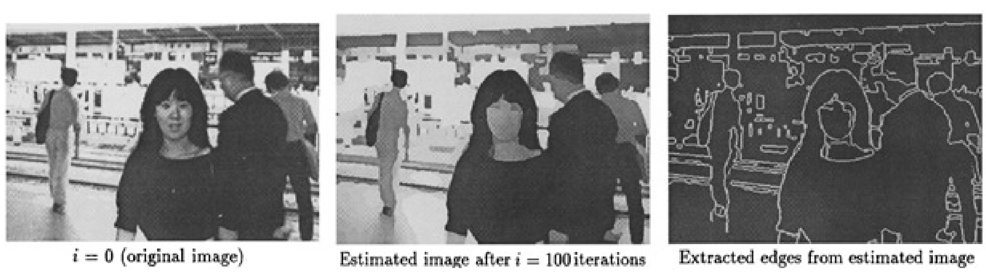

Figure 5: Estimated image and edge detection results obtained by the variational model

4 Automatic multi-scale image edge detection technique

Our main contribution in this computer vision field is described in this section. A novel automatic anisotropic diffusion-based multi-scale image edge detection framework was introduced in [15]. The multiscale image analysis used by this framework is based on a nonlinear anisotropic diffusion-based scale-space that was constructed by applying the finite difference method-based numerical approximation scheme of a novel well-posed second-order anisotropic diffusion model.

We have performed much research in the PDE-based image analysis area, many diffusion-based filtering models being developed by us [16, 17, 18]. The nonlinear second-order anisotropic diffusion-based filtering model proposed in [15] has the form:

$$\begin{cases} \frac{\partial u}{\partial t} - \beta\delta(\|\nabla u\|)\text{div}(\varphi(\|\nabla u_\sigma\|))\nabla u) + \alpha(u - u_0) = 0,\ \forall(x,y) \in \Omega \\ u(0,x,y) = u_0(x,y),\ \forall(x,y) \in \Omega \\ u(t,x,y) = 0,\ \forall(x,y) \in \partial\Omega \end{cases} \tag{16}$$

where $u_0 \in L^2(\Omega), u_\sigma = u * G_\sigma,\ G_\sigma(x,y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$ and the diffusivity function is

$$\varphi : [0,\infty) \to [0,\infty) : \varphi(s) = \xi \sqrt[3]{\frac{\lambda}{|\eta s^2 + \gamma|}} \tag{17}$$

The function of the term having the role of controlling the speed of this diffusion procedure has the form:

$$\delta : [0,\infty) \to [0,\infty) : \delta(s) = \left(\frac{\varepsilon s^\tau + \zeta}{v}\right)^{\frac{1}{\tau+1}} \tag{18}$$

The following finite difference method-based explicit iterative numerical approximation algorithm is constructed for solving the PDE model [15, 19]:

$$u_{i,j}^{n+1} = u_{i,j}^n(1-\alpha) + u_{i,j}^0\alpha + \beta\delta_{i,j}\begin{pmatrix} \varphi_{i+\frac{1}{2},j}((G_\sigma * u)_{i+1,j}^n - (G_\sigma * u)_{i,j}^n) \\ -\varphi_{i-\frac{h}{2},j}((G_\sigma * u)_{i,j}^n - (G_\sigma * u)_{i-1,j}^n) \\ +\varphi_{i,j+\frac{1}{2}}((G_\sigma * u)_{i,j+1}^n - (G_\sigma * u)_{i,j}^n) \\ -\varphi_{i,j-\frac{1}{2}}((G_\sigma * u)_{i,j}^n - (G_\sigma * u)_{i,j-1}^n) \end{pmatrix} \tag{19}$$

The scale-space representation is obtained as the next set of image subtraction results:

$$S = \{|u^\tau - u^0|, |u^{2\tau} - u^0|, ..., |u^{K\tau} - u^0|\} \tag{20}$$

where $K \geq 10$, time step $\tau \in [30, 50]$ and $k \in \{1, ..., K\}$.

The zero-crossing points of each $U^k = |u^{k\tau} - u^0|$, are then determined and binary image $E(U^k)$ is obtained. Only the zero-crossing points corresponding to the pixels characterized by a gradient magnitude exceeding a properly selected threshold value are kept in $E(U^k)$. We have:

$$\|\nabla U_{ij}^k\| \leq T_k \Rightarrow E(U^k)[i,j] := 0,\ \forall k \in \{1, ..., K\},\ i \in \{1, ..., I\},\ j \in \{1, ..., J\} \tag{21}$$

where $T_k = 2\mu(\|\nabla U^k\|)$ [15].

One discards the very small white spots in the binary image that could represent noise in the grayscale image. Next, one applies morphological operations on the obtained binary image to enhance the edge detection process [20]. So, the closing $E(U^k) := (E(U^k) \oplus Sq) \ominus Sq$, where Sq is a $[1 \times 1]$ structuring element, is performed first, then a morphological thinning process is performed on the closed image. The edge detection result at the k^{th} scale of S is thus obtained.

The edges extracted at multiple scales have to be combined, a fine-to-coarse edge tracking method being used for this task. At lower scales (small k values) the edge maps correspond to finer image details and some of their white pixels may not represent real edges, but some noise or clutter. At higher scales (large k values) they could miss some real boundaries and dislocate others. Our fine-to-coarse tracking algorithm considers all the edge (white) pixels of first edge image and track them through the scale-space. If they appear in a large enough number of next consecutive edge images, they are considered to belong to real boundaries and labeled as such, otherwise they are suppressed. Then, the tracking process continues the same way, by considering the unlabeled edge pixels of the second image and so on. It can be expressed as:

$$\begin{aligned} \forall k \in \left\{1, ..., \left[\frac{K}{2}\right]\right\},\ i \in \{1, ..., I\},\ j \in \{1, ..., J\} : E(U^k)[i,j] = \cdots \\ = E(U^{k+m})[i,j] = 1 \& m \geq \left[\frac{K}{2}\right] \Rightarrow E(S)[i,j] = 1 \end{aligned} \tag{22}$$

In order to address any possible edge displacement occuring at large scales, an edge pixel can be accepted as valid even if it is not found at the same location in the edge map of the next scale, but in a 4 or 8-neighborhood of it [15]. So, we have:

$$\exists a \in \{i-1, i, i+1\},\ b \in \{j-1, j, j+1\} : E(U^k)[i,j] = E(U^{k+1})[a,b] = 1, \quad \forall k \geq \text{high threshold} \tag{23}$$

This multi-scale edge extraction technique outperforms the classic edge detectors, achieving better performance metric values. scale, but it is slightly outperformed by the Canny edge detector, as shown by the method comparison results described by Fig. 6 and Table 1.

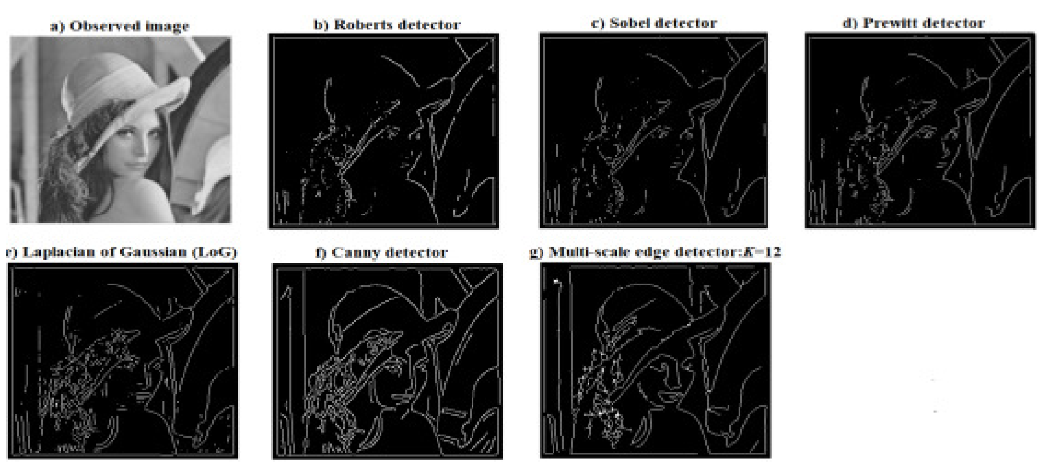

Figure 6: Edge detection results obtained by various models

Table 1. Method comparison: average PR and PSNR values obtained by several edge detectors

Edge detection approach	Average PR	Average PSNR (in dB)
The proposed multi-scale edge detector	11.0217	20.1742
Canny filter	12.1304	22.3817
Laplacian of Gaussian (Log)	9.3267	19.4837
Roberts operator	6.5486	15.6745
Sobel detector	7.4251	16.7802
Prewitt filter	8.5139	16.9837

5 Conclusions

An overview of nonlinear PDE-based edge detection techniques was described here. They are based on various PDE models representing anisotropic diffusion, directional diffusion, reaction diffusion and variational schemes. They achieve effective image edge detection results, but do not outperform the state of the art of this domain, which is the Canny filter.

Our main contribution in this image analysis field was also presented in this survey. Unlike other techniques, the proposed automatic anisotropic diffusion-based multi-scale image edge detection framework is based on a non-variational PDE model. Also the proposed well-posed PDE model was applied for the multi-scale image analysis. It outperforms other edge detectors but it is outperformed by the Canny operator.

The image segmentation and the object detection and tracking domains represent the main application areas of these edge detection approaches.

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