Bulletin of the *Transilvania* University of Braşov
Series III: Mathematics and Computer Science, Vol. 3(65), No. 1 - 2023, 157-170
https://doi.org/10.31926/but.mif.2023.3.65.1.12

RINGS IN WHICH NOT INVERTIBLE ELEMENTS ARE UNIQUELY CLEAN

Yonghua GUO[*,1] and Hua JIANG[2]

Abstract

We call a ring R a generalized uniquely clean (or GUC for short) if every not invertible element in R is uniquely clean. Let R be a ring. It is shown that R is GUC if and only if it is a local ring or a uniquely clean ring. Thus the GUC ring is a generalization of the local ring. Some basic properties of GUC rings are proved.

2000 *Mathematics Subject Classification:* 16U99, 16L99, 16S34.
Key words: strongly clean ring, local ring, uniquely clean ring, group ring

1 Introduction

Throughout this paper, R denotes an associative ring with identity 1, all modules are unital left R-modules. An element a in a ring R is strongly clean, if $a = e + u$ and $eu = ue$ where $e^2 = e \in R$ and u is a unit. A ring R is called strongly clean if every element is strongly clean. Strongly clean rings are studied by many authors (cf. [7, 8, 9, 10, 11, 15]). An element a in a ring R is uniquely clean, if $a = e + u$ where $e^2 = e \in R$ and u is a unit, and this representation is unique. A ring R is called uniquely clean if every element is uniquely clean. Uniquely clean rings were first studied by Anderson and Camillo [1] in connection with commutative clean rings. Uniquely clean rings are always strongly clean by [14, Lemma 4]. But the uniquely clean element need not be

[*] *Corresponding author*, School of Mathematics and Statistics, Guangxi Normal University Guilin 541006, China, e-mail: yhguo@189.cn

School of Mathematics and Statistics, Guangxi Normal University Guilin 541006, China, e-mail: huajiangj@qq.com

strongly clean (cf. [12, Example 1.1]). Dorin Andrica and Grigore Călugăreanu discussed uniquely clean 2×2 invertible integral matrices in [3]. Recently, the rings whose clean elements are uniquely clean (CUC for short) are completely characterized by Grigore Călugăreanu and Yiqiang Zhou in [4]. It is clear that CUC ring need not be strongly clean. Thus, a natural problem is what uniquely clean subset of a ring R such that R is strongly clean. The class of the rings described by the title is a partial answer of the question.

For a ring R, $J(R)$ denotes the Jacobson radical of R and $U(R)$ is the group of units of R. We write $ucn(R)$ for the set of uniquely clean elements in R. The center of R is denoted by $Z(R)$. Other standard terminology and notation in rings and modules we refer the reader to [2].

2 Generalized uniquely clean rings and examples

Definition 2.1. A ring R is called generalized uniquely clean (or GUC for short) if every not invertible element in R is uniquely clean.

It is clear that uniquely clean rings are GUC rings. The following lemma will be used.

Lemma 2.2. Let R be a ring and $a \in R$. If a is uniquely clean, then so is $1 - a$.

Proof. Suppose that $a \in R$ is uniquely clean. Then there exists $e^2 = e \in R$ and $u \in U(R)$ such that $a = e + u$. Hence $1 - a = (1 - e) + (-u) \in R$. Suppose $1 - a = f + v$ where $f^2 = f \in R$ and $v \in U(R)$. Then $a = (1 - f) - v$. Since a is uniquely clean, there is $1 - f = e$ and then $1 - e = f$. □

Since 0 is uniquely clean in a GUC ring R and hence 1 is uniquely clean in R by Lemma 2.2.

Lemma 2.3. Every idempotent in a GUC ring R is central. Moreover, R is strongly clean.

Proof. If $e^2 = e \in R$, then e is uniquely clean (if e is not invertible, then it is uniquely clean; if e is a unit, then $e = 1$ is uniquely clean by Lemma 2.2). Therefore $e \in Z(R)$ by [12, Corollary 2.8]. □

Lemma 2.4. R_i is a ring for all $i \in I$. If $\prod_{i\in I} R_i$ is GUC, then each R_i is GUC.

Proof. Obviously each R_i is clean. For every element $a \notin U(R_i)$, there is an element $(1, \ldots, a, 1, \ldots) \in \prod_{i\in I} R_i$ that is not a unit. Because $\prod_{i\in I} R_i$ is GUC,

$(1,\ldots,a,1,\ldots)$ is uniquely clean in $\prod_{i\in I} R_i$. Then a is uniquely clean in R_i (otherwise, if a has two different clean decompositions at least, then $(1,\ldots,a,1,\ldots)$ has two different clean decompositions at least, a contradiction). $\square$

The converse of the Lemma 2.4 is false. For example, clearly $\mathbb{Z}_2$ and $\mathbb{Z}_3$ are GUC. But $\mathbb{Z}_2\times\mathbb{Z}_3$ is not GUC. Because $(0,2)$ is not invertible in $\mathbb{Z}_2\times\mathbb{Z}_3$ and $(0,2)=(1,0)+(1,2)=(1,1)+(1,1)$ are two different clean decompositions in $\mathbb{Z}_2\times\mathbb{Z}_3$.

Theorem 2.5. Let I be an index set with $|I|\geqslant 2$. If R_i is a ring for all $i\in I$, then $\prod_{i\in I} R_i$ is GUC if and only if each R_i $(i\in I)$ is uniquely clean.

Proof. First we know if each R_i is uniquely clean, then $\prod_{i\in I} R_i$ is uniquely clean by [14, Example 3] and then $\prod_{i\in I} R_i$ is GUC. Conversely, assume $\prod_{i\in I} R_i$ is GUC and R_j is not uniquely clean for a $j\in I$. Then there exists a is not uniquely clean in R_j. Hence $(0,\ldots,0,a,0,\ldots)$ is not uniquely clean in $\prod_{i\in I} R_i$. But $(0,\ldots,0,a,0,\ldots)$ is not invertible in $\prod_{i\in I} R_i$ and then $(0,\ldots,0,a,0,\ldots)$ is uniquely clean, a contradiction. Thus each R_i $(i\in I)$ is uniquely clean. $\square$

Corollary 2.6. Let R be a ring and $0\neq e^2=e\in R$. If R is GUC, then so is eRe. Specifically, if e is not trivial, then eRe is uniquely clean.

Proof. Because R is GUC, $R=eRe\oplus(1-e)R(1-e)$ clearly for $0\neq e^2=e$. Hence eRe is GUC by Lemma 2.4. Specifically, if e is not trivial, then eRe is uniquely clean by Theorem 2.5. $\square$

Corollary 2.7. Every GUC ring R is directly finite ($\forall a,\ b\in R,\ ab=1$ implies $ba=1$).

Proof. Assume $ab=1$, then $baba=b(ab)a=ba$. Then ba is an idempotent and ba is central by Lemma 2.3. So $ba=ba(ab)=a(ba)b=1$. $\square$

Example 2.8. Let R be a ring. If R is local then R is GUC.

Proof. Suppose ring R is local. Then R is clean and has only two idempotents that 0 and 1 by[14, Lemma 14]. So clearly, R is strongly clean. For every element $a\in R$ and $a\notin U(R)$, a has only a clean expression that $a=1+(a-1)$ where $a-1\in U(R)$ (Because R is local). Hence a is uniquely clean. $\square$

Proposition 2.9. For a ring R, the following statements are equivalent:

(1) R is a local ring;

(2) R is a GUC ring and 0 and 1 are the only idempotents in R;

(3) R is a strongly clean ring and 0 and 1 are the only idempotents in R;

(4) R is a clean ring and 0 and 1 are the only idempotents in R.

Proof. (1) $\Rightarrow$ (2) If R is local, then R is GUC by Example 2.8.

(2) $\Rightarrow$ (3) By Lemma 2.3.

(3) $\Rightarrow$ (4) It is clear.

(4) $\Rightarrow$ (1) By [14, Lemma 14]. □

Now we can give the structure theorem of GUC rings.

Theorem 2.10. For a ring R, the following statements are equivalent:

(1) R is a GUC ring;

(2) R is a local ring or R is a uniquely clean ring.

Proof. (1) $\Rightarrow$ (2) If a GUC ring R is not a local ring, then there exists idempotent e which is not trivial such that $R = eRe \oplus (1-e)R(1-e)$. Hence eRe and $(1-e)R(1-e)$ are uniquely clean by Corollary 2.6. So R is uniquely clean by [14, Example 3].

(2) $\Rightarrow$ (1) If R is a uniquely clean ring, obviously R is GUC. If R is local, then R is GUC by Example 2.8. □

Corollary 2.11. If R is GUC, then $R/J(R)$ is a Boolean ring or a division ring.

Proof. Let R is GUC, then R is uniquely clean or local. Then $R/J(R)$ is a Boolean ring by [14, Theorem 20] or a division ring. □

Corollary 2.12. If R is GUC, then every nonzero homomorphism image of R is GUC.

Proof. Let R is GUC, then R is local or uniquely clean. Because every nonzero homomorphism image of uniquely clean ring (local ring) is again uniquely clean (local ring) by [14, Theorem 22]. Hence every nonzero homomorphism image of R is GUC. □

We know that *units lift* module an ideal I of a ring R if whenever $\bar{u} \in U(R/I)$ there exists $v \in U(R)$ such that $\bar{v} = \bar{u}$. Similarly, we say that *idempotent lift* module an ideal I of a ring R if whenever $\bar{a}^2 = \bar{a} \in R/I$ there exists $e^2 = e \in R$ such that $\bar{e} = \bar{a} \in R/I$.

Lemma 2.13. Let R be a ring. Then units lift module $J(R)$.

Proof. Let $\bar{R} = R/J(R)$. For every $\bar{a} \in U(\bar{R})$, there exists $\bar{b} \in \bar{R}$ such that $\bar{a}\bar{b} = \bar{1}$. Let $\bar{a} = a + J(R)$ and $\bar{b} = b + J(R)$, where $a, b \in R$. So $1 - ab \in J(R)$ and hence $1 - ab$ is quasi-regular (Because each element in $J(R)$ is quasi-regular). So ab is invertible. Hence $a \in U(R)$. □

Proposition 2.14. Let R be a GUC ring and $\bar{R} = R/J(R)$. The following hold:

(1) For any $e^2 = e \in R$, $f^2 = f \in R$, $u \in U(R)$, if $e \neq f$, then $e + f \neq 1 + u$;

(2) For any $\bar{e}^2 = \bar{e} \in \bar{R}$, $\bar{f}^2 = \bar{f} \in \bar{R}$, $\bar{u} \in U(\bar{R})$, if $\bar{e} \neq \bar{f}$, then $\bar{e} + \bar{f} \neq \bar{1} + \bar{u}$.

Proof. (1) Assume $e^2 = e \in R$, $f^2 = f \in R$, $u \in U(R)$ and $e \neq f$, $e + f = 1 + u$. Then $e = (1 - f) + u = (1 - e) + (2e - 1)$. If e is a unit, then $e = 1$. Hence $f - u = 1 - e = 0$. So $f = u$ and then $f = 1 = e$, a contradiction.

(2) Given $\bar{e}^2 = \bar{e} \in \bar{R}$, $\bar{f}^2 = \bar{f} \in \bar{R}$, $\bar{u} \in U(\bar{R})$ and $\bar{e} \neq \bar{f}$. Because idempotents lift module $J(R)$ [14, Lemma 17], e and f are idempotents. Hence for any $u \in U(R)$, $e + f \neq 1 + u$ from (1). Then $\bar{e} + \bar{f} \neq \bar{1} + \bar{u}$ by Lemma 2.13. □

Chen et al. [6] call a ring *uniquely strongly clean* (or USC for short) if every element can be written uniquely as the sum of an idempotent and a unit that commute.

Example 2.15. $\mathbb{Z}_{(3)}$ is a GUC ring that is not uniquely clean and is not USC.

Proof. We know $\mathbb{Z}_{(3)} = \{a/b | a, b \in \mathbb{Z}, (3, b) = 1\}$ and $\mathbb{Z}_{(3)}$ is local. Hence $\mathbb{Z}_{(3)}$ is GUC. Clearly, $7/2 = 0 + 7/2 = 1 + 5/2$ are different clean decompositions in $\mathbb{Z}_{(3)}$. Thus $\mathbb{Z}_{(3)}$ is not uniquely clean and is not USC. □

A ring R is called a *semipotent* ring if every right (left) ideal $I \not\subseteq J(R)$ of R contains a nonzero idempotent.

Proposition 2.16. If R is a clean ring, then R is a semipotent ring.

Proof. If R is clean and $T \not\subseteq J(R)$ is a right (or left) ideal of R. Then there exists $0 \neq e^2 = e$ [14, Lemma 17]. Hence R is a semipotent ring. □

Proposition 2.17. Let R be a GUC ring and let $n = 1 + 1 + \cdots + 1$ (The sum of n ones and 1 is the identity of R). Then $n \in J(R)$ or $n \in U(R)$.

Proof. Suppose R is a GUC ring. If R is local then clearly $n \in J(R)$ or $n \in U(R)$ for each $n \in R$. If R is uniquely clean then $2 \in J(R)$ by [14, Lemma 18]. Thus $2n \in J(R)$ for each $n \in R$. Then $2n - 1 \in U(R)$. Therefore $n \in J(R)$ or $n \in U(R)$ for each $n \in R$. □

Example 2.18. An $n \times n$ matrix ring is never GUC for any $n>1$.

Proof. Assume a $n \times n$ matrix ring $S = \mathbb{M}_n(R)$ is GUC where R is a ring. We see that $a = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \in R$ is an idempotent. Hence a is central by Lemma 2.3. But

$$\begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} = 0,$$

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

a contradiction. □

We can find that a triangular matrix ring $\mathbb{T}_n(R)$ is never GUC for any $n>1$ where R is a ring by Example 2.18.

If R is a ring and $\alpha : R \to R$ is a ring endomorphism. let $R[[x,\alpha]]$ denote the ring of *skew formal power series* over R and the multiplication defined by $xr = \alpha(r)x$ for all $r \in R$. Let $R[[x]]$ be the ring of *formal power series* over R.

Proposition 2.19. Let R be a ring and $\alpha : R \to R$ is a ring endomorphism. Then $R[[x,\alpha]]$ is a GUC ring if and only if R is a GUC ring and $e = \alpha(e)$ for all $e^2 = e \in R$.

Proof. It is well known that the uniquely clean ring holds true by [14, Example 9]. It is well known that the situation of local ring is true. Hence we can get the result. □

Corollary 2.20. Let R be a ring, then the power series ring $R[[x]]$ is GUC if and only if R is GUC.

A ring R is called *left quasi-duo* (respectively *right quasi-duo*) if every maximal left (right) ideal of R is an ideal.

Proposition 2.21. Every GUC ring is left and right quasi-duo.

Proof. If R is uniquely clean, then R is left and right quasi-duo by [14, Proposition 23]. If R is local, then $J(R)$ is the only max left or right ideal of R and $J(R)$ is an ideal. Hence every GUC ring is left and right quasi-duo. □

3 Uniquely clean elements of GUC rings

We know that not invertible elements in a GUC ring are uniquely clean. But also there are some invertible elements in a GUC ring are uniquely clean. In this section, the set of uniquely clean elements in GUC rings was given.

According to [12, Definition 2.1], for a ring R,

$$ucn_0(R) = \left\{e + j|\ e^2 = e \in Z(R),\ j \in J(R)\right\}.$$

It is clear that $ucn_0(R) \subseteq ucn(R)$ by [12, Lemma 3.5].

Proposition 3.1. Let R be a GUC ring. Then idempotents lift uniquely module $J(R)$.

Proof. Because R is clean, idempotents lift module $J(R)$. Hence if $a^2 - a \in J(R)$, then there exists $e^2 = e \in R$ such that $e - a \in J(R)$. Assume $f - a \in J(R)$ where $f^2 = f \in R$. If $a \notin U(R)$, then $-a$ is uniquely clean and then $-a = (1 - e) - [1 - (e - a)] = (1 - f) - [1 - (f - a)]$. Note that $1 - (e - a)$ and $1 - (f - a)$ are units in R because $e - a \in J(R)$ and $f - a \in J(R)$. Hence $e = f$. If $a \in U(R)$, because $e - a \in J(R)$, then $ea^{-1} - 1 \in J(R)$ and then $1 - ea^{-1} \in J$. So $ea^{-1} = 1 - (1 - ea^{-1}) \in U(R)$. Similarly, we have $fa^{-1} \in U(R)$. Hence e and f are units in R. Thus $e = f = 1$. □

Lemma 3.2. If R is a local ring, then $ucn(R) = ucn_0(R)$.

Proof. Let R be a local ring and $a \in R$ is uniquely clean. There are two cases:

(i) If $a \notin U(R)$, then $a \in J(R)$ and then $a = 0 + a \in ucn_0R$;

(ii) If $a \in U(R)$, then $a - 1 \in J(R)$. Otherwise, $a = 0 + a = 1 + (a - 1)$ are two different clean decompositions. But a is uniquely clean, a contradiction. Hence $a = 1 + (a - 1) \in ucn_0(R)$.

Hence $a \in ucn_0(R)$ and then $ucn(R) = ucn_0(R)$. □

Lemma 3.3. If R is a uniquely clean ring, then $ucn(R) = ucn_0(R)$.

Proof. Let R be a uniquely clean ring with radical $J = J(R)$, then idempotents lift module $J(R)$ and R/J is Boolean by [14, Theroem 20]. Hence $ucn(R/J) = ucn_0(R/J)$. So $ucn(R) = ucn_0(R)$ by [12, Proposition 3.6]. □

Proposition 3.4. If R is a GUC ring, then $ucn(R) = ucn_0(R)$.

Proof. This can be concluded by Theorem 2.10, Lemma 3.2 and Lemma 3.3. □

Theorem 3.5. The following statements are equivalent for a ring R with radical $J = J(R)$:

(1) R is GUC;

(2) R/J is GUC and idempotents lift uniquely module J;

(3) R/J is GUC, idempotents lift module J and idempotents in R are central;

(4) For every not invertible element $a \in R$, there exists a unique central idempotent $e \in R$ such that $e - a \in J$.

Proof. (1) $\Rightarrow$ (2) It is easily proved by Corollary 2.12 and Proposition 3.1.

(2) $\Rightarrow$ (3) For every $e^2 = e \in R, r \in R$, we have $[e+(er-ere)]^2 = [e+(er-ere)]$ and $\bar{e}\bar{r} = \bar{r}\bar{e}$ where $\bar{e} = e + J, \bar{r} = r + J$. Then $\overline{er} = \overline{ere}$ and then $er - ere \in J$. Hence $e = e + (er - ere)$ by idempotents lift uniquely module J. So $er = ere$. Similarly, we have $re = ere$. Hence $er = re$.

(3) $\Rightarrow$ (4) For every $a \in R$ and $a \notin U(R)$, we have $\bar{a} = a + J \notin U(R/J)$. Because R/J is GUC, $\bar{a} \in ucn(R/J) = ucn_0(R/J)$ by Proposition 3.4. Thus there exists $(\bar{e})^2 = \bar{e} \in R/J$ such that $\bar{a} = \bar{e}$ and then $e - a \in J$. Assume $f - a \in J$ where $f^2 = f \in R$. Then $e - f = e - a - (f - a) \in J$. Clearly $e(1 - f)$ is an idempotent of R and $e(1 - f) = (e - f)(1 - f) \in J(R)$. Thus $e(1 - f) = 0$. So $e = ef$. Similarly, we have $f = ef$. Hence $e = f$.

(4) $\Rightarrow$ (1) For every $a \in R$ and $a \notin U(R)$, there exists a unique central idempotent $e \in R$ such that $e - a \in J$. Thus there exists $j \in J$ such that $a = e + j \in ucn_0(R) \subseteq ucn(R)$. Hence R is GUC.

□

Let M be a left R-module. It was proved in [6, Example 9] that $\alpha \in End(_RM)$ is USC if and only if there exist a unique decomposition $M = P \oplus Q$ where P, Q are α-invariant and $\alpha \mid_P$ and $(1-\alpha) \mid_Q$ are isomorphisms. So we can immediately come to the following example.

Example 3.6. Let M be a left R-module such that $End(_RM)$ is a GUC ring and $\alpha \in End(_RM)$. The following are equivalent:

(1) $\alpha \notin U(End(_RM))$;

(2) There exists a unique decomposition $M = P \oplus Q$ where P, Q are α-invariant and $\alpha\mid_P$ and $(1-\alpha)\mid_Q$ are isomorphisms.

Let A and B be rings. Then a bimodule ${}_AV_B$ is called *indecomposable bimodule* if both ${}_AV$ and V_B are indecomposable.

Example 3.7. Let $R = \begin{pmatrix} A & V \\ 0 & B \end{pmatrix}$, where A and B are GUC rings and ${}_AV_B$ is an indecomposable bimodule. The following are equivalent:

(1) R is strongly clean;

(2) If $1-a \in ucn(A), b \in ucn(B), v \in V$ and $1-a = (1-e)+j_1, b = f+j_2$ where $e^2 = e \in A$, $f^2 = f \in B$, $j_1 \in J(A)$, $j_2 \in J(B)$, then there exists $x \in V$ such that $x = ex + xf$ and $ev - vf = xb - ax$.

Proof. (1) $\Rightarrow$ (2) Let $\begin{pmatrix} a & v \\ 0 & b \end{pmatrix} \in R$ and $1-a \in ucn(A), b \in ucn(B)$. And there exists $e^2 = e \in A, f^2 = f \in B, j_1 \in J(A), j_2 \in J(B)$ such that $1-a = (1-e)+j_1, b = f+j_2$. Hence $a = e+(-j_1)$ and a is uniquely clean. Clearly, there exists $x \in V$ and $x = ex+xf$. Then $\begin{pmatrix} 1-e & x \\ 0 & 1-f \end{pmatrix}\begin{pmatrix} 1-e & x \\ 0 & 1-f \end{pmatrix} = \begin{pmatrix} 1-e & 2x-ex-xf \\ 0 & 1-f \end{pmatrix} = \begin{pmatrix} 1-e & x \\ 0 & 1-f \end{pmatrix}$. Hence $\begin{pmatrix} 1-e & x \\ 0 & 1-f \end{pmatrix}$ is an idempotent element in R. We have $2e-1-j_1 \in U(A), 2f-1+j_2 \in U(B)$. So $\begin{pmatrix} 2e-1-j_1 & v-x \\ 0 & 2f-1+j_2 \end{pmatrix} \in U(R)$ and then $\begin{pmatrix} a & v \\ 0 & b \end{pmatrix} = \begin{pmatrix} 1-e & x \\ 0 & 1-f \end{pmatrix} + \begin{pmatrix} 2e-1-j_1 & v-x \\ 0 & 2f-1+j_2 \end{pmatrix}$ is a clean decomposition of $\begin{pmatrix} a & v \\ 0 & b \end{pmatrix}$. Because $1-a \in ucn(A)$ and $b \in ucn(B)$, a and b are uniquely clean. Thus the clean decomposition of $\begin{pmatrix} a & v \\ 0 & b \end{pmatrix}$ whose idempotent part is $\begin{pmatrix} 1-e & * \\ 0 & 1-f \end{pmatrix}$. R is strongly clean. So we can let $\begin{pmatrix} a & v \\ 0 & b \end{pmatrix} = \begin{pmatrix} 1-e & x \\ 0 & 1-f \end{pmatrix} + \begin{pmatrix} 2e-1-j_1 & v-x \\ 0 & 2f-1+j_2 \end{pmatrix}$ is a strongly clean decomposition.

We have $\begin{pmatrix} 1-e & x \\ 0 & 1-f \end{pmatrix}\begin{pmatrix} 2e-1-j_1 & v-x \\ 0 & 2f-1+j_2 \end{pmatrix}$

$= \begin{pmatrix} (1-e)(2e-1-j_1) & (1-e)(v-x)+x(2f-1+j_2) \\ 0 & (1-f)(2f-1+j_2) \end{pmatrix}$ and

$$\begin{pmatrix} 2e-1-j_1 & v-x \\ 0 & 2f-1+j_2 \end{pmatrix}\begin{pmatrix} 1-e & x \\ 0 & 1-f \end{pmatrix}$$
$= \begin{pmatrix} (2e-1-j_1)(1-e) & (2e-1-j_1)x+(v-x)(1-f) \\ 0 & (2f-1+j_2)(1-f) \end{pmatrix}$. Then $(1-e)(v-x)+x(2f-1+j_2) = (2e-1-j_1)x+(v-x)(1-f)$ and then $ev-vf = xb-ax$.

(2) $\Rightarrow$ (1) For each $r = \begin{pmatrix} a & v \\ 0 & b \end{pmatrix} \in R$. We will prove R is strongly clean in four cases:

(i) If $a \in U(A)$ and $b \in U(B)$, then $r \in U(R)$ and then r is strongly clean.

(ii) If $a \notin U(A)$ and $b \notin U(B)$, then $1-a \in ucn(A)$ and $b \in ucn(B)$. Hence $1-a = (1-e)+j_1$ and $b = f+j_2$ where $e^2 = e \in A$, $f^2 = f \in B$, $j_1 \in J(A)$ and $j_2 \in J(B)$ by Proposition 3.4. Hence $a = e+(-j_1) = e-j_1$. So there exists $x \in V$ such that $x = ex+xf$ and $ev-vf = xb-ax$ by (2). Hence $\begin{pmatrix} 1-e & x \\ 0 & 1-f \end{pmatrix}\begin{pmatrix} 1-e & x \\ 0 & 1-f \end{pmatrix} = \begin{pmatrix} 1-e & 2x-ex-xf \\ 0 & 1-f \end{pmatrix} = \begin{pmatrix} 1-e & x \\ 0 & 1-f \end{pmatrix}$ and $\begin{pmatrix} 2e-1-j_1 & v-x \\ 0 & 2f-1+j_2 \end{pmatrix}\begin{pmatrix} 1-e & x \\ 0 & 1-f \end{pmatrix}$
$= \begin{pmatrix} 1-e & x \\ 0 & 1-f \end{pmatrix}\begin{pmatrix} 2e-1-j_1 & v-x \\ 0 & 2f-1+j_2 \end{pmatrix}$.

Clearly $2e-1-j_1 \in U(A)$ and $2f-1+j_2 \in U(B)$. Thus $r = \begin{pmatrix} 1-e & x \\ 0 & 1-f \end{pmatrix} + \begin{pmatrix} 2e-1-j_1 & v-x \\ 0 & 2f-1+j_2 \end{pmatrix}$ is strongly clean.

(iii) If $a \in U(A)$ and $b \notin U(B)$, there are two cases. If $1-a \notin U(A)$, then the proof is the same as (ii). If $1-a \in U(A)$, then $\begin{pmatrix} a & v \\ 0 & b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & s \end{pmatrix} + \begin{pmatrix} a-1 & v \\ 0 & b-s \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & s \end{pmatrix} + \begin{pmatrix} a & v \\ 0 & b-s \end{pmatrix}$ where $s^2 = s \in B$ and $b-s \in U(B)$ are two clean decompositions. Because V is an indecomposable bimodule, 0 and 1 are the only idempotents in $End(_AV)$. Let $\rho(b) : v \mapsto vb$ where $b \in B$ and $v \in V$. Then clearly $\rho(b) \in End(_AV)$. Because $\rho(s)\rho(s) = \rho(s)$, $\rho(s)$ is an idempotent element in $End(_AV)$. Thus $\rho(s) = 0$ or $\rho(s) = 1$. Hence $vs = 0$ or $vs = v$. So $r = \begin{pmatrix} 0 & 0 \\ 0 & s \end{pmatrix} + \begin{pmatrix} a & v \\ 0 & b-s \end{pmatrix}$ or $r = \begin{pmatrix} 1 & 0 \\ 0 & s \end{pmatrix} + \begin{pmatrix} a-1 & v \\ 0 & b-s \end{pmatrix}$ is a strongly clean decomposition. Hence r is strongly clean.

(iv) If $a \notin U(A)$ and $b \in U(B)$, there also two cases. If $1-b \notin U(B)$, then $b \in ucn(B)$ and $1 - a \in ucn(A)$. Then the proof is the same as (ii). If $1 - b \in U(B)$, then $\begin{pmatrix} a & v \\ 0 & b \end{pmatrix} = \begin{pmatrix} m & o \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} a - m & v \\ 0 & b - 1 \end{pmatrix} = \begin{pmatrix} m & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a - m & v \\ 0 & b \end{pmatrix}$ are two clean decompositions. Because V is an indecomposable bimodule, 0 and 1 are the only idempotents in $End(V_B)$. Let $\lambda(a) : v \mapsto av$ where $a \in A$ and $v \in V$. Then clearly $\lambda(a) \in End(V_B)$. Because $\lambda(m)\lambda(m) = \lambda(m)$, $\lambda(m)$ is an idempotent element in $End(V_B)$. Thus $\lambda(m) = 0$ or $\lambda(m) = 1$. Hence $mv = 0$ or $mv = v$. So $r = \begin{pmatrix} m & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} a - m & v \\ 0 & b \end{pmatrix}$ or $r = \begin{pmatrix} m & o \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} a - m & v \\ 0 & b - 1 \end{pmatrix}$ is a strongly clean decomposition. Hence r is strongly clean.

Thus we have shown the result. □

4 GUC group rings

We know that GUC rings include uniquely clean rings and local rings. In addition, there are some results on group rings of uniquely rings and local rings are proved in [5] and [13] respectively. Corresponding, we can also obtain some conclusions about group rings of GUC rings.

If G is a group and p is a prime number, $g \in G$ is called a *p-torsion* element if the order of g is p^k for some $k \geq 0$. If every element is p-torsion, then G is called a *p-group*. A group G is called *locally finite* if every finitely generated subgroup is finite. For convenience, all p in this paper are prime numbers.

Lemma 4.1. ([13, Theorem])Let A be a ring and G a group.

(1) If AG is local then A is local, G is a p-group and $p \in J(A)$;

(2) If A is local, G is a locally finite p-group and $p \in J(A)$ then AG is local;

(3) If G is abelian then AG is local if and only if A is local, G is a p-group and $p \in J(A)$.

Proposition 4.2. Let R be a ring and let G be a group. If RG is GUC then R is GUC, G is a p-group and $p \in J(R)$.

Proof. If RG is GUC, then R is GUC by Corollary 2.12. We know RG is uniquely clean or local. If RG is local, then G is a p-group and $p \in J(R)$ by Lemma 4.1. If RG is uniquely clean then G is a 2-group by [5, Theorem 5] and $2 \in J(R)$ by [14, Lemma 18]. □

Lemma 4.3. Let R be a ring with $R/J(R) \cong \mathbb{Z}_2$ and let G be a locally finite group. Then RG is GUC if and only if G is a 2-group.

Proof. If RG is uniquely clean the result follows from [5, Theorem 5]. If RG is local, then G is a p-group and $p \in J(R)$ by Lemma 4.1. Because $R/J(R) \cong \mathbb{Z}_2$, R is uniquely clean by [14, Theorem 15]. We know if $p \neq 2$ then $p \notin J(R)$ by the proof of Proposition 2.17. Hence $p = 2$. Thus we can get the result in conjunction with the previous proof. For the converse, R is uniquely clean by $R/J(R) \cong \mathbb{Z}_2$. Hence RG is uniquely clean by [5, Theorem 12] and then RG is GUC. □

Proposition 4.4. Let R be a ring and let G be a locally finite 2-group. Then RG is GUC if and only if R is GUC and $2 \in J(R)$.

Proof. It is easily proved by [5, Theorem 12] and Lemma 4.1. □

Let G be a group. Let $G^{(1)}$ be the derived subgroup of G and for each $i \geqslant 0$, $G^{(i+1)}$ denote the derived subgroup of $G^{(i)}$. Thus the series $G = G^{(0)} \supseteq G^{(1)} \supseteq G^{(2)} \supseteq \cdots$ is called the derived series of G. If $G^{(n)} = 1$ for some n then G is called *solvable*.

By the proved of [5, Theorem 13], we know a solvable 2-group is a locally finite 2-group. Hence we can draw the following inference.

Corollary 4.5. Let R be a ring and let G be a solvable 2-group. Then RG is GUC if and only if R is GUC and $2 \in J(R)$.

Similar to [5, Theorem 12], we can obtain a theorem on group rings of GUC rings.

Theorem 4.6. Let R be a ring and let G be a locally finite group. Then RG is GUC if and only if R is GUC, G is a p-group and $p \in J(R)$.

Proof. If RG is GUC the result follows from Proposition 4.2. Conversely, if R is GUC, there are two cases. If R is local then RG is local by Lemma 4.1, and thus RG is GUC . If R is uniquely clean then $p = 2$ by the proof of Proposition 2.17. Hence RG is uniquely clean by [5, Theorem 12]. Thus RG is GUC. This fully supports the conclusion. □

Example 4.7. Let C_n denote the cyclic group of order n. If $n \geq 3$ is odd and R is a Boolean ring, then RC_n is not GUC.

Proof. Write $C_n = \{1, g, g^2, \cdots, g^{n-1}\}$. If $1 + g^{n-1}$ is not invertible, then there exists $b = r_0 + r_1 g + \cdots + r_{n-1} g^{n-1} \in RC_n$ such that $(1 + g^{n-1})b = 1$. Then

$(1+g^{n-1})b = (1+g^{n-1})(r_0 + r_1 g + \cdots + r_{n-1}g^{n-1}) = (r_0 + r_1) + (r_1 + r_2)g + \cdots + (r_{n-2} + r_{n-1})g^{n-2} + (r_{n-1} + r_0)g^{n-1} = 1$. Hence $r_{n-1} = -r_0 = -r_{n-2}$, $r_0 + r_1 = 1$, $r_1 = r_3 = \cdots = r_{n-2}$. So $r_0 = r_{n-2} = r_1$. Thus $2r_0 = 1$. This is in contradiction with the condition that R is Boolean. Hence $1 + g^{n-1} \notin U(RC_n)$. Let $x = g + g^2 + \cdots + g^{n-1}$. Then $x^2 = x$ and $x + g^{n-1} \in U(RC_n)$ by the proved of [14, Proposition 24]. Because $(1+g+g^2+\cdots+g^{n-1})^2 = (1+g+g^2+\cdots+g^{n-1}) + (g+g^2+\cdots+g^{n-1}+1) + \cdots + (g^{n-1}+1+g+\cdots+g^{n-2}) = n(1+g+\cdots+g^{n-1}) = (1+g+\cdots+g^{n-1})$, $(1+g+\cdots+g^{n-1})$ is an idempotent in RC_n. It is easy to prove that $g^{n-1} \in U(RC_n)$. Thus $1 + g^{n-1} = (1+g+\cdots+g^{n-1}) + (x + g^{n-1})$ are different clean decompositions in RC_n. Hence RC_n is not GUC. □

We conclude with

Questions 4.8. Let R be a ring. Is it ture that R is a GUC ring if for every not invertible element $a \in R$ there exists a unique idempotent $e \in R$ such that $e - a \in J$ (cf. Theorem 3.5 (4))?

ACKNOWLEDGEMENTS

The authors are very grateful to the referee for the careful reading of the manuscript and for his/her valuable comments and suggestions that have largely improved the presentation. This research was partially supported by the science and technology project of Guangxi (Guike AD21220114).

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