NOTES ON SYMMETRIC BI-\((\alpha, \alpha)\)-DERIVATIONS IN RINGS

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Abstract

Let \( R \) be a prime ring with center \( Z \), \( I \) a nonzero ideal of \( R \) and \( D : R \times R \rightarrow R \) a symmetric bi-\((\alpha, \alpha)\)-derivation and \( d \) be the trace of \( D \). In the present paper, we have considered the following conditions: i) \( [d(x), x]_{\alpha, \alpha} = 0 \), ii) \( [d(x), x]_{\alpha, \alpha} \subseteq C_{\alpha, \alpha} \), iii) \( d(x), x \) \( _{\alpha, \alpha} = 0 \), iv) \( D_1 (d_2 (x), x) = 0 \), v) \( d_1 (d_2 (x)) = f(x) \), for all \( x, y \in I \), where \( D_1 \) and \( D_2 \) are two symmetric bi-\((\alpha, \alpha)\)-derivations, \( d_1, d_2 \) are the traces of \( D_1, D_2 \) respectively, \( B : R \times R \rightarrow R \) is a symmetric bi-additive mapping, \( f \) is the trace of \( B \).

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1 Introduction

Throughout \( R \) will represent an assosiative ring with center \( Z \). A ring \( R \) is said to be prime if \( xRy = (0) \) implies that either \( x = 0 \) or \( y = 0 \) and semiprime if \( xRx = (0) \) implies that \( x = 0 \), where \( x, y \in R \). A prime ring is obviously semiprime. For any \( x, y \in R \), the symbol \( [x, y] \) stands for the commutator \( xy - yx \) and the symbol \( xoy \) stands for the commutator \( xy + yx \). A mapping \( F \) from \( R \) to \( R \) is called centralizing on \( S \) if \( [F(x), x] \in Z \), for all \( x \in S \) and is called commuting on \( S \) if \( [F(x), x] = 0 \), for all \( x \in S \). An additive mapping \( d : R \rightarrow R \) is called a derivation if \( d(xy) = d(x)y + xd(y) \) holds for all \( x, y \in R \).

The study of centralizing and commuting mappings on prime rings was initiated by the result of Posner \([4]\) which states that the existence of a nonzero centralizing derivation on a prime ring implies that the ring has to be commutative. Through the years, a lot work has been done in this subject by a number of authors. The concept of commuting mappings is closely connected to the notion of bi-derivations. Symmetric bi-derivation has been introduced by Maksa

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in [3] and Vukman [6] investigated symmetric bi-derivations on rings with centralizing mappings. A mapping $D(.,.) : R \times R \to R$ is said to be symmetric if $D(x,y) = D(y,x)$ for all $x,y \in R$. A mapping $d : R \to R$ is called the trace of $D(.,.)$ if $d(x) = D(x,x)$ for all $x \in R$. It is obvious that if $D(.,.)$ is bi-additive (i.e., additive in both arguments), then the trace $d$ of $D(.,.)$ satisfies the identity $d(x + y) = d(x) + d(y) + 2D(x,y)$, for all $x,y \in R$. If $D(.,.)$ is bi-additive and symmetric mapping satisfies

$$D(xy, z) = D(x, z)y + xD(y, z)$$

and

$$D(x, yz) = D(x, y)z + yD(x, z),$$

for all $x, y, z \in R$ called symmetric bi-derivation. Besides, many mathematicians showed that symmetric bi-derivations are related to general solutions of some functional equations.

Inspired by the definition symmetric bi-derivation, we introduce the notion of symmetric bi-$(\alpha, \alpha)$-derivation as follow:

Let $\alpha$ be an any automorphism of $R$. A bi-additive mapping $D(.,.) : R \times R \to R$ is said to be symmetric bi-$(\alpha, \alpha)$-derivation if it satisfies the identities

$$D(xy, z) = D(x, z)\alpha(y) + \alpha(x)D(y, z)$$

and

$$D(x, yz) = D(x, y)\alpha(z) + \alpha(y)D(x, z),$$

for all $x, y, z \in R$. Of course a symmetric bi-$(1,1)$-derivation where $1$ is the identity map on $R$ is symmetric bi-derivation. For any $x, y \in R$, we set $[x,y]_{\alpha,\alpha} = x\alpha(y) - \alpha(y)x$. We set $C_{\alpha,\alpha} = \{ c \in R \mid c\alpha(x) = \alpha(x)c, \text{ for all } x \in R \}$ and call this set the $(\alpha, \alpha)$-center of $R$. In particular, $C_{1,1} = Z$. It can be given $(\alpha, \alpha)$-centralizing (resp. $(\alpha, \alpha)$-commuting) on $R$ by the similarly definition centralizing (resp. commuting).

The purpose of this paper can be regarded as a contribution to the theory of centralizing and commuting symmetric bi-$(\alpha, \alpha)$-derivation. We obtained Vukman’s result for a nonzero ideal of $R$ with $D$ a symmetric bi-$(\alpha, \alpha)$-derivation in [6, Theorem 2].

Throughout the paper, we denote a symmetric bi-$(\alpha, \alpha)$-derivation $D : R \times R \to R$ and $d$ be the trace od $D$. For $x, y \in R$, $(x, y)_{\alpha,\alpha}$ will denote the Jordan commutator $x\alpha(y) + \alpha(y)x$ and make some extensive use of the basic commutator identities:

$$[x, yz] = y[x, z] + [x, y]z$$
$$[xy, z] = [x, z]y + x[y, z]$$
$$[xy, z]_{\alpha,\alpha} = x[y, z]_{\alpha,\alpha} + [x, \alpha(z)]y = x[y, \alpha(z)] + [x, z]_{\alpha,\alpha}y$$
$$[x, yz]_{\alpha,\alpha} = \alpha(y)[x, z]_{\alpha,\alpha} + [x, y]_{\alpha,\alpha}\alpha(z)$$
$$(xy, z)_{\alpha,\alpha} = x(y, z)_{\alpha,\alpha} - [x, \alpha(z)]y = x[y, \alpha(z)] + (x, z)_{\alpha,\alpha}y$$
$$(x, yz)_{\alpha,\alpha} = \alpha(y)(x, z)_{\alpha,\alpha} + [x, y]_{\alpha,\alpha}\alpha(z) = (x, y)_{\alpha,\alpha}\alpha(z) - \alpha(y)[x, z]_{\alpha,\alpha}. $$
2 Results

Lemma 1. [1, Lemma 3.1] Let $R$ be a 2–torsion free semiprime ring and $I$ a nonzero ideal of $R$. If $a, b \in R$ such that $axb + bxa = 0$ for all $x \in I$, then $axb = 0 = bxa$ for all $x \in I$.

Lemma 2. [2, Lemma 2 (b)] If $R$ be a semiprime ring, then the center of a nonzero ideal of $R$ is contained the center of $R$.

Lemma 3. Let $R$ be a prime ring, $I$ a nonzero ideal of $R$ and $a, b \in R$. If $aIb = (0)$, then $a = 0$ or $b = 0$.

Proof. We get $axb = 0$, for all $x \in I$. Replacing $x$ by $xr$, $r \in R$ in this equation, we have $axrb = 0$, for all $x \in I$, $r \in R$.

That is $axRb = (0)$. Since $R$ is a prime ring, we have $ax = 0$ or $b = 0$. In the former case, we get $ax = 0$, for all $x \in I$. Replacing $x$ by $rx$, $r \in R$ in last equation, we have $aRx = (0)$.

Since $I$ a nonzero ideal of $R$ and $R$ is a prime ring, we have $a = 0$. We conclude that $a = 0$ or $b = 0$.

Lemma 4. Let $R$ be a prime ring, $I$ a nonzero ideal of $R$ and $d$ a nonzero derivation of $R$. If $a \in R$ such that $ad(x) = 0$ for all $x \in I$, then $a = 0$.

Proof. Replacing $x$ by $xs$, $s \in R$ in the hypothesis, we have $axd(s) = 0$, for all $x \in I$, $s \in R$.

By Lemma 3 and $d \neq 0$, we obtain that $a = 0$.

Lemma 5. Let $R$ be a prime ring. If a nonzero ideal of $R$ is in the center of $R$, then $R$ is a commutative ring.

Proof. By the hypothesis, we get $[x, r] = 0$, for all $x \in I$, $r \in R$.

Replacing $x$ by $sx$, $s \in R$ in this equation and using this, we obtain that $[s, r]x = 0$, for all $x \in I$, $r \in R$.

Thus, $[R, R]I = (0)$. Multiplying this equation on the right by $[R, R]$, we have $[R, R]I[R, R] = (0)$. By Lemma 3, we conclude that $R$ is a commutative ring.

The proof is completed.

Lemma 6. Let $R$ be a prime ring and $I$ a nonzero ideal of $R$. If $a \in R$ such that $aD(x, y) = 0$ ($D(x, y)a = 0$) for all $x, y \in I$, then $a = 0$ or $D = 0$. 


Proof. Replacing $x$ by $xr, r \in R$ in the hypothesis, we have

$$a\alpha(x)D(r, y) = 0, \text{ for all } x, y \in I, r \in R.$$ 

That is,

$$aVD(r, y) = (0), \text{ for all } y \in I, r \in R,$$

where $\alpha(I) = V$ is ideal of $R.$ By Lemma 3, we see that

$$a = 0 \text{ or } D(r, y) = 0, \text{ for all } x \in I, r \in R.$$ 

Let $D(r, y) = 0,$ for all $y \in I, r \in R.$ Taking $y$ by $ys, s \in R$ in this equation and using this, we get

$$\alpha(y)D(r, s) = 0, \text{ for all } y \in I, r, s \in R$$

and so, $VD(r, s) = (0),$ for all $r, s \in R.$ Again by Lemma 3, we conclude that $D(r, s) = (0),$ for all $r, s \in R.$ This completes the proof.

Using the similar arguments, we prove that $D(x, y)a = 0$ for all $x, y \in I,$ then $a = 0$ or $D = 0.$

\begin{lemma}
Let $R$ be a prime ring and $I$ a nonzero ideal of $R.$ If $D(x, y) = 0$ for all $x, y \in I,$ then $D = 0.$
\end{lemma}

Proof. Replacing $x$ by $xr, r \in R$ in the hypothesis, we see that

$$\alpha(x)D(r, y) = 0, \text{ for all } x, y \in I, r \in R.$$ 

This implies that

$$VD(r, y) = 0, \text{ for all } y \in I, r \in R,$$

where $\alpha(I) = V$ is ideal of $R.$ By Lemma 3, we have

$$D(r, y) = 0, \text{ for all } x \in I, r \in R.$$ 

Writting $y$ by $ys, s \in R$ in this equation and using this, we arrive at $D = 0.$

\begin{lemma}
Let $R$ be a 2–torsion free prime ring and $I$ a nonzero ideal of $R.$ If $d(x) = 0$ for all $x \in I,$ then $D = 0.$
\end{lemma}

Proof. Taking $x$ by $x + y$ in the hypothesis and using this, we get

$$0 = d(x + y) = d(x) + d(y) + 2D(x, y)$$

and so, $2D(x, y) = 0,$ for all $x, y \in I.$ By Lemma 7, we get $D = 0.$

\begin{lemma}
Let $R$ be a 2–torsion free prime ring and $I$ a nonzero ideal of $R.$ If $D(x, y) \subseteq C_{\alpha, \alpha}$ for all $x, y \in I,$ then $D = 0$ or $R$ is commutative ring.
\end{lemma}
Proof. By the hypothesis, we have
\[ [D(x, y), r]_{\alpha, \alpha} = 0, \text{for all } x, y \in I, r \in R. \]

Taking \( xt, t \in I \) instead of \( x \) in this equation and using this, we find that
\[ D(x, y)[\alpha(t), \alpha(r)] + [\alpha(x), \alpha(r)]D(x, y) = 0, \text{for all } x, y, t \in I, r \in R. \]
Replacing \( r \) by \( x \) in this equation, we get
\[ D(x, y)[\alpha(t), \alpha(x)] = 0, \text{for all } x, y, t \in I. \]

Taking \( st, s \in I \) instead of \( t \) and using this, we have
\[ D(x, y)\alpha(s)[\alpha(t), \alpha(x)] = 0, \text{for all } x, y, t, s \in I. \]

We obtain that
\[ D(x, y)V[\alpha(t), \alpha(x)] = (0), \text{for all } x, y, t \in I, \]
where \( \alpha(I) = V \) is ideal of \( R \). By Lemma 3, we obtain that
\[ D(x, y) = 0 \text{ or } [\alpha(t), \alpha(x)] = 0, \text{for all } x, y, t, s \in I. \]

Let \( K = \{ x \in I \mid D(x, y) = 0, \text{for all } y \in I \} \) and \( L = \{ x \in I \mid [\alpha(t), \alpha(x)] = 0, \text{for all } t \in I \} \) of additive subgroups of \( I \). Moreover, \( I \) is the set-theoretic union of \( K \) and \( L \). But a group can not be the set-theoretic union of two proper subgroups, hence \( K = I \) or \( L = I \). In the former case, we get \( D = 0 \) by Lemma 7. In the latter case, \( [\alpha(I), \alpha(I)] = (0) \). We have \([V, V] = (0)\). That is \( V \subseteq Z \) by Lemma 2, and so \( R \) is a commutative ring by Lemma 5. This completes the proof.

The following theorem gives a generalization of Posner’s well known result [4, Lemma 3] and an extension of [6, Theorem 1].

**Theorem 1.** Let \( R \) be a 2–torsion free prime ring, \( I \) a nonzero ideal of \( R \) and \( D, d \) a symmetric bi-\((\alpha, \alpha)\)-derivation and the trace of \( D \), respectively. If \([d(x), x]_{\alpha, \alpha} = 0, \text{for all } x \in I, \) then \( D = 0 \).

**Proof.** By the hypothesis, we have
\[ [d(x), x]_{\alpha, \alpha} = 0, \text{for all } x \in I. \quad (1) \]
A linearization of (1) yields that
\[ [d(x), y]_{\alpha, \alpha} + [d(y), x]_{\alpha, \alpha} + 2[D(x, y), x]_{\alpha, \alpha} + 2[D(x, y), y]_{\alpha, \alpha} = 0, \text{for all } x, y \in I. \quad (2) \]
Replacing \( x \) by \(-x\) in (2), we obtain that
\[ [d(x), y]_{\alpha, \alpha} - [d(y), x]_{\alpha, \alpha} + 2[D(x, y), x]_{\alpha, \alpha} - 2[D(x, y), y]_{\alpha, \alpha} = 0, \text{for all } x, y \in I. \quad (3) \]
Comparing (2) and (3), using 2-torsion freeness of \( R \), we get
\[
[d(x), y]_{\alpha, \alpha} + 2[D(x, y), x]_{\alpha, \alpha} = 0, \text{ for all } x, y \in I. \tag{4}
\]
Replacing \( y \) by \( xy \) in (4) and using the hypothesis, we see that
\[
0 = [d(x), xy]_{\alpha, \alpha} + 2[D(x, xy), x]_{\alpha, \alpha}
= \alpha(x)[d(x), y]_{\alpha, \alpha} + [d(x), x]_{\alpha, \alpha}\alpha(y) + 2[d(x)\alpha(y) + \alpha(x)D(x, y), x]_{\alpha, \alpha}
= \alpha(x)[d(x), y]_{\alpha, \alpha} + 2d(x)[\alpha(y), \alpha(x)] + 2[d(x), x]_{\alpha, \alpha}\alpha(y) + 2\alpha(x)[D(x, y), x]_{\alpha, \alpha}.
\]
By (4) and using 2-torsion freeness of \( R \), we get
\[
d(x)[\alpha(y), \alpha(x)] = 0, \text{ for all } x, y \in I.
\]
Again replacing \( y \) by \( yz, z \in I \) in the last equation and using this, we have
\[
d(x)\alpha(y)[\alpha(z), \alpha(x)] = 0, \text{ for all } x, y, z \in I,
\]
and so
\[
d(x)\alpha(z)[\alpha(x)] = (0), \text{ for all } x, z \in I,
\]
where \( \alpha(I) = V \) is ideal of \( R \). By Lemma 3, we get either \( d(x) = 0 \) or \( [V, \alpha(x)] = (0) \) for each \( x \in I \). By Lemma 2, we have \( d(x) = 0 \) or \( \alpha(x) \in Z \) for each \( x \in I \). Since \( \alpha \) is automorphism of \( R \), we obtain that \( d(x) = 0 \) or \( x \in Z \) for each \( x \in I \).

Let \( x \in Z, y \notin Z \). Then \( x + y \notin Z \) and \( -y \notin Z \). Also, \( d(x + y) = 0 \). Then we get
\[
0 = d(x + y) = d(x) + 2D(x, y).
\]
Taking \( x \) by \( -x \) in this equation, we have
\[
0 = d(x) - 2D(x, y).
\]
Comparing the last two equations, we arrive at \( d(x) = 0 \), for all \( x \in Z \). Hence we obtain that \( d(x) = 0 \), for all \( x \in I \), and so, \( D = 0 \) by Lemma 8. This completes the proof.

The following theorem is a generalization of [6, Theorem 2] and [4, Theorem 2].

**Theorem 2.** Let \( R \) be a 2 and 3-torsion free prime ring, \( I \) a nonzero ideal of \( R \) and \( D, d \) a symmetric bi-(\( \alpha, \alpha \))-derivation and the trace of \( D \), respectively. If \( [d(x), x]_{\alpha, \alpha} \subseteq C_{\alpha, \alpha} \) for all \( x \in I \), then \( D = 0 \).

**Proof.** Linearizing \( [d(x), x]_{\alpha, \alpha} \in C_{\alpha, \alpha} \), we get
\[
[d(x), y]_{\alpha, \alpha} + [d(y), x]_{\alpha, \alpha} + 2[D(x, y), x]_{\alpha, \alpha} + 2[D(x, y), y]_{\alpha, \alpha} \in C_{\alpha, \alpha}.
\tag{5}
\]
Taking \( x \) by \( -x \) in (5), we have
\[
[d(x), y]_{\alpha, \alpha} - [d(y), x]_{\alpha, \alpha} + 2[D(x, y), x]_{\alpha, \alpha} - 2[D(x, y), y]_{\alpha, \alpha} \in C_{\alpha, \alpha}.
\tag{6}
\]
Notes on symmetric bi-(α, α)- derivations in rings

Using (5) and (6) and since \( R \) is a \( 2 \)-torsion free, we get
\[
[d(x), y]_{\alpha, \alpha} + 2[D(x, y), x]_{\alpha, \alpha} \in C_{\alpha, \alpha} \text{ for all } x, y \in I. \tag{7}
\]
Replacing \( y \) by \( x^2 \) in (7) and using the hypothesis, we find that
\[
[d(x), x^2]_{\alpha, \alpha} + 2[D(x, x^2), x]_{\alpha, \alpha} = \alpha(x)[d(x), x]_{\alpha, \alpha} + [d(x), x]_{\alpha, \alpha} \alpha(x) + 2[d(x), x]_{\alpha, \alpha} \alpha(x) + 2\alpha(x)[d(x), x]_{\alpha, \alpha} \in C_{\alpha, \alpha}.
\]
Using (7) and the assumption that \( R \) is a \( 2, 3 \)-torsion free ring, we get
\[\alpha(x)[d(x), x]_{\alpha, \alpha} \in C_{\alpha, \alpha} \text{ for all } x \in I.\]
Commuting this term with \( y \) and using \([d(x), x]_{\alpha, \alpha} \in C_{\alpha, \alpha}\), we have
\[\alpha(x)[\alpha(y)]d(x), x]_{\alpha, \alpha} = 0, \text{ for all } x, y \in I.\]
Writing \( y \) by \( yz, z \in I \) in this equation, we get
\[\alpha(x)[\alpha(y)]\alpha(z)[d(x), x]_{\alpha, \alpha} = 0, \text{ for all } x, y, z \in I.\]
That is,
\[\alpha(x), \alpha(y)]V[d(x), x]_{\alpha, \alpha} = 0, \text{ for all } x, y \in I,\]
where \( \alpha(I) = V \) is ideal of \( R \). By Lemma 3, we arrive at
\[\alpha(x), V] = 0 \text{ or } [d(x), x]_{\alpha, \alpha} = 0, \text{ for each } x \in I.\]
By Lemma 2, we have
\[\alpha(x) \in Z \text{ or } [d(x), x]_{\alpha, \alpha} = 0, \text{ for each } x \in I.\]
If \( \alpha(x) \in Z \), then \([d(x), x]_{\alpha, \alpha} = d(x)\alpha(x) - \alpha(x)d(x) = 0. \) Thus we obtain that
\[\alpha(x), x]_{\alpha, \alpha} = 0, \text{ for all } x \in I, \text{ for any cases. By Theorem 1, we obtain that } D = 0.\]
This completes the proof. \( \square \)

**Theorem 3.** Let \( R \) be a \( 2 \)-torsion free prime ring, \( I \) a nonzero ideal of \( R \) and \( D \) a symmetric bi-(\( \alpha, \alpha \))-derivation and the trace of \( D \), respectively. If \([d(x), x]_{\alpha, \alpha} = 0, \text{ for all } x \in I, \text{ then } D = 0.\)

**Proof.** By the hypothesis, we have
\[[d(x), x]_{\alpha, \alpha} = 0, \text{ for all } x \in I. \tag{8}\]
A linearization of this equation yields that
\[[d(x), y]_{\alpha, \alpha} + (d(y), x)_{\alpha, \alpha} + 2(D(x, y), x)_{\alpha, \alpha} + 2(D(x, y), y)_{\alpha, \alpha} = 0, \text{ for all } x, y \in I. \tag{9}\]
Replacing \( x \) by \(-x \) in (9), we obtain that
\[[d(x), y]_{\alpha, \alpha} - (d(y), x)_{\alpha, \alpha} + 2(D(x, y), x)_{\alpha, \alpha} - 2(D(x, y), y)_{\alpha, \alpha} = 0, \text{ for all } x, y \in I. \tag{10}\]
Comparing (9) and (10) and using 2–torsion freeness of $R$, we get
\[(d(x),y)_{\alpha,\alpha} + 2(D(x,y),x)_{\alpha,\alpha} = 0, \text{ for all } x,y \in I.\] (11)

Replacing $y$ by $yx$ in (11) and using the hypothesis, we see that
\[0 = (d(x),yx)_{\alpha,\alpha} + 2(D(x,yx),x)_{\alpha,\alpha} = (d(x),y)_{\alpha,\alpha} \alpha(x) - \alpha(y)[d(x),x]_{\alpha,\alpha} + 2(D(x,y)\alpha(x) + \alpha(y)d(x),x)_{\alpha,\alpha} = (d(x),y)_{\alpha,\alpha} \alpha(x) - \alpha(y)[d(x),x]_{\alpha,\alpha} + 2(D(x,y),x)_{\alpha,\alpha} \alpha(x) - 2[\alpha(y),\alpha(x)]d(x).\]

By (11), we get
\[\alpha(y)[d(x),x]_{\alpha,\alpha} + 2[\alpha(y),\alpha(x)]d(x) = 0, \text{ for all } x,y \in I.\] (12)

Again replacing $y$ by $ry, r \in R$ in the last equation and using this, we have
\[2[\alpha(r),\alpha(x)]\alpha(y)d(x) = 0, \text{ for all } x,y \in I, r \in R.\]

Using 2–torsion freeness of $R$, we have
\[[\alpha(r),\alpha(x)]\alpha(y)d(x) = 0, \text{ for all } x,y \in I, r \in R.\]

Since $\alpha$ is a automorphism of $R$, we have
\[[r,\alpha(x)]Vd(x) = (0), \text{ for all } x,y \in I, r \in R,\]

where $\alpha(I) = V$ is a ideal of $R$. By Lemma 3 and Lemma 2, we obtain that
\[\alpha(x) \in Z \text{ or } d(x) = 0, \text{ for each } x \in I.\]

If $\alpha(x) \in Z$, then $(d(x),x)_{\alpha,\alpha} = d(x)\alpha(x) + \alpha(x)d(x) = 2d(x)\alpha(x) = 0$, and so, $d(x)\alpha(x) = 0$. Again using $\alpha(x) \in Z$, we obtain that
\[x = 0 \text{ or } d(x) = 0, \text{ for each } x \in I.\]

If $x = 0$, then $d(x) = D(x,x) = 0(0,0) = 0$. Thus we find that $d(x) = 0$ for any cases, and so, $D = 0$ by Lemma 8. This completes the proof.

**Theorem 4.** Let $R$ be a 2–torsion free prime ring, $I$ a nonzero ideal of $R$ and $D_1,d_1$ a symmetric bi-(\(\alpha,\alpha\))-derivation, $D_2,d_2$ symmetric bi-derivation and the traces of $D_1,D_2$ respectively. If $D_1(d_2(x),x) = 0$, for all $x \in I$, then either $D_1 = 0$ or $D_2 = 0$.

**Proof.** Linearizing of the hypothesis, we get
\[D_1(d_2(y),x) + 2D_1(D_2(x,y),x) + D_1(d_2(x),y) + 2D_1(D_2(x,y),y) = 0.\] (13)

Substituing in (13) $x$ by $-x$, we have
\[-D_1(d_2(y),x) + 2D_1(D_2(x,y),x) + D_1(d_2(x),y) - 2D_1(D_2(x,y),y) = 0.\] (14)
Comparing (13) and (14) and using 2–torsion free, we obtain that
\[ D_1(d_2(x), y) + 2D_1(D_2(x, y), x) = 0, \text{ for all } x, y \in I. \] (15)
Replacing \( y \) by \( xy \) in (15), using this equation and the hypothesis, we see that
\[
0 = D_1(d_2(x), xy) + 2D_1(D_2(x, xy), x) \\
= D_1(d_2(x), x)\alpha(y) + \alpha(x)D_1(d_2(x), y) + 2D_1(D_2(x, x)y + xD_2(x, y), x) \\
= D_1(d_2(x), x)\alpha(y) + \alpha(x)D_1(d_2(x), y) + 2D_1(d_2(x), x)\alpha(y) \\
+ 2\alpha(d_2(x))D_1(x, y) + 2\alpha(x)D_1(D_2(x, y), x) + 2d_1(x)\alpha(D_2(x, y))
\]
and so
\[
\alpha(d_2(x))D_1(x, y) + d_1(x)\alpha(D_2(x, y)) = 0, \text{ for all } x, y \in I. \] (16)
Taking \( y \) by \( yx \) in (16), we have
\[
0 = \alpha(d_2(x))D_1(x, yx) + d_1(x)\alpha(D_2(x, yx)) \\
= \alpha(d_2(x))D_1(x, y)\alpha(x) + \alpha(d_2(x))\alpha(y)d_1(x) \\
+ d_1(x)\alpha(y)d_2(x) + d_1(x)\alpha(D_2(x, y))\alpha(x),
\]
and so
\[
\alpha(d_2(x))\alpha(y)d_1(x) + d_1(x)\alpha(y)\alpha(d_2(x)) = 0, \text{ for all } x, y \in I.
\]
That is,
\[
\alpha(d_2(x))yd_1(x) + d_1(x)y\alpha(d_2(x)) = 0, \text{ for all } x \in I, y \in V, \] (17)
where \( y \in \alpha(I) = V \) is a ideal of \( R \). If \( d_1(x) = 0 \) or \( d_2(x) = 0 \), for all \( x \in I \), then we get the required result by Lemma 8. Now, we assume that \( d_1 \) and \( d_2 \) are both different from zero. Hence there exist elements \( x_1, x_2 \in I \) such that \( d_1(x_1) \neq 0 \) and \( d_2(x_2) \neq 0 \). It follows \( d_1(x_2) = 0 \) and \( d_2(x_1) = 0 \) from (17) and Lemma 1. Since \( d_1(x_2) = 0 \), the equation (16) reduces to \( \alpha(d_2(x_2))D_1(x_2, y) = 0 \). Now, we define that \( F : R \to R, F(y) = D_1(x_2, y) \). It is clear that \( F \) is a derivation. By Lemma 4 and \( d_2(x_2) \neq 0 \), we find that \( D_1(x_2, y) = 0 \). In particular, we get \( D_1(x_2, x_1) = 0 \). Similarly we see that \( D_2(x_2, x_1) = 0 \) holds as well.

Let us write \( y \) for \( x_1 + x_2 \). Then
\[
d_1(y) = d_1(x_1 + x_2) = d_1(x_1) + d_1(x_2) + 2D_1(x_1, x_2) = d_1(x_1) \neq 0
\]
and
\[
d_2(y) = d_2(x_1 + x_2) = d_2(x_1) + d_2(x_2) + 2D_2(x_1, x_2) = d_2(x_2) \neq 0.
\]
That is \( d_1(y) \) and \( d_2(y) \) are not from zero. But they cannot be both different from zero according to (17) and Lemma 1. It is a contradiction. Hence we must have \( d_1(x) = 0 \) or \( d_2(x) = 0 \), for all \( x \in I \), and so, \( D_1 = 0 \) or \( D_2 = 0 \).
The following theorem gives a generalization of Posner’s result [4, Theorem 1] and a extension of [6, Theorem 5].

**Theorem 5.** Let $R$ be a 2,3–torsion free prime ring, $I$ a nonzero ideal of $R$ and $D_1, d_1$ a symmetric bi-$(\alpha, \alpha)$-derivation, $D_2, d_2$ symmetric bi-derivation and the traces of $D_1, D_2$ respectively. If $B : R \times R \to R$ a symmetric bi-additive mapping such that $d_1(d_2(x)) = f(x)$, for all $x \in I$, where $f$ is the trace of $B$, then either $D_1 = 0$ or $D_2 = 0$.

**Proof.** The linearization of the hypothesis, we have

\[2d_1(D_2(x, y)) + D_1(d_2(x), d_2(y)) + 2D_1(d_2(x), D_2(x, y)) + 2D_1(d_2(y), D_2(x, y)) = B(x, y).\]  

(18)

Taking $x$ by $-x$ in (18), we get

\[2d_1(D_2(x, y)) + D_1(d_2(x), d_2(y)) - 2D_1(d_2(x), D_2(x, y)) - 2D_1(d_2(y), D_2(x, y)) = -B(x, y).\]  

(19)

Comparing (18) and (19) and using 2–torsion free, we obtain that

\[2d_1(D_2(x, y)) + D_1(d_2(x), d_2(y)) = 0, \text{ for all } x, y \in I.\]  

(20)

Using (20) in (18), we arrive at

\[2D_1(d_2(x), D_2(x, y)) + 2D_1(d_2(y), D_2(x, y)) = B(x, y), \text{ for all } x, y \in I.\]

Taking $x$ by $y$ in this equation and using $d_1(d_2(x)) = f(x)$, we have

\[2D_1(d_2(x), D_2(x, x)) + 2D_1(d_2(x), D_2(x, x)) = B(x, x)\]

\[4d_1(d_2(x)) = f(x) = d_1(d_2(x))\]

\[3d_1(d_2(x)) = 0.\]

Since $R$ is 3–torsion free, we get

\[d_1(d_2(x)) = 0, \text{ for all } x \in I.\]

On the other hand, again comparing (18) and (19) and using $d_1(d_2(x)) = 0$, for all $x \in I$, we find that

\[2D_1(d_2(x), D_2(x, y)) + 2D_1(d_2(y), D_2(x, y)) = B(x, y), \text{ for all } x, y \in I.\]  

(21)

Replacing $x$ by $2x$ in (21), we see that

\[8D_1(d_2(x), D_2(x, y)) + 2D_1(d_2(y), D_2(x, y)) = B(x, y), \text{ for all } x, y \in I.\]  

(22)

By (21) and (22), we get

\[6D_1(d_2(x), D_2(x, y)) = 0\]

and so

\[D_1(d_2(x), D_2(x, y)) = 0, \text{ for all } x, y \in I.\]  

(23)
Replacing $y$ by $yx$ in (23) and using this, $d_1(d_2(x)) = 0$, we see that

$$0 = D_1(d_2(x), D_2(x, yx)) = D_1(d_2(x), D_2(x, y)x + yd_2(x))$$

$$= D_1(d_2(x), D_2(x, y))\alpha(x) + \alpha(D_2(x, y))D_1(d_2(x), x)$$

$$+ D_1(d_2(x), y)\alpha(d_2(x)) + \alpha(y)D_1(d_2(x), d_2(x))$$

and so

$$\alpha(D_2(x, y))D_1(d_2(x), x) + D_1(d_2(x), y)\alpha(d_2(x)) = 0, \text{ for all } x, y \in I. \quad (24)$$

Let in $y$ by $xy$ in (24) and using this, we have

$$0 = \alpha(D_2(x, xy))D_1(d_2(x), x) + D_1(d_2(x), xy)\alpha(d_2(x))$$

$$= \alpha(x)\alpha(D_2(x, y))D_1(d_2(x), x) + \alpha(d_2(x))\alpha(y)D_1(d_2(x), x)$$

$$+ D_1(d_2(x), x)\alpha(y)\alpha(d_2(x)) + \alpha(x)D_1(d_2(x), y)\alpha(d_2(x))$$

and so

$$\alpha(d_2(x))\alpha(y)D_1(d_2(x), x) + D_1(d_2(x), x)\alpha(y)\alpha(d_2(x)) = 0, \text{ for all } x, y \in I.$$

That is,

$$\alpha(d_2(x))yD_1(d_2(x), x) + D_1(d_2(x), x)y\alpha(d_2(x)) = 0, \text{ for all } x \in I, y \in V, \quad (25)$$

where $y \in \alpha(I) = V$ is ideal of $R$. If $D_1(d_2(x), x) \neq 0$, for some $x \in I$, then $d_2(x) = 0$ by Lemma 1, and so $D_1(d_2(x), x) = 0$, a contrary to the assumption $D_1(d_2(x), x) \neq 0$. Hence, $D_1(d_2(x), x) = 0$, for all $x \in I$, and so $D_1 = 0$ or $D_2 = 0$ by Theorem 4. We get the required result. \qed

3 Conclusion

The present study has shown some essential properties of a nonzero ideals of a prime rings with symmetric bi-$(\alpha, \alpha)$-derivations. In future research, some well-known results in symmetric bi-derivations can be applied to symmetric bi-$(\alpha, \beta)$-derivations. Besides, the findings herein could help to uncover properties of symmetric bi-$(\alpha, \alpha)$-derivations in Lie ideals or square-closed Lie ideals.

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References


