Bulletin of the *Transilvania* University of Braşov
Series III: Mathematics and Computer Science, Vol. 3(65), No. 1 - 2023, 145-156
https://doi.org/10.31926/but.mif.2023.3.65.1.11

NOTES ON SYMMETRIC BI-(α, α)-DERIVATIONS IN RINGS

Öznur GÖLBAŞI[1] **and Emine Koç SÖGÜTCÜ** [*,2]

Abstract

Let R be a prime ring with center Z, I a nonzero ideal of R and $D : R \times R \to R$ a symmetric bi–(α, α)-derivation and d be the trace of D. In the present paper, we have considered the following conditions: i) $[d(x), x]_{\alpha,\alpha} = 0$, ii)$[d(x), x]_{\alpha,\alpha} \subseteq C_{\alpha,\alpha}$, iii)$(d(x), x)_{\alpha,\alpha} = 0$, iv)$D_1(d_2(x), x) = 0$, v)$d_1(d_2(x)) = f(x)$, for all $x, y \in I$,where D_1 and D_2 are two symmetric bi-(α, α)-derivations, d_1, d_2 are the traces of D_1, D_2 respectively, $B : R \times R \to R$ is a symmetric bi-additive mapping, f is the trace of B.

2000 *Mathematics Subject Classification:* 16W25.
Key words: derivations, symmetric bi-derivations, symmetric bi-(α, α)-derivations, commutativity, prime rings.

1 Introduction

Throughout R will represent an assosiative ring with center Z. A ring R is said to be prime if $xRy = (0)$ implies that either $x = 0$ or $y = 0$ and semiprime if $xRx = (0)$ implies that $x = 0$, where $x, y \in R$. A prime ring is obviously semiprime. For any $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$ and the symbol xoy stands for the commutator $xy + yx$. A mapping F from R to R is called centralizing on S if $[F(x), x] \in Z$, for all $x \in S$ and is called commuting on S if $[F(x), x] = 0$, for all $x \in S$. An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$.

The study of centralizing and commuting mappings on prime rings was initiated by the result of Posner [4] which states that the existence of a nonzero centralizing derivation on a prime ring implies that the ring has to be commutative. Through the years, a lot work has been done in this subject by a number of authors. The concept of commuting mappings is closely connected to the notion of bi-derivations. Symmetric bi-derivation has been introduced by Maksa

[1]Sivas Cumhuriyet University, Faculty of Science, Department of Mathematics, Sivas - Türkiye, e-mail: ogolbasi@cumhuriyet.edu.tr
[2*] *Corresponding author*, Sivas Cumhuriyet University, Faculty of Science, Department of Mathematics, Sivas - Türkiye, e-mail: eminekoc@cumhuriyet.edu.tr

in [3] and Vukman [6] investigated symmetric bi-derivations on rings with centralizing mappings. A mapping $D(.,.) : R \times R \to R$ is said to be symmetric if $D(x,y) = D(y,x)$ for all $x, y \in R$. A mapping $d : R \to R$ is called the trace of $D(.,.)$ if $d(x) = D(x,x)$ for all $x \in R$. It is obvious that if $D(.,.)$ is bi-additive (i.e., additive in both arguments), then the trace d of $D(.,.)$ satisfies the identity $d(x+y) = d(x) + d(y) + 2D(x,y)$, for all $x, y \in R$. If $D(.,.)$ is bi-additive and symmetric mapping satisfies

$$D(xy,z) = D(x,z)y + xD(y,z)$$

and

$$D(x,yz) = D(x,y)z + yD(x,z),$$

for all $x, y, z \in R$ called symmetric bi-derivation. Besides, many mathematicians showed that symmetric bi-derivations are related to general solutions of some functional equations.

Inspired by the definition symmetric bi-derivation, we introduce the notion of symmetric bi-(α,α)-derivation as follow:

Let α be an any automorphism of R. A bi-additive mapping $D(.,.) : R \times R \to R$ is said to be symmetric bi-(α,α)-derivation if it satisfies the identities

$$D(xy,z) = D(x,z)\alpha(y) + \alpha(x)D(y,z)$$

and

$$D(x,yz) = D(x,y)\alpha(z) + \alpha(y)D(x,z),$$

for all $x, y, z \in R$. Of course a symmetric bi-$(1,1)-$derivation where 1 is the identity map on R is symmetric bi-derivation. For any $x, y \in R$, we set $[x,y]_{\alpha,\alpha} = x\alpha(y) - \alpha(y)x$. We set $C_{\alpha,\alpha} = \{c \in R \mid c\alpha(x) = \alpha(x)c,$ for all $x \in R\}$ and call this set the $(\alpha,\alpha)-$center of R. In particular, $C_{1,1} = Z$. It can be given $(\alpha,\alpha)-$centralizing (resp. $(\alpha,\alpha)-$commuting) on R by the similarly definition centralizing (resp. commuting).

The purpose of this paper can be regarded as a contribution to the theory of centralizing and commuting symmetric bi-(α,α)-derivation. We obtained Vukman's result for a nonzero ideal of R with D a symmetric bi-(α,α)-derivation in [6, Theorem 2].

Throughout the paper, we denote a symmetric bi-(α,α)-derivation $D : R \times R \to R$ and d be the trace od D. For $x, y \in R$, $(x,y)_{\alpha,\alpha}$ will denote the Jordan commutator $x\alpha(y) + \alpha(y)x$ and make some extensive use of the basic commutator identities:

$[x,yz] = y[x,z] + [x,y]z$
$[xy,z] = [x,z]y + x[y,z]$
$[xy,z]_{\alpha,\alpha} = x[y,z]_{\alpha,\alpha} + [x,\alpha(z)]y = x[y,\alpha(z)] + [x,z]_{\alpha,\alpha}y$
$[x,yz]_{\alpha,\alpha} = \alpha(y)[x,z]_{\alpha,\alpha} + [x,y]_{\alpha,\alpha}\alpha(z)$
$(xy,z)_{\alpha,\alpha} = x(y,z)_{\alpha,\alpha} - [x,\alpha(z)]y = x[y,\alpha(z)] + (x,z)_{\alpha,\alpha}y$
$(x,yz)_{\alpha,\alpha} = \alpha(y)(x,z)_{\alpha,\alpha} + [x,y]_{\alpha,\alpha}\alpha(z) = (x,y)_{\alpha,\alpha}\alpha(z) - \alpha(y)[x,z]_{\alpha,\alpha}$.

2 Results

Lemma 1. *[1, Lemma 3.1]Let R be a $2-$torsion free semiprime ring and I a nonzero ideal of R. If $a,b \in R$ such that $axb + bxa = 0$ for all $x \in I$, then $axb = 0 = bxa$ for all $x \in I$.*

Lemma 2. *[2, Lemma 2 (b)]If R be a semiprime ring, then the center of a nonzero ideal of R is contained the center of R.*

Lemma 3. *Let R be a prime ring, I a nonzero ideal of R and $a,b \in R$. If $aIb = (0)$, then $a = 0$ or $b = 0$.*

Proof. We get
$$axb = 0, \text{ for all } x \in I.$$
Replacing x by xr, $r \in R$ in this equation, we have
$$axrb = 0, \text{ for all } x \in I,\ r \in R.$$
That is $axRb = (0)$. Since R is a prime ring, we have $ax = 0$ or $b = 0$. In the former case, we get $ax = 0$, for all $x \in I$. Replacing x by $rx, r \in R$ in last equation, we have $aRx = (0)$. Since I a nonzero ideal of R and R is a prime ring, we have $a = 0$. We conclude that $a = 0$ or $b = 0$. □

Lemma 4. *Let R be a prime ring, I a nonzero ideal of R and d a nonzero derivation of R. If $a \in R$ such that $ad(x) = 0$ for all $x \in I$, then $a = 0$.*

Proof. Replacing x by $xs, s \in R$ in the hypothesis, we have
$$axd(s) = 0, \text{ for all } x \in I, s \in R.$$
By Lemma 3 and $d \neq 0$, we obtain that $a = 0$. □

Lemma 5. *Let R be a prime ring. If a nonzero ideal of R is in the center of R, then R is a commutative ring.*

Proof. By the hypothesis, we get
$$[x, r] = 0, \text{ for all } x \in I, r \in R.$$
Replacing x by $sx, s \in R$ in this equation and using this, we obtain that
$$[s, r]x = \text{ for all } x \in I, r \in R.$$
Thus, $[R, R]I = (0)$. Multiplying this equation on the right by $[R, R]$, we have $[R, R]I[R, R] = (0)$. By Lemma 3, we conclude that R is a commutative ring. The proof is completed. □

Lemma 6. *Let R be a prime ring and I a nonzero ideal of R. If $a \in R$ such that $aD(x, y) = 0$ $(D(x, y)a = 0)$ for all $x, y \in I$, then $a = 0$ or $D = 0$.*

Proof. Replacing x by $xr, r \in R$ in the hypothesis, we have

$$a\alpha(x)D(r,y) = 0, \text{ for all } x, y \in I, r \in R.$$

That is,

$$aVD(r,y) = (0), \text{ for all } y \in I, r \in R,$$

where $\alpha(I) = V$ is ideal of R. By Lemma 3, we see that

$$a = 0 \text{ or } D(r,y) = 0, \text{ for all } x \in I, r \in R.$$

Let $D(r,y) = 0$, for all $y \in I, r \in R$. Taking y by $ys, s \in R$ in this equation and using this, we get

$$\alpha(y)D(r,s) = 0, \text{ for all } y \in I, r, s \in R$$

and so, $VD(r,s) = (0)$, for all $r, s \in R$. Again by Lemma 3, we conclude that $D(r,s) = (0)$, for all $r, s \in R$. This completes the proof.

Using the similar arguments, we prove that $D(x,y)a = 0$ for all $x, y \in I$, then $a = 0$ or $D = 0$. □

Lemma 7. *Let R be a prime ring and I a nonzero ideal of R. If $D(x,y) = 0$ for all $x, y \in I$, then $D = 0$.*

Proof. Replacing x by $xr, r \in R$ in the hypothesis, we see that

$$\alpha(x)D(r,y) = 0, \text{ for all } x, y \in I, r \in R.$$

This implies that

$$VD(r,y) = 0, \text{ for all } y \in I, r \in R,$$

where $\alpha(I) = V$ is ideal of R. By Lemma 3, we have

$$D(r,y) = 0, \text{ for all } x \in I, r \in R.$$

Writting y by $ys, s \in R$ in this equation and using this, we arrive at $D = 0$. □

Lemma 8. *Let R be a $2-$torsion free prime ring and I a nonzero ideal of R. If $d(x) = 0$ for all $x \in I$, then $D = 0$.*

Proof. Taking x by $x + y$ in the hypothesis and using this, we get

$$0 = d(x+y) = d(x) + d(y) + 2D(x,y)$$

and so, $2D(x,y) = 0$, for all $x, y \in I$. By Lemma 7, we get $D = 0$. □

Lemma 9. *Let R be a $2-$torsion free prime ring and I a nonzero ideal of R. If $D(x,y) \subseteq C_{\alpha,\alpha}$ for all $x, y \in I$, then $D = 0$ or R is commutative ring.*

Proof. By the hypothesis, we have

$$[D(x,y),r]_{\alpha,\alpha}=0, \text{for all } x,y\in I, r\in R.$$

Taking $xt, t\in I$ instead of x in this equation and using this, we find that

$$D(x,y)[\alpha(t),\alpha(r)]+[\alpha(x),\alpha(r)]D(x,y)=0, \text{for all } x,y,t\in I, r\in R.$$

Replacing r by x in this equation, we get

$$D(x,y)[\alpha(t),\alpha(x)]=0, \text{for all } x,y,t\in I.$$

Taking $st, s\in I$ instead of t and using this, we have

$$D(x,y)\alpha(s)[\alpha(t),\alpha(x)]=0, \text{for all } x,y,t,s\in I.$$

We obtain that

$$D(x,y)V[\alpha(t),\alpha(x)]=(0), \text{for all } x,y,t\in I,$$

where $\alpha(I)=V$ is ideal of R. By Lemma 3, we obtain that

$$D(x,y)=0 \text{ or } [\alpha(t),\alpha(x)]=0, \text{for all } x,y,t,s\in I.$$

Let $K=\{x\in I \mid D(x,y)=0$, for all $y\in I\}$ and $L=\{x\in I \mid [\alpha(t),\alpha(x)]=0$, for all $t\in I\}$ of additive subgroups of I. Morever, I is the set-theoretic union of K and L. But a group can not be the set-theoretic union of two proper subgroups, hence $K=I$ or $L=I$. In the former case, we get $D=0$ by Lemma 7. In the latter case, $[\alpha(I),\alpha(I)]=(0)$. We have $[V,V]=(0)$. That is $V\subseteq Z$ by Lemma 2, and so R is a commutative ring by Lemma 5. This completes the proof. □

The following theorem gives a generalization of Posner's well known result [4, Lemma 3] and a extension of [6, Theorem 1].

Theorem 1. *Let R be a $2-$torsion free prime ring, I a nonzero ideal of R and D,d a symmetric bi-(α,α)-derivation and the trace of D, respectively. If $[d(x),x]_{\alpha,\alpha}=0$, for all $x\in I$, then $D=0$.*

Proof. By the hypothesis, we have

$$[d(x),x]_{\alpha,\alpha}=0, \text{for all } x\in I. \tag{1}$$

A linearization of (1) yields that

$$[d(x),y]_{\alpha,\alpha}+[d(y),x]_{\alpha,\alpha}+2[D(x,y),x]_{\alpha,\alpha}+2[D(x,y),y]_{\alpha,\alpha}=0, \text{ for all } x,y\in I. \tag{2}$$

Replacing x by $-x$ in (2), we obtain that

$$[d(x),y]_{\alpha,\alpha}-[d(y),x]_{\alpha,\alpha}+2[D(x,y),x]_{\alpha,\alpha}-2[D(x,y),y]_{\alpha,\alpha}=0, \text{ for all } x,y\in I. \tag{3}$$

Comparing (2) and (3), using $2-$torsion freeness of R, we get

$$[d(x),y]_{\alpha,\alpha}+2[D(x,y),x]_{\alpha,\alpha}=0, \text{ for all } x,y\in I. \tag{4}$$

Replacing y by xy in (4) and using the hypothesis, we see that

$$\begin{aligned}0&=[d(x),xy]_{\alpha,\alpha}+2[D(x,xy),x]_{\alpha,\alpha}\\&=\alpha(x)[d(x),y]_{\alpha,\alpha}+[d(x),x]_{\alpha,\alpha}\alpha(y)+2[d(x)\alpha(y)+\alpha(x)D(x,y),x]_{\alpha,\alpha}\\&=\alpha(x)[d(x),y]_{\alpha,\alpha}+2d(x)[\alpha(y),\alpha(x)]+2[d(x),x]_{\alpha,\alpha}\alpha(y)+2\alpha(x)[D(x,y),x]_{\alpha,\alpha}.\end{aligned}$$

By (4) and using $2-$torsion freeness of R, we get

$$d(x)[\alpha(y),\alpha(x)]=0, \text{ for all } x,y\in I.$$

Again replacing y by $yz, z\in I$ in the last equation and using this, we have

$$d(x)\alpha(y)[\alpha(z),\alpha(x)]=0, \text{ for all } x,y,z\in I,$$

and so

$$d(x)V[\alpha(z),\alpha(x)]=(0), \text{ for all } x,z\in I,$$

where $\alpha(I)=V$ is ideal of R.By Lemma 3, we get either $d(x)=0$ or $[V,\alpha(x)]=(0)$ for each $x\in I$. By Lemma 2, we have $d(x)=0$ or $\alpha(x)\in Z$ for each $x\in I$. Since α is automorphism of R, we obtain that $d(x)=0$ or $x\in Z$ for each $x\in I$.

Let $x\in Z, y\notin Z$. Then $x+y\notin Z$ and $-y\notin Z$. Also, $d(x+y)=0$. Then we get

$$0=d(x+y)=d(x)+2D(x,y).$$

Taking x by $-x$ in this equation, we have

$$0=d(x)-2D(x,y).$$

Comparing the last two equations, we arrive at $d(x)=0$, for all $x\in Z$. Hence we obtain that $d(x)=0$, for all $x\in I$, and so, $D=0$ by Lemma 8. This completes the proof. □

The following theorem is a generalization of [6, Theorem 2] and [4, Theorem 2].

Theorem 2. *Let R be a 2 and $3-$torsion free prime ring, I a nonzero ideal of R and D,d a symmetric bi-(α,α)-derivation and the trace of D, respectively. If $[d(x),x]_{\alpha,\alpha}\subseteq C_{\alpha,\alpha}$ for all $x\in I$, then $D=0$.*

Proof. Linearizing $[d(x),x]_{\alpha,\alpha}\in C_{\alpha,\alpha}$, we get

$$[d(x),y]_{\alpha,\alpha}+[d(y),x]_{\alpha,\alpha}+2[D(x,y),x]_{\alpha,\alpha}+2[D(x,y),y]_{\alpha,\alpha}\in C_{\alpha,\alpha}. \tag{5}$$

Taking x by $-x$ in (5), we have

$$[d(x),y]_{\alpha,\alpha}-[d(y),x]_{\alpha,\alpha}+2[D(x,y),x]_{\alpha,\alpha}-2[D(x,y),y]_{\alpha,\alpha}\in C_{\alpha,\alpha}. \tag{6}$$

Using (5) and (6) and since R is a $2-$torsion free, we get

$$[d(x),y]_{\alpha,\alpha}+2[D(x,y),x]_{\alpha,\alpha}\in C_{\alpha,\alpha} \text{ for all } x,y\in I. \tag{7}$$

Replacing y by x^2 in (7) and using the hypothesis, we find that

$$\begin{aligned}&[d(x),x^2]_{\alpha,\alpha}+2[D(x,x^2),x]_{\alpha,\alpha}\\ &=\alpha(x)[d(x),x]_{\alpha,\alpha}+[d(x),x]_{\alpha,\alpha}\alpha(x)+2[d(x),x]_{\alpha,\alpha}\alpha(x)+2\alpha(x)[d(x),x]_{\alpha,\alpha}\in C_{\alpha,\alpha}.\end{aligned}$$

Using (7) and the assumptations that R is a $2,3-$torsion free ring, we get

$$\alpha(x)[d(x),x]_{\alpha,\alpha}\in C_{\alpha,\alpha} \text{ for all } x\in I.$$

Commuting this term with y and using $[d(x),x]_{\alpha,\alpha}\in C_{\alpha,\alpha}$, we have

$$[\alpha(x),\alpha(y)][d(x),x]_{\alpha,\alpha}=0, \text{ for all } x,y\in I.$$

Writing y by $yz, z\in I$ in this equation, we get

$$[\alpha(x),\alpha(y)]\alpha(z)[d(x),x]_{\alpha,\alpha}=0, \text{ for all } x,y,z\in I.$$

That is,

$$[\alpha(x),\alpha(y)]V[d(x),x]_{\alpha,\alpha}=(0), \text{ for all } x,y\in I,$$

where $\alpha(I)=V$ is ideal of R. By Lemma 3, we arrive at

$$[\alpha(x),V]=(0) \text{ or } [d(x),x]_{\alpha,\alpha}=0, \text{ for each } x\in I.$$

By Lemma 2, we have

$$\alpha(x)\in Z \text{ or } [d(x),x]_{\alpha,\alpha}=0, \text{ for each } x\in I.$$

If $\alpha(x)\in Z$, then $[d(x),x]_{\alpha,\alpha}=d(x)\alpha(x)-\alpha(x)d(x)=0$. Thus we obtain that $[d(x),x]_{\alpha,\alpha}=0$, for all $x\in I$, for any cases. By Theorem 1, we obtain that $D=0$. This completes the proof. □

Theorem 3. *Let R be a $2-$torsion free prime ring, I a nonzero ideal of R and D,d a symmetric bi-(α,α)-derivation and the trace of D, respectively. If $(d(x),x)_{\alpha,\alpha}=0$, for all $x\in I$, then $D=0$.*

Proof. By the hypothesis, we have

$$(d(x),x)_{\alpha,\alpha}=0, \text{for all } x\in I. \tag{8}$$

A linearization of this equation yields that

$$(d(x),y)_{\alpha,\alpha}+(d(y),x)_{\alpha,\alpha}+2(D(x,y),x)_{\alpha,\alpha}+2(D(x,y),y)_{\alpha,\alpha}=0, \text{ for all } x,y\in I. \tag{9}$$

Replacing x by $-x$ in (9), we obtain that

$$(d(x),y)_{\alpha,\alpha}-(d(y),x)_{\alpha,\alpha}+2(D(x,y),x)_{\alpha,\alpha}-2(D(x,y),y)_{\alpha,\alpha}=0, \text{ for all } x,y\in I. \tag{10}$$

Comparing (9) and (10) and using 2−torsion freeness of R, we get

$$(d(x), y)_{\alpha,\alpha} + 2(D(x, y), x)_{\alpha,\alpha} = 0, \text{ for all } x, y \in I. \quad (11)$$

Replacing y by yx in (11) and using the hypothesis, we see that

$$\begin{aligned} 0 &= (d(x), yx)_{\alpha,\alpha} + 2(D(x, yx), x)_{\alpha,\alpha} \\ &= (d(x), y)_{\alpha,\alpha}\alpha(x) - \alpha(y)[d(x), x]_{\alpha,\alpha} + 2(D(x, y)\alpha(x) + \alpha(y)d(x), x)_{\alpha,\alpha} \\ &= (d(x), y)_{\alpha,\alpha}\alpha(x) - \alpha(y)[d(x), x]_{\alpha,\alpha} + 2(D(x, y), x)_{\alpha,\alpha}\alpha(x) - 2[\alpha(y), \alpha(x)]d(x). \end{aligned}$$

By (11), we get

$$\alpha(y)[d(x), x]_{\alpha,\alpha} + 2[\alpha(y), \alpha(x)]d(x) = 0, \text{ for all } x, y \in I. \quad (12)$$

Again replacing y by $ry, r \in R$ in the last equation and using this, we have

$$2[\alpha(r), \alpha(x)]\alpha(y)d(x) = 0, \text{ for all } x, y \in I, r \in R.$$

Using 2−torsion freeness of R, we have

$$[\alpha(r), \alpha(x)]\alpha(y)d(x) = 0, \text{ for all } x, y \in I, r \in R.$$

Since α is a automorphism of R, we have

$$[r, \alpha(x)]Vd(x) = (0), \text{ for all } x, y \in I, r \in R,$$

where $\alpha(I) = V$ is a ideal of R. By Lemma 3 and Lemma 2, we obtain that

$$\alpha(x) \in Z \text{ or } d(x) = 0, \text{ for each } x \in I.$$

If $\alpha(x) \in Z$, then $(d(x), x)_{\alpha,\alpha} = d(x)\alpha(x) + \alpha(x)d(x) = 2d(x)\alpha(x) = 0$, and so, $d(x)\alpha(x) = 0$. Again using $\alpha(x) \in Z$, we obtain that

$$x = 0 \text{ or } d(x) = 0, \text{ for each } x \in I.$$

If $x = 0$, then $d(x) = D(x, x) = D(0, 0) = 0$. Thus we find that $d(x) = 0$ for any cases, and so, $D = 0$ by Lemma 8. This completes the proof. □

Theorem 4. *Let R be a 2−torsion free prime ring, I a nonzero ideal of R and D_1, d_1 a symmetric bi-(α,α)-derivation, D_2, d_2 symmetric bi-derivation and the traces of D_1, D_2 respectively. If $D_1(d_2(x), x) = 0$, for all $x \in I$, then either $D_1 = 0$ or $D_2 = 0$.*

Proof. Linearizing of the hypothesis, we get

$$D_1(d_2(y), x) + 2D_1(D_2(x, y), x) + D_1(d_2(x), y) + 2D_1(D_2(x, y), y) = 0. \quad (13)$$

Substituing in (13) x by $-x$, we have

$$-D_1(d_2(y), x) + 2D_1(D_2(x, y), x) + D_1(d_2(x), y) - 2D_1(D_2(x, y), y) = 0. \quad (14)$$

Comparing (13) and (14) and using 2−torsion free, we obtain that

$$D_1(d_2(x),y)+2D_1(D_2(x,y),x)=0, \text{ for all } x,y\in I. \tag{15}$$

Replacing y by xy in (15), using this equation and the hypothesis, we see that

$$\begin{aligned}0&=D_1(d_2(x),xy)+2D_1(D_2(x,xy),x)\\&=D_1(d_2(x),x)\alpha(y)+\alpha(x)D_1(d_2(x),y)+2D_1(D_2(x,x)y+xD_2(x,y),x)\\&=D_1(d_2(x),x)\alpha(y)+\alpha(x)D_1(d_2(x),y)+2D_1(d_2(x),x)\alpha(y)\\&+2\alpha(d_2(x))D_1(x,y)+2\alpha(x)D_1(D_2(x,y),x)+2d_1(x)\alpha(D_2(x,y))\end{aligned}$$

and so

$$\alpha(d_2(x))D_1(x,y)+d_1(x)\alpha(D_2(x,y))=0, \text{ for all } x,y\in I. \tag{16}$$

Taking y by yx in (16), we have

$$\begin{aligned}0&=\alpha(d_2(x))D_1(x,yx)+d_1(x)\alpha(D_2(x,yx))\\&=\alpha(d_2(x))D_1(x,y)\alpha(x)+\alpha(d_2(x))\alpha(y)d_1(x)\\&+d_1(x)\alpha(y)\alpha(d_2(x))+d_1(x)\alpha(D_2(x,y))\alpha(x),\end{aligned}$$

and so

$$\alpha(d_2(x))\alpha(y)d_1(x)+d_1(x)\alpha(y)\alpha(d_2(x))=0, \text{ for all } x,y\in I.$$

That is,

$$\alpha(d_2(x))yd_1(x)+d_1(x)y\alpha(d_2(x))=0, \text{ for all } x\in I,\ y\in V, \tag{17}$$

where $y\in\alpha(I)=V$ is a ideal of R. If $d_1(x)=0$ or $d_2(x)=0$, for all $x\in I$, then we get the required result by Lemma 8. Now, we assume that d_1 and d_2 are both different from zero. Hence there exist elements $x_1,x_2\in I$ such that $d_1(x_1)\neq 0$ and $d_2(x_2)\neq 0$. It follows $d_1(x_2)=0$ and $d_2(x_1)=0$ from (17) and Lemma 1. Since $d_1(x_2)=0$, the equation (16) reduces to $\alpha(d_2(x_2))D_1(x_2,y)=0$. Now, we define that $F:R\to R, F(y)=D_1(x_2,y)$. It is clear that F is a derivation. By Lemma 4 and $d_2(x_2)\neq 0$, we find that $D_1(x_2,y)=0$. In particular, we get $D_1(x_2,x_1)=0$. Similarly we see that $D_2(x_2,x_1)=0$ holds as well.

Let us write y for x_1+x_2. Then

$$d_1(y)=d_1(x_1+x_2)=d_1(x_1)+d_1(x_2)+2D_1(x_1,x_2)=d_1(x_1)\neq 0$$

and

$$d_2(y)=d_2(x_1+x_2)=d_2(x_1)+d_2(x_2)+2D_2(x_1,x_2)=d_2(x_2)\neq 0.$$

That is $d_1(y)$ and $d_2(y)$ are not from zero. But they cannot be both different from zero according to (17) and Lemma 1. It is a contradiction. Hence we must have $d_1(x)=0$ or $d_2(x)=0$,for all $x\in I$, and so, $D_1=0$ or $D_2=0$. □

The following theorem gives a generalization of Posner's result [4, Theorem 1] and a extension of [6, Theorem 5].

Theorem 5. *Let R be a $2,3-$torsion free prime ring, I a nonzero ideal of R and D_1, d_1 a symmetric bi-(α,α)-derivation, D_2, d_2 symmetric bi-derivation and the traces of D_1, D_2 respectively. If $B : R \times R \to R$ a symmetric bi-additive mapping such that $d_1(d_2(x)) = f(x)$, for all $x \in I$, where f is the trace of B, then either $D_1 = 0$ or $D_2 = 0$.*

Proof. The linearization of the the hypothesis, we have

$$2d_1(D_2(x,y))+D_1(d_2(x),d_2(y))+2D_1(d_2(x),D_2(x,y))+2D_1(d_2(y),D_2(x,y)) = B(x,y). \tag{18}$$

Taking x by $-x$ in (18), we get

$$2d_1(D_2(x,y))+D_1(d_2(x),d_2(y))-2D_1(d_2(x),D_2(x,y))-2D_1(d_2(y),D_2(x,y)) = -B(x,y). \tag{19}$$

Comparing (18) and (19) and using $2-$torsion free, we obtain that

$$2d_1(D_2(x,y)) + D_1(d_2(x),d_2(y)) = 0, \text{ for all } x,y \in I. \tag{20}$$

Using (20) in (18), we arrive at

$$2D_1(d_2(x),D_2(x,y)) + 2D_1(d_2(y),D_2(x,y)) = B(x,y), \text{ for all } x,y \in I.$$

Taking x by y in this equation and using $d_1(d_2(x)) = f(x)$, we have

$$\begin{aligned} 2D_1(d_2(x),D_2(x,x)) + 2D_1(d_2(x),D_2(x,x)) &= B(x,x) \\ 4d_1(d_2(x)) &= f(x) = d_1(d_2(x)) \\ 3d_1(d_2(x)) &= 0. \end{aligned}$$

Since R is $3-$torsion free, we get

$$d_1(d_2(x)) = 0, \text{ for all } x \in I.$$

On the other hand, again comparing (18) and (19) and using $d_1(d_2(x)) = 0$, for all $x \in I$, we find that

$$2D_1(d_2(x),D_2(x,y)) + 2D_1(d_2(y),D_2(x,y)) = B(x,y), \text{ for all } x,y \in I. \tag{21}$$

Replacing x by $2x$ in (21), we see that

$$8D_1(d_2(x),D_2(x,y)) + 2D_1(d_2(y),D_2(x,y)) = B(x,y), \text{ for all } x,y \in I. \tag{22}$$

By (21) and (22), we get

$$6D_1(d_2(x),D_2(x,y)) = 0$$

and so

$$D_1(d_2(x),D_2(x,y)) = 0, \text{ for all } x,y \in I. \tag{23}$$

Replacing y by yx in (23) and using this, $d_1(d_2(x)) = 0$, we see that

$$\begin{aligned} 0 &= D_1(d_2(x), D_2(x, yx)) = D_1(d_2(x), D_2(x, y)x + yd_2(x)) \\ &= D_1(d_2(x), D_2(x, y))\alpha(x) + \alpha(D_2(x, y))D_1(d_2(x), x) \\ &+ D_1(d_2(x), y)\alpha(d_2(x)) + \alpha(y)D_1(d_2(x), d_2(x)) \end{aligned}$$

and so

$$\alpha(D_2(x, y))D_1(d_2(x), x) + D_1(d_2(x), y)\alpha(d_2(x)) = 0, \text{ for all } x, y \in I. \tag{24}$$

Let in y by xy in (24) and using this, we have

$$\begin{aligned} 0 &= \alpha(D_2(x, xy))D_1(d_2(x), x) + D_1(d_2(x), xy)\alpha(d_2(x)) \\ &= \alpha(x)\alpha(D_2(x, y))D_1(d_2(x), x) + \alpha(d_2(x))\alpha(y)D_1(d_2(x), x) \\ &+ D_1(d_2(x), x)\alpha(y)\alpha(d_2(x)) + \alpha(x)D_1(d_2(x), y)\alpha(d_2(x)) \end{aligned}$$

and so

$$\alpha(d_2(x))\alpha(y)D_1(d_2(x), x) + D_1(d_2(x), x)\alpha(y)\alpha(d_2(x)) = 0, \text{ for all } x, y \in I.$$

That is,

$$\alpha(d_2(x))yD_1(d_2(x), x) + D_1(d_2(x), x)y\alpha(d_2(x)) = 0, \text{ for all } x \in I, \ y \in V, \tag{25}$$

where $y \in \alpha(I) = V$ is ideal of R. If $D_1(d_2(x), x) \neq 0$, for some $x \in I$, then $d_2(x) = 0$ by Lemma 1, and so $D_1(d_2(x), x) = 0$, a contrary to the assumption $D_1(d_2(x), x) \neq 0$. Hence, $D_1(d_2(x), x) = 0$, for all $x \in I$, and so $D_1 = 0$ or $D_2 = 0$ by Theorem 4. We get the required result. □

3 Conclusion

The present study has shown some essential properties of a nonzero ideals of a prime rings with symmetric bi-(α,α)-derivations. In future research, some well-known results in symmetric bi-derivations can be applied to symmetric bi-(α,β)-derivations. Besides, the findings herein could help to uncover properties of symmetric bi-(α,α)-derivations in Lie ideals or square-closed Lie ideals.

4 Acknowledgement

The authors wishes to thank the referee for his valuable comments and suggestion.

References

[1] Bell, H.E., *Some commutativity results involving derivations,* Trends in Theory of Rings and Modules, S. T. Rivzi, S. M. A. Zaidi (Eds), Anayama Publisher, New Delhi, India, (2005).

[2] Daif, M.N. and Bell, H.E.: *Remarks on derivations on semiprime rings*, INternational Journal of Mathematics and Mathematical Sciences **15** (1992), Article ID 863506.

[3] Maksa, G., *A remark on symmetric biadditive functions having nonnegative diagolization,* Glasnik Math. **15(35)**, (1980), 279-282.

[4] Posner, E.C., *Derivations in prime rings*, Proc. Amer. Math. Soc. **8** (1957), 1093-1100.

[5] Samman, M.S. and Thaheem, A.B., *Derivations on semiprime rings,* International Journal of Pure and Apllied Mathematics, **5**), (2003), no. 4, 465-472.

[6] Vukman, J., *On symmetric bi-derivations in prime and semiprime rings,* Aequationes Math. **38**, (1989), 245-254.