Bulletin of the *Transilvania* University of Braşov
Series III: Mathematics and Computer Science, Vol. 3(65), No. 1 - 2023, 171-184
https://doi.org/10.31926/but.mif.2023.3.65.1.13

OPERATORS ON REGULAR RINGS OF LEAVITT PATH ALGEBRAS

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Abstract

In [8, Theorem 1], Jain and Prasad obtained a kind of symmetry of regular rings which is interesting and useful in the theory of shorted operators (cf. [9]). We show that this symmetry property indeed holds for endomorphism rings of Leavitt path algebras. Using this property, we analyze a (strong/weak) regular inverse of an element of the regular the endomorphism ring A of the Leavitt path algebra $L := L_K(E)$ (viewed as a right L-module). We also introduce some partial orders on the endomorphism ring A of the Leavitt path algebra L and investigate the behavior of regular elements in A.

2000 *Mathematics Subject Classification:* 06A06, 16E50, 16S50.
Key words: Leavitt path algebra, von Neumann regular ring, unit-regular ring, associated elements, partial order, shorted operator.

1 Introduction

If E is a directed graph and K is a field, $L_K(E)$ denotes the Leavitt path algebra of E over K. $L_K(E)$ was introduced independently by Abrams and Aranda Pino [1], and by Ara, Moreno and Pardo [5], using different approaches. Since then, these algebras garnered significant interest and attention of ring theorists and operator algebraists, among others. Particular attention has been given to understanding basic algebra data: ideal structure, primeness, local chain conditions, socle, etc.

Let E be a graph and K be a field. G. Aranda Pino, K. M. Rangaswamy and M. Siles Molina [6] studied conditions on a graph E which are necessary and sufficient for the endomorphism ring A of the Leavitt path algebra $L := L_K(E)$ considered as a right L-module to be von Neumann regular (recall that a ring R is von Neumann regular if for every $a \in R$ there exists $b \in R$ such that $a = aba$).

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The algebra L embeds in A and $A = L$ if the graph E has finitely many vertices. The authors of [6] state that their focus is on the case when the graph E has infinitely many vertices since some earlier works in the literature (for instance, [2]) contain necessary and sufficient conditions on E for L to be von Neumann regular, and they show in [6, Theorem 3.5] that, if E is a row-finite graph, A is von Neumann regular if and only if E is acyclic and every infinite path ends in a sink (equivalently, L is left and right self-injective and von Neumann regular if and only if L is semisimple right L-module). In [8], Jain and Prasad obtained an interesting symmetry property for von Neumann regular rings: Let R be a ring and $a, b \in R$. If $a+b$ is regular, then $aR \oplus bR = (a+b)R$ (equivalently, $Ra \oplus Rb = R(a+b)$) if and only if $aR \cap bR = 0 = Ra \cap Rb$. The authors of [13] introduced symmetry property of endomorphism ring of the module. It is quite natural to ask if this symmetry property holds in the case of endomorphism rings of Leavitt path algebras. Let E be an arbitrary graph, K be any field, A be the endomorphism ring of $L = L_K(E)$ as a right $L_K(E)$-module and $f, g \in A$. If $f + g$ is a regular element, then, by [8], $fA \oplus gA = (f + g)A$ (equivalently, $Af \oplus Af = A(f + g)$) if and only if $fA \cap gA = (0) = Af \cap Ag$. Similarly, we have $\lambda_{f(x)}L \oplus \lambda_{g(x)}L = \lambda_{f(x)+g(x)}L$, for all $x \in L$ (equivalently, $L\lambda_{f(x)} \oplus L\lambda_{g(x)} = L\lambda_{f(x)+g(x)}$, for all $x \in L$) if and only if $\lambda_{f(x)}L \cap \lambda_{g(x)}L = (0) = L\lambda_{f(x)} \cap L\lambda_{g(x)}$ for all $x \in L$, under the assumption regularity of $f+g$. We also prove that, $f+g$ is a regular element in A, if $fA \oplus gA = (f+g)A$ (or equivalently, $Af \oplus Af = A(f+g)$), then $\lambda_{f(x)}L \oplus \lambda_{g(x)}L = \lambda_{f(x)+g(x)}L$, for all $x \in L$ (or equivalently, $L\lambda_{f(x)} \oplus L\lambda_{g(x)} = L\lambda_{f(x)+g(x)}$, for all $x \in L$) , and if $fA \cap gA = (0) = Af \cap Ag$, then $\lambda_{f(x)}L \cap \lambda_{g(x)}L = (0) = L\lambda_{f(x)} \cap L\lambda_{g(x)}$, for all $x \in L$.

Shorted operators associated to positive semidefinite Hermitian matrices were introduced by Anderson [3]; they correspond to impedance matrices of electrical networks with some ports shorted out. Let S denote the set of positive semidefinite Hermitian $n \times n$ matrices, and let e be an idempotent $n \times n$ complex matrix. For $a \in S \backslash eSe^*$, there is a unique shorted operator of a corresponding to the subspace $e\mathbb{C}^n$, the formulas for which were given by Anderson and Trapp [4]. In [9], the authors showed that the above shorted operator is permutation equivalent to $af_a^+ f$, where $f = e^*$ and f_a^+ is the a-weighted Moore-Penrose inverse of f. They also obtained this result from an analysis of a partial order $\leq^{\oplus}$ on $\mathbb{M}_n(\mathbb{C})$, by proving that if the partially ordered set $C = \{s \in eSe^* : s \leq^{\oplus} a\}$ has maximal (meaning maximal proper) elements, then $af_a^+ f$ is the unique maximal element of C. In the literature, many of the papers involve extensions of these ideas to elements of an arbitrary (von Neumann) regular ring R; specialization to the case in which $R = \mathbb{M}_n(\mathbb{C})$ yields the above results. The relation $\leq^{\oplus}$ on R is defined as follows: $\leq^{\oplus}$ if and only if $bR = aR \oplus (b - a)R$. Now we recall again [8, Theorem 1]. The latter condition is right-left symmetric by [8, Theorem 1]. Moreover, as proved in [8, Remark 1], $\leq^{\oplus}$ coincides with a differently defined partial order introduced by Hartwig [7]. Hartwig-Luh showed that, when R is a regular ring, the statement (2) is equivalent to the statement (3) in [8, Theorem 1] with the additional hypothesis that $a \in bRb$ (see [12, page 5]). For any two elements a, b in a von Neumann regular ring R, the relations $\leq^{\oplus}$, $\leq^{-}$ and $\leq^{s}$ on R are also defined as follows:

1. $a \leq^{\oplus} b$ if $bR = aR \oplus (b-a)R$.

2. $a \leq^{-} b$ if there exists an $x \in R$ such that $ax = bx$ and $xa = xb$, where $axa = a$, and called it that a is less than or equal to b under the minus partial order.

3. $a \leq^{s} b$, if $aR \subseteq bR$ and $Ra \subseteq Rb$.

In the present paper, we will introduce the relations $\leq^{\oplus}$, $\leq^{-}$ and $\leq^{s}$ on a von Neumann regular endomorphism ring A of Leavitt path algebra $L_K(E)$ and investigate the behavior of regular elements in A.

2 Definitions and preliminaries

We recall some graph-theoretic concepts, the definition and standard examples of the Leavitt path algebras.

Definition 1. A (directed) graph $E = (E^0, E^1, r, s)$ consist of two set E^0 and E^1 (with no restriction on their cardinals) together with maps $r, s : E^1 \to E^0$. The elements of E^0 are called vertices and the elements of E^1 edges. For $e \in E^1$, the vertices $s(e)$ and $r(e)$ are called the source and range of e. If $s^{-1}(v)$ is a finite set for every $v \in E^0$, then the graph is called row-finite. If E^0 is finite and E is row finite, then E^1 must necessarily be finite as well; in this case we say simply that E is finite.

A vertex which emits (receives) no edges is called a sink (source). A vertex v is called an infinite emitter if s^{-1} is an infinite set. A path α in a graph E is a finite sequence of edges $\alpha = e_1 ... e_n$ such that $r(e_i) = s(e_{i+1})$ for $1 \leq i \leq n-1$. In this case, $s(\alpha) = s(e_1)$ and $r(\alpha) = r(e_n)$ are the source and range of α, respectively, and n is the length of α. We view the elements of E^0 as paths of length 0.

If α is a path in E, with $v = r(\alpha) = s(\alpha)$ and $s(e_i) \neq s(e_j)$ for every $i \neq j$, then α is a called a cycle. A graph which contains no cycles is called acyclic.

Definition 2. *(The Leavitt Path Algebras of Arbitrary Graph)*

For an arbitrary graph E and a field K, the Leavitt path K-algebra of E, denoted by $L_K(E)$, is the K-algebra generated by the set $E^0 \cup E^1 \cup \{e^* | e \in E^1\}$ with the following relations,

(1) $v_i v_j = \delta_{v_i, v_j} v_i$ for every $v_i, v_j \in E^0$

(2) $s(e)e = e = er(e)$ for all $e \in E^1$.

(3) $r(e)e^* = e^* = e^* s(e)$ for all $e \in E^1$.

(4) (CK1) $e^* f = \delta_{e,f} r(e)$ for all $e, f \in E^1$.

(5) (CK2)$v = \sum_{\{e \in E^1, s(e) = v\}} ee^*$ for every $v \in E^0$ that is neither a sink nor an infinite emitter.

The first three relations are the path algebra relations. The last two are the so-called Cuntz-Krieger relations. We let $r(e^*)$ denote $s(e)$, and we let $s(e^*)$ denote $r(e)$. If $\alpha = e_1...e_n$ is a path in E, we write α^* for the element $e_n^*...e_1^*$ of $L_K(E)$. With this notation, the Leavitt path algebra $L_K(E)$ can be viewed as a $K-$vector space span of $\{pq^* \mid p,\, q \text{ are paths in } E\}$.

We remark that the Leavitt path algebras that we look at will not necessary have a unit. If E is a graph and K is a field, the Leavitt path algebra $L_K(E)$ is unital if and only if the vertex set E^0 is finite, in which case $\sum_{v\in E^0} v = 1_{L_K(E)}$. However, every Leavitt path algebra does have a set of local units (A set of local units for a ring R is a set $S \subseteq R$ of commuting idempotents with the property that for any $x \in R$ there exists $t \in S$ such that $tx = xt = x$. If R is a ring with a set of local units S, then for any finite number of elements $x_1, ..., x_n \in R$, there exists $t \in S$ such that $tx_i = x_i t = x_i$ for all $1 \leq i \leq n$.)

Let E be a graph with the field K and the Leavitt path algebra $L := L_K(E)$, A the unital ring $End(L_L)$ and L be identified with a subring of A. Let $\Phi : L \to End(L_L)$ be a monomorphism of rings such that $x \mapsto \lambda_x$, where $\lambda_x : L \to L$ is a left multiplication by x, i.e. for every $y \in L$, $\lambda_x(y) := xy$ which is a homomorphism of right L-modules. The map Φ is also a monomorphism because given a nonzero $x \in L$ there exists an idempotent $u \in L$ such that $xu = x$, hence $0 \neq x = \lambda_x(u)$.

Throughout the paper, we will assume that E is an arbitrary graph, K is any field and A is the endomorphism ring of $L := L_K(E)$ as a right L-module.

3 Main results

An element $a \in R$ is called regular if $axa = a$ for some $x \in R$ and x is called an inverse of a. We will denote an arbitrary regular inverse of a by $a^{(1)}$. An element $a \in R$ is called weakly regular if $xax = x$ for some $x \in R$, and x is called a weak regular inverse of a. We will denote a weak regular inverse of a by $a^{(2)}$. If $axa = a$ and $xax = x$, then x is called a strong von Neumann inverse of a. We will denote a strong regular inverse of a by $a^{(1,2)}$.

The regularity of $Hom_R(M,N)$, where M and N right-R modules, was introduced by Kasch and Mader in [10] to extend the notion of the regularity of a ring to $Hom_R(M,N)$. Recall that $f \in Hom_R(M,N)$ is called regular if $f = fgf$ for some $g \in Hom_R(N,M)$. The module $Hom_R(M,N)$ is said to be regular if each $f \in Hom_R(M,N)$ is regular. (see also some result of $Hom_R(M,N)$ in [14], [15] and [16]).

Let E be an arbitrary graph, K be any field, A be the endomorphism ring of $L := L_K(E)$ as a right L-module. By [6, Proposition 3.1], if x is a regular inverse of $a \in A$, then, choosing an idempotent $u \in L$ satisfying $ua = a = au$ so that $\lambda_a = \lambda_{ua}$, there is a regular inverse $\lambda_{f(u)}$ of λ_a in L.

If $a \in A$ is a regular element, then λ_a is the regular element in L by [6, Proposition 3.1].

Lemma 1. *1. For any $a, b \in L$, $\lambda_{a+b} = \lambda_a + \lambda_b$.*

2. If $f, g \in A$ such that $\lambda_{f(x)} \in \{\lambda_{g(x)}{}^{(1)}\}$ for all $x \in L$, then $f \in \{g^{(1)}\}$.

3. If $f, g \in A$ such that $f + g$ is a regular element, then $\lambda_{f(x)+g(x)}$ is regular element in L, for any $x \in L$.

Proof. (1) For every $x \in L$, we get

$$\begin{aligned}\lambda_{a+b}(x) &= (a+b)x \\ &= ax + bx \\ &= \lambda_a(x) + \lambda_b(x) \\ &= (\lambda_a + \lambda_b)(x),\end{aligned}$$

which implies $\lambda_{a+b} = \lambda_a + \lambda_b$.

(2) Let $\lambda_{f(x)} \in \{\lambda_{g(x)}{}^{(1)}\}$ for all $x \in L$. Then $\lambda_{g(x)}\lambda_{f(x)}\lambda_{g(x)} = \lambda_{g(x)}$, for all $x \in L$. So $\lambda_{g(x)f(x)g(x)} = \lambda_{g(x)}$, and hence $g(x)f(x)g(x) = f(x)$ for all $x \in L$.

(3) Take $x \in L$. Since $f + g$ is a regular element in A, there is an $h \in A$ such that $f + g = (f + g)h(f + g)$. Hence,

$$\begin{aligned}\lambda_{f(x)+g(x)} &= \lambda_{(f(x)+g(x))h(x)(f(x)+g(x))} \\ &= \lambda_{f(x)+g(x)}\lambda_{h(x)}\lambda_{f(x)+g(x)}.\end{aligned}$$

Thus $\lambda_{f(x)+g(x)}$ is regular element in L for all $x \in L$. □

In [8, Theorem 1], the authors obtained a kind of symmetry of regular rings which is interesting and useful in the theory of shorted operators (cf. [9] and [13]). So we have this symmetry property indeed holds for endomorphism rings of Leavitt path algebras:

Corollary 1. Let $f, g \in A$. If $f + g$ is a regular element, then the following are equivalent :

1. $fA \oplus gA = (f + g)A$
2. $Af \oplus Ag = A(f + g)$
3. $fA \cap gA = (0) = Af \cap Ag$.

Similarly, we have the following equivalent implications under the assumption regularity of $f + g$ by Lemma 1:

4. $\lambda_{f(x)}L \oplus \lambda_{g(x)}L = \lambda_{f(x)+g(x)}L$, for all $x \in L$,
5. $L\lambda_{f(x)} \oplus L\lambda_{g(x)} = L\lambda_{f(x)+g(x)}$, for all $x \in L$,
6. $\lambda_{f(x)}L \cap \lambda_{g(x)}L = (0) = L\lambda_{f(x)} \cap L\lambda_{g(x)}$, for all $x \in L$,

Theorem 1. *Let $f, g \in A$. Assume that $f+g$ is a regular element.*

1. *If $fA \oplus gA = (f+g)A$ (or equivalently, $Af \oplus Af = A(f+g)$), then $\lambda_{f(x)}L \oplus \lambda_{g(x)}L = \lambda_{f(x)+g(x)}L$, for all $x \in L$ (or equivalently, $L\lambda_{f(x)} \oplus L\lambda_{g(x)} = L\lambda_{f(x)+g(x)}$, for all $x \in L$) .*

2. *If $fA \cap gA = (0) = Af \cap Ag$, then $\lambda_{f(x)}L \cap \lambda_{g(x)}L = (0) = L\lambda_{f(x)} \cap L\lambda_{g(x)}$, for all $x \in L$.*

Proof. Let $Af \oplus Ag = A(f+g)$. Since $f+g$ is a regular element, by Lemma 1, we get $\lambda_{f(x)+g(x)}$ is regular in L. Hence there exists an $h \in A$ such that $\lambda_{f(x)+g(x)} = \lambda_{f(x)+g(x)}\lambda_{h(x)}\lambda_{f(x)+g(x)}$ for all $x \in L$. Write $f = r(f+g)$ and $g = s(f+g)$ for some $r, s \in A$. Clearly,

$$\begin{aligned}\lambda_{f(x)} &= \lambda_{r(x)(f(x)+g(x))}\\ &= \lambda_{r(x)}\lambda_{f(x)+g(x)}\end{aligned}$$

and

$$\begin{aligned}\lambda_g(x) &= \lambda_{s(x)(f(x)+g(x))}\\ &= \lambda_{s(x)}\lambda_{f(x)+g(x)}.\end{aligned}$$

They imply that

$$\begin{aligned}\lambda_{f(x)}\lambda_{h(x)}\lambda_{f(x)+g(x)} &= \lambda_{r(x)}\lambda_{f(x)+g(x)}\lambda_{h(x)}\lambda_{f(x)+g(x)}\\ &= \lambda_{r(x)}\lambda_{f(x)+g(x)}\\ &= \lambda_{f(x)}\end{aligned}$$

and

$$\begin{aligned}\lambda_{g(x)}\lambda_{h(x)}\lambda_{f(x)+g(x)} &= \lambda_{s(x)}\lambda_{f(x)+g(x)}\lambda_{h(x)}\lambda_{f(x)+g(x)}\\ &= \lambda_{s(x)}\lambda_{f(x)+g(x)}\\ &= \lambda_{g(x)}.\end{aligned}$$

Note that $\lambda_{f(x)} \in \lambda_{f(x)}L$ and $\lambda_{g(x)} \in \lambda_{g(x)}L$. Since $Af \oplus Ag = A(f+g)$ and $Af \cap Ag = (0)$, by Lemma 1,

$$\lambda_{f(x)} = \lambda_{f(x)}\lambda_{h(x)}\lambda_{f(x)+g(x)} = \lambda_{f(x)}\lambda_{h(x)}\lambda_{f(x)} + \lambda_{f(x)}\lambda_{h(x)}\lambda_{g(x)}$$

which implies

$$\lambda_{f(x)} = \lambda_{f(x)}\lambda_{h(x)}\lambda_{f(x)}$$

$$0 = \lambda_{f(x)}\lambda_{h(x)}\lambda_{g(x)}$$

$$\lambda_{g(x)} = \lambda_{g(x)}\lambda_{h(x)}\lambda_{f(x)+g(x)} = \lambda_{g(x)}\lambda_{h(x)}\lambda_{f(x)} + \lambda_{g(x)}\lambda_{h(x)}\lambda_{g(x)},$$

and so

$$\lambda_{g(x)} = \lambda_{g(x)}\lambda_{h(x)}\lambda_{g(x)}$$

$$0 = \lambda_{g(x)}\lambda_{h(x)}\lambda_{f(x)}.$$

Hence,

$$\begin{aligned}\lambda_{f(x)+g(x)} &= \lambda_{f(x)} + \lambda_{g(x)}\\ &= \lambda_{f(x)}\lambda_{h(x)}\lambda_{f(x)} + \lambda_{f(x)}\lambda_{h(x)}\lambda_{g(x)} + \lambda_{g(x)}\lambda_{h(x)}\lambda_{f(x)} + \lambda_{g(x)}\lambda_{h(x)}\lambda_{g(x)}\\ &= (\lambda_{f(x)} + \lambda_{g(x)})\lambda_{h(x)}\lambda_{f(x)} + (\lambda_{f(x)} + \lambda_{g(x)})\lambda_{h(x)}\lambda_{g(x)}\\ &= (\lambda_{f(x)} + \lambda_{g(x)})(\lambda_{h(x)}\lambda_{f(x)} + \lambda_{h(x)}\lambda_{g(x)}),\end{aligned}$$

which implies $\lambda_{f(x)+g(x)}L = \lambda_{f(x)}L + \lambda_{g(x)}L$.

Now we show that $\lambda_{f(x)}L \cap \lambda_{g(x)}L = 0$. Let $y \in \lambda_{f(x)}L \cap \lambda_{g(x)}L$. Then $y = \lambda_{f(x)}k = \lambda_{g(x)}l$ for some $k, l \in L$. Thus

$$\begin{aligned} y &= \lambda_{f(x)}k \\ &= \lambda_{f(x)}\lambda_{h(x)}\lambda_{f(x)}k \\ &= \underbrace{\lambda_{f(x)}\lambda_{h(x)}\lambda_{g(x)}}_{0}l \\ &= 0 \end{aligned}$$

and so $\lambda_{f(x)}L \cap \lambda_{g(x)}L = 0$. Finally, $\lambda_{f(x)}L \oplus \lambda_{g(x)}L = \lambda_{f(x)+g(x)}L$. $\square$

According to [8] and [9], for any two elements a, b in a von Neumann regular ring R, the relations $\leq^{\oplus}$, $\leq^{-}$ and $\leq^{s}$ on R are defined as follows:

$a \leq^{\oplus} b$ if and only if $bR = aR \oplus (b-a)R$, and called it the direct sum partial order.

$a \leq^{-} b$ if there exists an $x \in R$ such that $ax = bx$ and $xa = xb$, where $axa = a$, and called it that a is less than or equal to b under the minus partial order.

$a \leq^{s} b$, if $aR \subseteq bR$ and $Ra \subseteq Rb$.

It is easy to see that $\leq^{s}$ is pre-order and that $a \leq^{-} b$ implies $a \leq^{s} b$.

Proposition 1. *If A is regular and $f, g \in A$, then the following conditions hold.*

1. *If $f \leq^{\oplus} g$, then $\lambda_{f(x)} \leq^{\oplus} \lambda_{g(x)}$ for all $x \in L$*
2. *If $f \leq^{-} g$, then $\lambda_{f(x)} \leq^{-} \lambda_{g(x)}$ for all $x \in L$.*
3. *If $f \leq^{s} g$, then $\lambda_{f(x)} \leq^{s} \lambda_{g(x)}$ for all $x \in L$.*

Proof. (1) If $f \leq^{\oplus} g$ then we have $gA = fA \oplus (g-f)A$. By Theorem 1 and Lemma 1,

$$\lambda_{f(x)}L \oplus \lambda_{f(x)-g(x)}L = \lambda_{g(x)}L$$

which implies

$$\lambda_{f(x)}L \oplus (\lambda_{f(x)} - \lambda_{g(x)})L = \lambda_{g(x)}L$$

for any $x \in L$. Hence, $\lambda_{f(x)} \leq^{\oplus} \lambda_{g(x)}$ for all $x \in L$.

(2) By the hypothesis, there is an $h \in A$ such that $fh = gh$ and $hf = hg$, where $fhf = f$. Take any $x \in L$. It is easy to see that $\lambda_{f(x)h(x)} = \lambda_{g(x)h(x)}$ implies $\lambda_{f(x)}\lambda_{h(x)} = \lambda_{g(x)}\lambda_{h(x)}$, and $\lambda_{h(x)f(x)} = \lambda_{h(x)g(x)}$ implies $\lambda_{h(x)}\lambda_{f(x)} = \lambda_{h(x)}\lambda_{g(x)}$, where $\lambda_{f(x)} = \lambda_{f(x)h(x)f(x)} = \lambda_{f(x)}\lambda_{h(x)}\lambda_{f(x)}$. So $\lambda_{f(x)} \leq^{-} \lambda_{g(x)}$ for any $x \in L$.

(3) Let $x \in L$. Since $f \leq^{s} g$ we have $fA \subseteq gA$ and $Af \subseteq Ag$, so there exist an $h \in A$ such that $f = gh$. We show, $\lambda_{f(x)} \leq^{s} \lambda_{g(x)}$ for all $x \in L$. Take any $y \in \lambda_{f(x)}L$. Then $y = \lambda_{f(x)}z$ for some $z \in L$. Clearly,

$$\begin{aligned} y &= \lambda_{f(x)}z \\ &= f(\lambda_x)z \\ &= g(\lambda_x)h(\lambda_x)z \\ &= \lambda_{g(x)}\lambda_{h(x)}z. \end{aligned}$$

Therefore, $y \in \lambda_{g(x)}L$ and so $\lambda_{f(x)}L \subseteq \lambda_{g(x)}L$. Similarly, $\lambda_{g(x)}L \subseteq \lambda_{f(x)}L$. Thus $\lambda_{f(x)} \leq^s \lambda_{g(x)}$ for all $x \in L$. $\square$

Remark 1. *If A is regular and $a, b \in A$, then each of the statements (1)-(3) in Corollary 1 is equivalent to*

(1) $f \leq^- f + g$

and each of the statements (4)-(6) in Corollary 1 is equivalent to

(2) $\lambda_{f(x)} \leq^- \lambda_{f(x)+g(x)}$, *for all $x \in L$.*

Let us continue with a study of the minus partial order in terms on a (strong/weak) regular inverse of an element of the regular ring A.

Theorem 2. *If A is regular and $f, g \in A$, then the following conditions are equivalent*

1. $f \leq^- g$
2. $\lambda_{f(x)} \leq^- \lambda_{g(x)}$, *for all $x \in L$.*
3. $\{g^{(1)}\} \subseteq \{f^{(1)}\}$
4. $\{{\lambda_{g(x)}}^{(1)}\} \subseteq \{{\lambda_{f(x)}}^{(1)}\}$, *for all $x \in L$.*
5. $\{g^{(1,2)}\} \subseteq \{f^{(1)}\}$
6. $\{{\lambda_{g(x)}}^{(1,2)}\} \subseteq \{{\lambda_{f(x)}}^{(1)}\}$, *for all $x \in L$.*

Proof. (1) $\Rightarrow$ (2) This follows from Proposition 1.

(1) $\Rightarrow$ (3) As $f \leq^- g$, there exists some $h \in A$ such that $fh = gh$ and $hf = hg$ where $fhf = f$. Clearly,

$$f(x) = f(x)h(x)f(x) = f(x)h(x)g(x) = g(x)h(x)f(x)$$

for any $x \in L$. For any $t \in \{g^{(1)}\}$ and for any $x \in L$, we have,

$$\begin{aligned} f(x)t(x)f(x) &= (f(x)h(x)g(x))t(x)(g(x)h(x)f(x)) \\ &= f(x)h(x)(g(x)t(x)g(x))h(x)f(x) \\ &= f(x)h(x)g(x)h(x)f(x) \\ &= f(x)h(x)f(x) = f(x), \end{aligned}$$

which implies $t \in \{f^{(1)}\}$, i.e., $\{g^{(1)}\} \subseteq \{f^{(1)}\}$.

(2) $\Rightarrow$ (4) Since $\lambda_{f(x)} \leq^- \lambda_{g(x)}$ for all $x \in L$, there exists a $\lambda_{h(x)} \in \{{\lambda_{f(x)}}^{(1)}\}$ such that $\lambda_{f(x)}\lambda_{h(x)} = \lambda_{g(x)}\lambda_{h(x)}$ and $\lambda_{h(x)}\lambda_{f(x)} = \lambda_{h(x)}\lambda_{g(x)}$. For any $z \in \{{\lambda_{g(x)}}^{(1)}\}$,

$$\begin{aligned} \lambda_{f(x)}z\lambda_{f(x)} &= \lambda_{f(x)}\lambda_{h(x)}\lambda_{g(x)}z\lambda_{g(x)}\lambda_{h(x)}\lambda_{f(x)} \\ &= \lambda_{f(x)}\lambda_{h(x)}\lambda_{g(x)}\lambda_{h(x)}\lambda_{f(x)} \\ &= \lambda_{f(x)}\lambda_{h(x)}\lambda_{f(x)} \\ &= \lambda_{f(x)} \end{aligned}$$

which implies $z \in \{{\lambda_{f(x)}}^{(1)}\}$. Thus $\{{\lambda_{g(x)}}^{(1)}\} \subseteq \{{\lambda_{f(x)}}^{(1)}\}$ for all $x \in L$.

(3) $\Rightarrow$ (4) Let $\{g^{(1)}\} \subseteq \{f^{(1)}\}$. Take any $\lambda_{h(x)} \in \{\lambda_{g(x)}{}^{(1)}\}$ for all $x \in L$. Then

$$\begin{aligned} \lambda_{g(x)} &= \lambda_{g(x)}\lambda_{h(x)}\lambda_{g(x)} \\ &= \lambda_{g(x)h(x)g(x)}, \end{aligned}$$

which implies $g(x) = g(x)h(x)g(x)$, i.e., $g = ghg$. Thus, $h \in \{g^{(1)}\}$. By the hypothesis, $h \in \{f^{(1)}\}$. Also, $f(x) = f(x)h(x)f(x)$ implying

$$\begin{aligned} \lambda_{f(x)} &= \lambda_{f(x)h(x)f(x)} \\ &= \lambda_{f(x)}\lambda_{h(x)}\lambda_{f(x)}, \end{aligned}$$

and so $\lambda_{h(x)} \in \{\lambda_{f(x)}{}^{(1)}\}$. Hence $\{\lambda_{g(x)}{}^{(1)}\} \subseteq \{\lambda_{f(x)}{}^{(1)}\}$ for all $x \in L$.

(3) $\Rightarrow$ (5) and (4) $\Rightarrow$ (6) are trivial.

(5) $\Rightarrow$ (1) Fix $h \in \{g^{(1,2)}\}$ and set $r = gh, s = hg$. By the hypothesis, $h \in \{f^{(1)}\}$. So we get

$$\begin{aligned} f(x)h(x)f(x) &= f(x) \\ g(x)h(x)g(x) &= g(x) \\ h(x)g(x)h(x) &= h(x) \end{aligned}$$

for all $x \in L$. On the other hand,

$$\begin{aligned} f(x) &= f(x)h(x)f(x) \\ &= f(x)h(x)g(x)h(x)f(x) \\ &= f(x)s(x)h(x)f(x) \\ &= f(x)h(x)r(x)f(x). \end{aligned}$$

Let $t = hfh$. Then, for any $x \in L$,

$$\begin{aligned} t(x) &= h(x)f(x)h(x) \\ &= h(x)g(x)h(x)f(x)h(x) \\ &= h(x)f(x)h(x)g(x)h(x) \\ &= h(x)g(x)h(x)f(x)h(x)g(x)h(x) \end{aligned}$$

which implies

$$\begin{aligned} t(x) &= h(x)r(x)f(x)s(x)h(x) \\ &= s(x)h(x)f(x)h(x)r(x). \end{aligned}$$

Since $h(x) = h(x)g(x)h(x) = h(x)r(x) = s(x)h(x)$ for all $x \in L$, we get

$$h = hr = sh \in \{g^{(1,2)}\} \subseteq \{f^{(1)}\},$$

$$f(x)s(x) = f(x)h(x)(r(x)f(x)s(x)) = f(x)h(x)f(x) = f(x)$$

and

$$r(x)f(x) = (r(x)f(x)s(x))h(x))f(x) = f(x)h(x)f(x) = f(x).$$

Thus, for all $x \in L$,

$$\begin{aligned} f(x)t(x)f(x) &= f(x)(h(x)f(x)h(x))f(x) \\ &= (f(x)h(x)f(x))h(x)f(x) \\ &= f(x)h(x)f(x) \\ &= f(x), \end{aligned}$$

$$\begin{aligned} f(x)t(x) &= f(x)(h(x)r(x)f(x)h(x)) \\ &= (f(x)h(x)r(x)f(x))h(x) \\ &= f(x)h(x) \\ &= r(x)f(x)h(x) \\ &= g(x)h(x)f(x)h(x) \\ &= g(x)t(x) \end{aligned}$$

and

$$\begin{aligned} t(x)f(x) &= (h(x)f(x)s(x)h(x))f(x) \\ &= h(x)(f(x)s(x)h(x)f(x)) \\ &= h(x)f(x) \\ &= h(x)f(x)s(x) \\ &= h(x)f(x)h(x)g(x) \\ &= t(x)g(x). \end{aligned}$$

Hence, $f \leq^- g$. □

It is well known that the minus partial order is a partial order on A, when A is regular.

Theorem 3. *If A is regular and $f, g \in A$, then the following conditions are equivalent.*

1. $f \leq^s g$
2. $\lambda_{f(x)} \leq^s \lambda_{g(x)}$ *for all* $x \in L$.
3. $f = gg^{(1)}f = fg^{(1)}g$
4. $\lambda_{f(x)} = \lambda_{g(x)}\lambda_{g(x)}{}^{(1)}\lambda_{f(x)} = \lambda_{f(x)}\lambda_{g(x)}{}^{(1)}\lambda_{g(x)}$ *for all* $x \in L$.
5. $fg^{(1)}g$ *is invariant under the all choices of* $g^{(1)}$.
6. $\lambda_{f(x)}\lambda_{g(x)}{}^{(1)}\lambda_{f(x)}$ *is invariant under the all choices of* $\lambda_{g(x)}{}^{(1)}$ *for all* $x \in L$.

Proof. (1) ⇒ (2) This follows from Proposition 1.

(1) ⇒ (3) Let $f \leq^s g$. Then $fA \subseteq gA$ and $Af \subseteq Ag$. There exist $h, t \in A$ such that $f = gh$ and $f = tg$. For all $x \in L$ and for all $g^{(1)} \in \{g^{(1)}\}$, we obtain

$$\begin{aligned} f(x) &= g(x)h(x) \\ &= (g(x)g^{(1)}(x)g(x))h(x) \\ &= g(x)g^{(1)}(x)f(x) \end{aligned}$$

which implies $f = gg^{(1)}f$ and

$$\begin{aligned} f(x) &= t(x)g(x) \\ &= t(x)(g(x)g^{(1)}(x)g(x)) \\ &= f(x)g^{(1)}(x)g(x) \end{aligned}$$

which implies $f = fg^{(1)}g$.

(3) $\Rightarrow$ (4) Let $f = gg^{(1)}f = fg^{(1)}g$. For all $x \in L$, we have

$$f(\lambda_x) = g(\lambda_x)g^{(1)}(\lambda_x)f(\lambda_x)$$

implying

$$\lambda_{f(x)} = \lambda_{g(x)}{\lambda_{g(x)}}^{(1)}\lambda_{f(x)}$$

and

$$f(\lambda_x) = f(\lambda_x)g^{(1)}(\lambda_x)g(\lambda_x)$$

implying

$$\lambda_{f(x)} = \lambda_{f(x)}{\lambda_{g(x)}}^{(1)}\lambda_{g(x)}.$$

(3) $\Rightarrow$ (5) Let $h, t \in \{g^{(1)}\}$ be arbitrary. For every $x \in L$, we obtain that

$$\begin{aligned} f(x)h(x)f(x) &= (f(x)g^{(1)}(x)g(x))h(x)(g(x)g^{(1)}(x)f(x)) \\ &= f(x)g^{(1)}(x)(g(x)h(x)g(x))g^{(1)}(x)f(x) \\ &= f(x)g^{(1)}(x)g(x)g^{(1)}(x)f(x) \\ &= (f(x)g^{(1)}(x)g(x))g^{(1)}(x)f(x) \\ &= f(x)g^{(1)}(x)f(x) \end{aligned}$$

which implies $fhf = fg^{(1)}f$ and

$$\begin{aligned} f(x)t(x)f(x) &= (f(x)g^{(1)}(x)g(x))t(x)(g(x)g^{(1)}(x)f(x)) \\ &= f(x)g^{(1)}(x)(g(x)t(x)g(x))g^{(1)}(x)f(x) \\ &= f(x)g^{(1)}(x)g(x)g^{(1)}(x)f(x) \\ &= (f(x)g^{(1)}(x)g(x))g^{(1)}(x)f(x) \\ &= f(x)g^{(1)}(x)f(x) \end{aligned}$$

which implies $ftf = fg^{(1)}f$. Hence $fg^{(1)}f$ is not depend on $g^{(1)}$.

(4) $\Rightarrow$ (6) Let $\lambda_{h(x)} \in \{{\lambda_{g(x)}}^{(1)}\}$ with $x \in L$. For every ${\lambda_{g(x)}}^{(1)} \in \{{\lambda_{g(x)}}^{(1)}\}$, we have $\lambda_{f(x)} = \lambda_{f(x)}{\lambda_{g(x)}}^{(1)}\lambda_{f(x)}$.

Then $(\lambda_{g(x)}{\lambda_{g(x)}}^{(1)})\lambda_{g(x)} = (\lambda_{g(x)}{\lambda_{g(x)}}^{(1)})\lambda_{f(x)}{\lambda_{g(x)}}^{(1)}\lambda_{g(x)}$, and so $\lambda_{f(x)} = \lambda_{g(x)}{\lambda_{g(x)}}^{(1)}\lambda_{f(x)}{\lambda_{g(x)}}^{(1)}\lambda_{g(x)}$. So, $\lambda_{f(x)} = \lambda_{g(x)}\lambda_{h(x)}\lambda_{f(x)}\lambda_{h(x)}\lambda_{g(x)}$.

Now,

$$\begin{aligned} \lambda_{f(x)}{\lambda_{g(x)}}^{(1)}\lambda_{f(x)} &= (\lambda_{g(x)}\lambda_{h(x)}\lambda_{f(x)}\lambda_{h(x)}\lambda_{g(x)}){\lambda_{g(x)}}^{(1)}(\lambda_{g(x)}\lambda_{h(x)}\lambda_{f(x)}\lambda_{h(x)}\lambda_{g(x)}) \\ &= \lambda_{g(x)}\lambda_{h(x)}\lambda_{f(x)}\lambda_{h(x)}\underbrace{(\lambda_{g(x)}{\lambda_{g(x)}}^{(1)}\lambda_{g(x)})}_{\lambda_{g(x)}}\lambda_{h(x)}\lambda_{f(x)}\lambda_{h(x)}\lambda_{g(x)} \\ &= \lambda_{g(x)}\lambda_{h(x)}\lambda_{f(x)}\lambda_{h(x)}\lambda_{g(x)}\lambda_{h(x)}\lambda_{f(x)}\lambda_{h(x)}\lambda_{g(x)} \end{aligned}$$

which does not depend on ${\lambda_{g(x)}}^{(1)}$.

(5) $\Rightarrow$ (6) Let $\lambda_{h(x)}, \lambda_{t(x)} \in \{{\lambda_{g(x)}}^{(1)}\}$ with $x \in L$ and let $fg^{(1)}f$ be invariant under the all choices of $\{g^{(1)}\}$. By Lemma 1, $h, t \in \{g^{(1)}\}$. Then

$$\begin{aligned} \lambda_{f(x)}\lambda_{h(x)}\lambda_{f(x)} &= f(\lambda_x)h(\lambda_x)f(\lambda_x) \\ &= f(\lambda_x)t(\lambda_x)f(\lambda_x) \\ &= \lambda_{f(x)}\lambda_{t(x)}\lambda_{f(x)}. \end{aligned}$$

Hence, $\lambda_{f(x)}{\lambda_{g(x)}}^{(1)}\lambda_{f(x)}$ is invariant under the all choices of $\{{\lambda_{g(x)}}^{(1)}\}$ for all $x \in L$.

(5) $\Rightarrow$ (1) Fix $h \in \{g^{(1)}\}$ and set

$$e_1 := gh, \quad e_2 := 1 - gh, \quad f_1 := hg, \quad f_2 := 1 - hg$$

Then $1 = e_1 + e_2$ and $1 = f_1 + f_2$ are two decomposition of identity of the endomorphism ring A of $L_K(E)$. If $g^{(1)} \in \{g^{(1)}\}$ then, for every $x \in L$,

$$\begin{aligned} f_1(x)g^{(1)}(x)e_1(x) &= (h(x)g(x))g^{(1)}(x)(g(x)h(x)) \\ &= h(x)(g(x)g^{(1)}(x)g(x))h(x) \\ &= h(x)g(x)h(x) \end{aligned}$$

so we have $f_1g^{(1)}e_1 = hgh$ and

$$g(x)(f_1(x)g^{(1)}(x)e_1(x))g(x) = g(x)(h(x)g(x)h(x))g(x).$$

Thus $g(x)g^{(1)}(x)g(x) = g(x)$ so $gg^{(1)}g = g$. Therefore $g^{(1)} \in \{g^{(1)}\}$ if and only if

$$g^{(1)} = \begin{bmatrix} hgh & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{bmatrix}$$

where $\alpha_{i,j} \in f_iAe_j$, $i, j = 1, 2$ are arbitrary. Now,

$$\begin{aligned} & f(x)g^{(1)}(x)f(x) \\ = \; & \begin{bmatrix} f(x)f_1(x) & f(x)f_2(x) \end{bmatrix} \begin{bmatrix} h(x)g(x)h(x) & \alpha_{12}(x) \\ \alpha_{21}(x) & \alpha_{22}(x) \end{bmatrix} \begin{bmatrix} e_1(x)f(x) \\ e_2(x)f(x) \end{bmatrix} \\ = \; & f(x)f_1(x)h(x)g(x)h(x)e_1(x)f(x) + f(x)f_1(x)\alpha_{12}(x)e_2(x)f(x) \\ & + f(x)f_2(x)\alpha_{21}(x)e_1(x)f(x) + f(x)f_2(x)\alpha_{22}(x)e_2(x)f(x) \\ = \; & f(x)h(x)g(x)h(x)f(x) + f(x)\alpha_{12}(x)f(x) + f(x)\alpha_{21}(x)f(x) \\ & + f(x)\alpha_{22}(x)f(x) \end{aligned}$$

does not depend on α_{12}, α_{21}, α_{22}. Setting $\alpha_{12} = \alpha_{22} = 0$, we have $f(x)\alpha_{21}(x)f(x) = 0$ for all $\alpha_{21} \in f_2Ae_1$ and for all $x \in L$. Then $f(x)f_2(x)t(x)e_1(x)f(x) = 0$ for all $t \in A$ and for all $x \in L$. Multiplying this equation by $e_1(x)$ from left and by $f_2(x)$ from the right, we get, for all $x \in L$

$$(e_1(x)f(x)f_2(x))t(x)(e_1(x)f(x)f_2(x)) = 0.$$

Since A is regular we can choose $t = (e_1ff_2)^{(1)} \in \{e_1ff_2^{(1)}\}$. Hence $e_1(x)f(x)f_2(x)$ for all $x \in L$. Similarly, $e_2(x)f(x)f_1(x) = 0$ and $e_2(x)f(x)f_2(x) = 0$, so we conclude, for all $x \in L$,

$$f(x) = e_1(x)f(x)f_1(x) = g(x)h(x)f(x)h(x)g(x)$$

which implies $f = e_1ff_1 = ghfhg$. Consequently, $fA \subseteq gA$ and $Af \subseteq Ag$.

(6) $\Rightarrow$ (2) This is similar to (5) $\Rightarrow$ (1). $\square$

References

[1] Abrams, G. and Aranda Pino, G., *The Leavitt path algebra of a graph*, J. Algebra, **293** (2005), no. 2, 319-334.

[2] Abrams, G. and Rangaswamy, K.M., *Regularity conditions for arbitrary Leavitt path algebras*, Algebr. Represent. Theory, **13** (2010), no. 3, 319-334.

[3] Anderson Jr., W.N., *Shorted operators*, SIAM J. Appl. Math. **20** (1971), 522-525.

[4] Anderson Jr., W.N. and Trapp, G.E., *Shorted operators II*, SIAM J. Appl. Math. **28** (1975), 60-71.

[5] Ara, P., Moreno, M.A. and Pardo, E., *Nonstable K-theory for graph algebras*, Algebr. Represent. Theory, **10** (2007), 157-178.

[6] Aranda Pino, G., Rangaswamy, K.M. and Siles Molina, M., *Endomorphism rings of Leavitt path algebras*, J. Pure Appl. Algebra, **219** (2015), 5330-5343.

[7] Hartwig, R.E., *How to partially order regular elements*, Math. Japon, **25** (1980), 1-13.

[8] Jain, S.K. and Prasad, K.M., *Right-left symmetry of $aR \oplus bR = (a+b)R$ in regular rings*, J. Pure Appl. Algebra, **133** (1998), no. 1-2, 141-142.

[9] Jain, S.K., Blackwood, B., Prasad, K.M. and Srivastava, A.K.,*Shorted operators relative to a partial order in a regular ring* , Comm. Algebra, **37** (2009), no. 11, 4141-4152.

[10] Kasch, F. and Mader, A., *Regularity and substructures of hom.*, Frontiers in Mathematics, Birkhäuser Verlag, Basel, 2009.

[11] Koşan, M.T., Quynh, T.C. and Şahinkaya S.,*On rings with associated elements*, Comm. Algebra, **45** (2017), no. 7, 2747-2756.

[12] Mitra, S.K., *The minus partial order and shorted matrix*, Linear Algebra Appl. **83** (1986), 1-27.

[13] Quynh, T.C., Abyzov, A. and Koşan, M.T., *On (unit-)regular morphisms*, Lobachevskii J. Math. **40** (2019), 2103–2110.

[14] Quynh, T.C. and Koşan, M.T., *On the substructures $\triangle$ and ∇* , Rocky Mountain J. Math. **45** (2015), no. 2, 661–674.

[15] Quynh, T.C., Koşan, M.T. and Hai, P.T., *A note on regular morphisms*, Ann. Univ. Sci. Budapest. Sect. Comput. **41** (2013), 249–260.

[16] Quynh, T.C., Koşan, M.T. and Thuyet, L.V., *On (semi)regular morphisms*, Comm. Algebra, **41** (2013), no. 8, 2933-2947.