

ON THE MERSENNE AND MERSENNE-LUCAS HYBRINOMIAL QUATERNIONS

Engin ESER¹, Bahar KULOĞLU² and Engin ÖZKAN^{*,3}

Abstract

In this paper, we introduce Mersenne and Mersenne-Lucas hybrinomial quaternions and present some of their properties. Some identities are derived for these polynomials. Furthermore, we give the Binet formulas, Catalan, Cassini, d'Ocagne identity and generating and exponential generating function of these hybrinomial quaternions.

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1 Introduction

Hybrid numbers were introduced by Özdemir in [11]. A hybrid number is given by

$$\mathbb{K} = \{ a + bi + c\varepsilon + dh : a, b, c, d \in \mathbb{R}, \\ i^2 = -1, \varepsilon^2 = 0, h^2 = 1, ih = hi = \varepsilon + i \}$$

where i, ε, h are hybrid units and their multiplication rules are in Table1. Recently, many researchers [3, 11, 12, 6, 7, 22, 20, 21, 23] have studied hybrid numbers. k - Fibonacci, k -Lucas, Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas hybrid numbers were studied, and some properties were given. In addition, the results of these studies were generalized by creating a Horadam hybrid number sequence. In addition, Kızılates, Szyanal-Liana, Kürüz and Wloch [10, 23, 17] worked

¹Department of Mathematics, Erzincan Binali Yıldırım University, Faculty of Art and Sciences, Erzincan, Turkey, e-mail:engineser1978@gmail.com

²Department of Mathematics, Graduate School of Natural and Applied Sciences, Erzincan Binali Yıldırım University, Yalnızbağ Campus, 24100, Erzincan, Türkiye, e-mail: bahar_kuloglu@hotmail.com

^{3*} *Corresponding author*, Department of Mathematics, Faculty of Arts and Sciences, Erzincan Binali Yıldırım University, Yalnızbağ Campus, 24100, Erzincan, Türkiye, e-mail: eozkan@erzincan.edu.tr

Table 1: Multiplication Rules

.	i	ε	h
i	-1	$1 - h$	$\varepsilon + i$
ε	$1 + h$	0	$-\varepsilon$
h	$-\varepsilon - i$	ε	1

on these hybrid number sequences and created their polynomials and named them as hybrinomial. Later, a study was carried out on their more general conditions which generalized the special polynomials of these number sequences to form hybrinomials. The Fibonacci and Lucas hybrinomials are defined, respectively, as:

$$FH_n(x) = F_n(x) + F_{n+1}(x)i + F_{n+2}(x)\varepsilon + F_{n+3}(x)h$$

and

$$LH_n(x) = L_n(x) + L_{n+1}(x)i + L_{n+2}(x)\varepsilon + L_{n+3}(x)h$$

with $FH_0(x) = i + x\varepsilon + (x^2 + 1)h$, $FH_1(x) = 1 + xi + (x^2 + 1)\varepsilon + (x^3 + 2x)h$, $LH_0(x) = 2 + xi + (x^2 + 2)\varepsilon + (x^3 + 3x)h$ and $LH_1(x) = x + (x^2 + 2)i + (x^3 + 3x)\varepsilon + (x^4 + 4x^2 + 2)h$.

For $n \geq 2$, there are recurrence relations between $FH_n(x)$ and $LH_n(x)$ as follows:

$$FH_n(x) = xFH_{n-1}(x) + FH_{n-2}(x)$$

and

$$LH_n(x) = xLH_{n-1}(x) + LH_{n-2}(x).$$

In [14], the Horadam polynomials, $h_n(x) = h_n(x; a, b; p(x), q(x))$, are given by

$$h_n(x) = p(x)h_{n-1}(x) + q(x)h_{n-2}(x), \quad n \geq 2 \quad (1)$$

with $h_1(x) = a$ and $h_2(x) = bx$.

Let's consider the characteristic equation $t^2 - p(x)t - q(x) = 0$ where its roots are

$$\alpha = \frac{p(x) + \sqrt{p^2(x) + 4q(x)}}{2} \quad \text{and} \quad \beta = \frac{p(x) - \sqrt{p^2(x) + 4q(x)}}{2}.$$

If we take $p(x) = 3x$, $q(x) = -2$, $h_1(x) = 2$ and $h_2(x) = 3x$, we get Mersenne polynomials,

$$M_n(x) = 3xM_{n-1}(x) - 2M_{n-2}(x), \quad n \geq 3$$

with $M_1(x) = 0$ and $M_2(x) = 1$ [9].

Mersenne polynomials were studied by many researchers [1, 5, 9].

Similarly, when $p(x) = 3x$, $q(x) = -2$, $h_1(x) = 0$ and $h_2(x) = 1$, we get Mersenne-Lucas polynomials,

$$m_n(x) = 3xm_{n-1}(x) - 2m_{n-2}(x), \quad n \geq 2$$

with $m_0(x) = 2$ and $m_1(x) = 3$.

For $n \geq 1$, the Horadam hybrinomials are given by

$$H_n(x) = h_n(x) + h_{n+1}(x)i + h_{n+2}(x)\varepsilon + h_{n+3}(x)h.$$

If we substitute $M_n(x)$ for $h_n(x)$, we get Mersenne hybrinomials. Similarly, if we substitute $m_n(x)$ for $h_n(x)$, we get Mersenne-Lucas hybrinomials. So that

$$\check{M}_n(x) = M_n(x) + M_{n+1}(x)i + M_{n+2}(x)\varepsilon + M_{n+3}(x)h \quad (2)$$

$$\check{m}_n(x) = m_n(x) + m_{n+1}(x)i + m_{n+2}(x)\varepsilon + m_{n+3}(x)h. \quad (3)$$

In 1866, Hamilton introduced the quaternions. And many researchers had studied them as extension of complex numbers [13, 14, 4, 15, 16, 8, 18, 24, 25].

Quaternions are used in different fields such as quantum physics and analysis. Quaternions, also called real quaternions, are defined as follows:

$$Q = z_0 + z_1i + z_2j + z_3k$$

where z_0, z_1, z_2 and z_3 are real numbers. Also i, j and k are the units of real quaternions which satisfy the following equalities

$$i^2 = j^2 = k^2 = ijk = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j \quad (4)$$

2 Mersenne Hybrinomial Quaternions

Now, we define Mersenne hybrinomial quaternions by using Mersenne polynomials and give some of their properties.

Definition 1. *The Mersenne hybrinomial quaternions denoted by $\hat{M}_n(x)$ are defined as follows*

$$\hat{M}_n(x) = \check{M}_n(x) + i\check{M}_{n+1}(x) + j\check{M}_{n+2}(x) + k\check{M}_{n+3}(x).$$

Furthermore, every Mersenne hybrinomial quaternions can be written as

$$\begin{aligned} \hat{M}_n(x) &= M_n(x) + iM_{n+1}(x) + \varepsilon M_{n+2}(x) + hM_{n+3}(x) \\ &+ i(M_{n+1}(x) + iM_{n+2}(x) + \varepsilon M_{n+3}(x) + hM_{n+4}(x)) \\ &+ j(M_{n+2}(x) + iM_{n+3}(x) + \varepsilon M_{n+4}(x) + hM_{n+5}(x)) \\ &+ k(M_{n+3}(x) + iM_{n+4}(x) + \varepsilon M_{n+5}(x) + hM_{n+6}(x)) \\ &= \check{M}_n(x) + i\check{M}_{n+1}(x) + \varepsilon\check{M}_{n+2}(x) + h\check{M}_{n+3}(x) \end{aligned}$$

where $\check{M}_n(x) = M_n(x) + iM_{n+1}(x) + jM_{n+2}(x) + kM_{n+3}(x)$ is the n th Mersenne quaternion polynomials.

Definition 2. Let $\hat{S}_n(x)$ and $\hat{K}_n(x)$ be the Mersenne hybrinomial quaternion such that

$$\begin{aligned}\hat{S}_n(x) &= \check{S}_n(x) + i\check{S}_{n+1}(x) + j\check{S}_{n+2}(x) + k\check{S}_{n+3}(x) \\ &= \tilde{S}_n(x) + i\tilde{S}_{n+1}(x) + \varepsilon\tilde{S}_{n+2}(x) + h\tilde{S}_{n+3}(x)\end{aligned}$$

and

$$\begin{aligned}\hat{K}_n(x) &= \check{K}_n(x) + i\check{K}_{n+1}(x) + j\check{K}_{n+2}(x) + k\check{K}_{n+3}(x) \\ &= \tilde{K}_n(x) + i\tilde{K}_{n+1}(x) + \varepsilon\tilde{K}_{n+2}(x) + h\tilde{K}_{n+3}(x).\end{aligned}$$

Then, their addition and subtraction are defined, respectively, by

$$\begin{aligned}\hat{S}_n(x) \mp \hat{K}_n(x) &= (\check{S}_n(x) \mp \check{K}_n(x)) + i(\check{S}_{n+1}(x) \mp \check{K}_{n+1}(x)) + j(\check{S}_{n+2}(x) \\ &\quad \mp \check{K}_{n+2}(x)) + k(\check{S}_{n+3}(x) \mp \check{K}_{n+3}(x)), \\ \hat{S}_n(x) \mp \hat{K}_n(x) &= (\tilde{S}_n(x) \mp \tilde{K}_n(x)) + i(\tilde{S}_{n+1}(x) \mp \tilde{K}_{n+1}(x)) \\ &\quad + \varepsilon(\tilde{S}_{n+2}(x) \mp \tilde{K}_{n+2}(x)) + h(\tilde{S}_{n+3}(x) \mp \tilde{K}_{n+3}(x)).\end{aligned}$$

Definition 3. Multiplication of the Mersenne hybrinomial quaternions is defined in terms of Mersenne hybrinomial as follows:

$$\begin{aligned}\hat{S}_n(x) \hat{K}_n(x) &= (\check{S}_n(x) \check{K}_n(x) - \check{S}_{n+1}(x) \check{K}_{n+1}(x) - \check{S}_{n+2}(x) \check{K}_{n+2}(x) - \check{S}_{n+3}(x) \check{K}_{n+3}(x)) \\ &\quad + i(\check{S}_n(x) \check{K}_{n+1}(x) + \check{S}_{n+1}(x) \check{K}_n(x) + \check{S}_{n+2}(x) \check{K}_{n+3}(x) - \check{S}_{n+3}(x) \check{K}_{n+2}(x)) \\ &\quad + j(\check{S}_n(x) \check{K}_{n+2}(x) - \check{S}_{n+1}(x) \check{K}_{n+3}(x) + \check{S}_{n+2}(x) \check{K}_n(x) + \check{S}_{n+3}(x) \check{K}_{n+1}(x)) \\ &\quad + k(\check{S}_n(x) \check{K}_{n+3}(x) + \check{S}_{n+1}(x) \check{K}_{n+2}(x) - \check{S}_{n+2}(x) \check{K}_{n+1}(x) + \check{S}_{n+3}(x) \check{K}_n(x))\end{aligned}$$

or it can be defined in terms of Mersenne quaternions as follows:

$$\begin{aligned}\hat{S}_n(x) \hat{K}_n(x) &= (\tilde{S}_n(x) \tilde{K}_n(x) - \tilde{S}_{n+1}(x) \tilde{K}_{n+1}(x) + \tilde{S}_{n+3}(x) \tilde{K}_{n+3}(x) + \tilde{S}_{n+1}(x) \tilde{K}_{n+2}(x) \\ &\quad + \tilde{S}_{n+2}(x) \tilde{K}_{n+1}(x)) \\ &\quad + i(\tilde{S}_n(x) \tilde{K}_{n+1}(x) + \tilde{S}_{n+1}(x) \tilde{K}_n(x) + \tilde{S}_{n+1}(x) \tilde{K}_{n+3}(x) - \tilde{S}_{n+3}(x) \tilde{K}_{n+1}(x)) \\ &\quad + \varepsilon(\tilde{S}_n(x) \tilde{K}_{n+2}(x) + \tilde{S}_{n+1}(x) \tilde{K}_{n+3}(x) + \tilde{S}_{n+2}(x) \tilde{K}_n(x) + \tilde{S}_{n+2}(x) \tilde{K}_{n+3}(x)) \\ &\quad + \varepsilon(-\tilde{S}_{n+3}(x) \tilde{K}_{n+1}(x) + \tilde{S}_{n+3}(x) \tilde{K}_{n+2}(x)) \\ &\quad + h(\tilde{S}_n(x) \tilde{K}_{n+3}(x) - \tilde{S}_{n+1}(x) \tilde{K}_{n+2}(x) + \tilde{S}_{n+2}(x) \tilde{K}_{n+1}(x) + \tilde{S}_{n+3}(x) \tilde{K}_n(x)).\end{aligned}$$

The scalar and vector parts of $\hat{S}_n(x)$ are denoted, respectively, by $S_{\hat{S}_n(x)} = \check{S}_n(x)$ and $V_{\hat{S}_n(x)} = i\check{S}_{n+1}(x) + j\check{S}_{n+2}(x) + k\check{S}_{n+3}(x)$.

So, $\hat{S}_n(x)$ can be given as $\hat{S}_n(x) = S_{\hat{S}_n(x)} + V_{\hat{S}_n(x)}$. Now, we also can define addition and subtraction of two the Mersenne hybrinomial quaternion sequences, respectively, as

$$\hat{S}_n(x) \mp \hat{K}_n(x) = \left(S_{\hat{S}_n(x)} \mp S_{\hat{K}_n(x)} \right) \mp \left(V_{\hat{S}_n(x)} \mp V_{\hat{K}_n(x)} \right)$$

and multiplication

$$\begin{aligned} & \hat{S}_n(x) \hat{K}_n(x) \\ &= S_{\hat{S}_n(x)} S_{\hat{K}_n(x)} - V_{\hat{S}_n(x)} V_{\hat{K}_n(x)} + S_{\hat{S}_n(x)} V_{\hat{K}_n(x)} + S_{\hat{K}_n(x)} V_{\hat{S}_n(x)} + V_{\hat{S}_n(x)} V_{\hat{K}_n(x)}. \end{aligned}$$

Definition 4. *The conjugate of Mersenne hybrinomial quaternion can be defined in three different types*

- i. *Quaternion conjugate, $\overline{\hat{M}_n(x)}$:* $\overline{\hat{M}_n(x)} = \overline{\hat{M}_n(x)} + i \overline{\hat{M}_{n+1}(x)} + \varepsilon \overline{\hat{M}_{n+2}(x)} + h \overline{\hat{M}_{n+3}(x)}$
- ii. *Hybrid conjugate, $\hat{M}_n(x)^C$:* $\hat{M}_n(x)^C = \tilde{M}_n(x) - i\tilde{M}_{n+1}(x) - \varepsilon\tilde{M}_{n+2}(x) - h\tilde{M}_{n+3}(x)$
- iii. *Total conjugate, $\hat{M}_n(x)^t$:* $\hat{M}_n(x)^t = \overline{\hat{M}_n(x)^C} = \overline{\tilde{M}_n(x)} - i\overline{\tilde{M}_{n+1}(x)} - \varepsilon\overline{\tilde{M}_{n+2}(x)} - h\overline{\tilde{M}_{n+3}(x)}$.

Theorem 1. *For $n \geq 1$ we have the following relations:*

- i. $3x\hat{M}_n(x) - 2\hat{M}_{n-1}(x) = \hat{M}_{n+1}(x)$
- ii.
$$\begin{aligned} & -2\hat{M}_n(x) - i\hat{M}_{n+1}(x) + 2j\hat{M}_{n+2}(x) - k\hat{M}_{n+3}(x) \\ &= \check{m}_{n+1}(x)(1 - 3i) + \check{m}_{n+5}(x)(1 + 3i) - 6k\check{M}_{n+3}(x) \end{aligned}$$

Proof. i.

$$\begin{aligned} 3x\hat{M}_n(x) - 2\hat{M}_{n-1}(x) &= 3x\check{M}_n(x) - 2\check{M}_{n-1}(x) + i(3x\check{M}_{n+1}(x) - 2\check{M}_n(x)) \\ &+ j(3x\check{M}_{n+2}(x) - 2\check{M}_{n+1}(x)) \\ &+ k(3x\check{M}_{n+3}(x) - 2\check{M}_{n+2}(x)) = \hat{M}_{n+1}(x). \end{aligned}$$

- ii.
$$\begin{aligned} & -2\hat{M}_n(x) - i\hat{M}_{n+1}(x) + 2j\hat{M}_{n+2}(x) - k\hat{M}_{n+3}(x) = -2\check{M}_n(x) + \check{M}_{n+2}(x) - 2\check{M}_{n+4}(x) + \check{M}_{n+6}(x) - 6k\check{M}_{n+3}(x) + 3i(\check{M}_{n+5}(x) - \check{M}_{n+1}(x)) \\ &= \check{m}_{n+1}(x) + \check{m}_{n+5}(x) + 3i(\check{m}_{n+5}(x) - \check{m}_{n+1}(x)) - 6k\check{M}_{n+3}(x) \\ &= (1 - 3i)\check{m}_{n+1}(x) + (1 + 3i)\check{m}_{n+5}(x) - 6k\check{M}_{n+3}(x) \end{aligned}$$

□

Theorem 2. For $\hat{M}_n(x)$ and $\hat{m}_n(x)$, we have

$$i. -2\hat{M}_{n-1}(x) + \hat{M}_{n+1}(x) = \hat{m}_n(x)$$

$$ii. \hat{M}_{n+2}(x) - 4\hat{M}_{n-2}(x) = 3\hat{m}_n(x)$$

Proof. i.

$$\begin{aligned} -2\hat{M}_{n-1}(x) + \hat{M}_{n+1}(x) &= (-2\check{M}_{n-1}(x) + \check{M}_{n+1}(x)) \\ &\quad + i(-2\check{M}_n(x) + \check{M}_{n+2}(x)) \\ &\quad + j(-2\check{M}_{n+1}(x) + \check{M}_{n+3}(x)) \\ &\quad + k(-2\check{M}_{n+2}(x) + \check{M}_{n+4}(x)) \\ &= \check{m}_n(x) + i\check{m}_{n+1}(x) + j\check{m}_{n+2}(x) + k\check{m}_{n+3}(x) \\ &= \hat{m}_n(x). \end{aligned}$$

ii.

$$\begin{aligned} \hat{M}_{n+2}(x) - 4\hat{M}_{n-2}(x) &= (\check{M}_{n+2}(x) - 4\check{M}_{n-2}(x)) \\ &\quad + i(\check{M}_{n+3}(x) - 4\check{M}_{n-1}(x)) \\ &\quad + j(\check{M}_{n+4}(x) - 4\check{M}_n(x)) \\ &\quad + k(\check{M}_{n+5}(x) - 4\check{M}_{n+1}(x)) \\ &= 3\check{m}_n(x) + 3i\check{m}_{n+1}(x) + 3j\check{m}_{n+2}(x) + 3k\check{m}_{n+3}(x) \\ &= \widehat{3m}_n(x). \end{aligned}$$

□

Theorem 3. The following relations are satisfied:

$$i. \hat{M}_n(x) + \overline{\hat{M}_n(x)} = 2\check{M}_n(x)$$

$$ii. \hat{M}_n(x) + \hat{M}_n(x)^C = 2\check{M}_n(x)$$

$$\begin{aligned} iii. \hat{M}_n(x) + \hat{M}_n(x)^t \\ = -2M_n(x) - 8M_{n+1}(x) + 2(\check{M}_{n+1}(x) + \check{M}_{n+2}(x) + \check{M}_{n+3}(x)). \end{aligned}$$

Proof. i.

$$\begin{aligned} \hat{M}_n(x) + \overline{\hat{M}_n(x)} &= (\check{M}_n(x) + i\check{M}_{n+1}(x) + j\check{M}_{n+2}(x) + k\check{M}_{n+3}(x)) \\ &\quad + (\check{M}_n(x) - i\check{M}_{n+1}(x) - j\check{M}_{n+2}(x) - k\check{M}_{n+3}(x)) \\ &= 2\check{M}_n(x) \end{aligned}$$

ii.

$$\begin{aligned} \hat{M}_n(x) + \hat{M}_n(x)^C &= (\check{M}_n(x) + \check{M}_n^C(x)) + i(\check{M}_{n+1}(x) + \check{M}_{n+1}^C(x)) \\ &\quad + j(\check{M}_{n+2}(x) + \check{M}_{n+2}^C(x)) \\ &\quad + k(\check{M}_{n+3}(x) + \check{M}_{n+3}^C(x)) \\ &= 2(M_n(x) + iM_{n+1}(x) + jM_{n+2}(x) + kM_{n+3}(x)) \\ &= 2\check{M}_n(x) \end{aligned}$$

iii.

$$\begin{aligned}
 \hat{M}_n(x) + \hat{M}_n(x)^t &= (\check{M}_n(x) + \check{M}_n^C(x)) + i(\check{M}_{n+1}(x) - \check{M}_{n+1}^C(x)) \\
 &+ j(\check{M}_{n+2}(x) - \check{M}_{n+2}^C(x)) + k(\check{M}_{n+3}(x) - \check{M}_{n+3}^C(x)) \\
 &= -2M_n(x) - 8M_{n+1}(x) \\
 &+ 2(\check{M}_{n+1}(x) + \check{M}_{n+2}(x) + \check{M}_{n+3}(x))
 \end{aligned}$$

□

Theorem 4. *The Binet's formulas for $\hat{M}_n(x)$ is given by*

$$\hat{M}_n(x) = \frac{\alpha^*(x)\alpha^{**}(x)\alpha^n(x) - \beta^*(x)\beta^{**}(x)\beta^n(x)}{\alpha(x) - \beta(x)}$$

where $\alpha^*(x) = 1 + i\alpha(x) + j\alpha^2(x) + k\alpha^3(x)$, $\beta^*(x) = 1 + i\beta(x) + j\beta^2(x) + k\beta^3(x)$,

$\alpha^{**}(x) = 1 + i\alpha(x) + \varepsilon\alpha^2(x) + h\alpha^3(x)$ and $\beta^{**}(x) = 1 + i\beta(x) + \varepsilon\beta^2(x) + h\beta^3(x)$.

Proof. Özkoç [15] gave the Binet's formula for (p, q) -Fibonacci quaternion polynomials. If $p = 3x$ and $q = -2$ are taken, Binet formula is found for Mersenne quaternion polynomials.

$$\tilde{M}_n(x) = \frac{\alpha^*(x)\alpha^n(x) - \beta^*(x)\beta^n(x)}{\alpha(x) - \beta(x)}.$$

From $\hat{M}_n(x) = \tilde{M}_n(x) + i\tilde{M}_{n+1}(x) + \varepsilon\tilde{M}_{n+2}(x) + h\tilde{M}_{n+3}(x)$, we obtain

$$\begin{aligned}
 \hat{M}_n(x) &= \frac{(\alpha^*(x)\alpha^n(x) + i\alpha^*(x)\alpha^{n+1}(x) + \varepsilon\alpha^*(x)\alpha^{n+2}(x) + h\alpha^*(x)\alpha^{n+3}(x))}{\alpha(x) - \beta(x)} \\
 &\quad - \frac{(\beta^*(x)\beta^n(x) + i\beta^*(x)\beta^{n+1}(x) + \varepsilon\beta^*(x)\beta^{n+2}(x) + h\beta^*(x)\beta^{n+3}(x))}{\alpha(x) - \beta(x)} \\
 &= \frac{\alpha^*(x)\alpha^n(x)(1 + i\alpha(x) + \varepsilon\alpha^2(x) + h\alpha^3(x))}{\alpha(x) - \beta(x)} \\
 &\quad - \frac{\beta^*(x)\beta^n(x)(1 + i\beta(x) + \varepsilon\beta^2(x) + h\beta^3(x))}{\alpha(x) - \beta(x)} \\
 &= \frac{\alpha^*(x)\alpha^{**}(x)\alpha^n(x) - \beta^*(x)\beta^{**}(x)\beta^n(x)}{\alpha(x) - \beta(x)}
 \end{aligned}$$

□

Theorem 5. *The generating function for $\hat{M}_n(x)$ is*

$$g\hat{M}_n(x) = \frac{\hat{M}_0(x) + t(\hat{M}_1(x) - 3x\hat{M}_0(x))}{1 - 3tx + 2t^2}.$$

Proof. The generating function $g\hat{M}_n(x)$ for $\hat{M}_n(x)$ is given

$$\begin{aligned} g\hat{M}_n(x) &= \sum_{n=0}^{\infty} \hat{M}_n(x) t^n = \hat{M}_0(x) + \hat{M}_1(x)t + \hat{M}_2(x)t^2 + \cdots + \hat{M}_n(x)t^n + \cdots \\ -3txg\hat{M}_n(x) &= -3tx\hat{M}_0(x) - 3x\hat{M}_1(x)t^2 - 3x\hat{M}_2(x)t^3 - \cdots - 3x\hat{M}_n(x)t^{n+1} + \cdots \\ 2t^2g\hat{M}_n(x) &= 2t^2\hat{M}_0(x) + 2\hat{M}_1(x)t^3 + 2\hat{M}_2(x)t^4 + \cdots + 2\hat{M}_n(x)t^{n+2} + \cdots \\ g\hat{M}_n(x) - 3txg\hat{M}_n(x) + 2t^2g\hat{M}_n(x) &= g\hat{M}_n(x) [1 - 3tx + 2t^2] \\ &= \hat{M}_0(x) + t [\hat{M}_1(x) - 3x\hat{M}_0(x)] + t^2 [\hat{M}_2(x) - 3x\hat{M}_1(x) + 2\hat{M}_0(x)] \\ &\quad + t^3 [\hat{M}_3(x) - 3x\hat{M}_2(x) + 2\hat{M}_1(x)] + \cdots \end{aligned}$$

From Theorem 1 (i), we get

$$\begin{aligned} g\hat{M}_n(x) [1 - 3tx + 2t^2] &= \hat{M}_0(x) + t(\hat{M}_1(x) - 3x\hat{M}_0(x)) \\ g\hat{M}_n(x) &= \frac{\hat{M}_0(x) + t(\hat{M}_1(x) - 3x\hat{M}_0(x))}{1 - 3tx + 2t^2}. \end{aligned}$$

□

Theorem 6. For $n \geq 0$ and $\hat{M}_n(x)$, we have the following binomial sum formula for odd and even terms, respectively,

$$\begin{aligned} \hat{M}_{2n}(x) &= \sum_{m=0}^n \binom{n}{m} (-2)^{n-m} (3x)^m \hat{M}_m(x) \\ \hat{M}_{2n+1}(x) &= \sum_{m=0}^n \binom{n}{m} (-2)^{n-m} (3x)^m \hat{M}_{m+1}(x) \end{aligned}$$

Proof. From the Binet's formula, we get

$$\begin{aligned} &\sum_{m=0}^n \binom{n}{m} (-2)^{n-m} (3x)^m \frac{\alpha^*(x) \alpha^{**}(x) \alpha^m(x) - \beta^*(x) \beta^{**}(x) \beta^m(x)}{\alpha(x) - \beta(x)} \\ &= \frac{\alpha^*(x) \alpha^{**}(x)}{\alpha(x) - \beta(x)} \sum_{m=0}^n \binom{n}{m} (-2)^{n-m} (3x\alpha(x))^m \\ &\quad - \frac{\beta^*(x) \beta^{**}(x)}{\alpha(x) - \beta(x)} \sum_{m=0}^n \binom{n}{m} (-2)^{n-m} (3x\beta(x))^m \\ &= \frac{\alpha^*(x) \alpha^{**}(x)}{\alpha(x) - \beta(x)} (-2 + 3x\alpha(x))^n - \frac{\beta^*(x) \beta^{**}(x)}{\alpha(x) - \beta(x)} (-2 + 3x\beta(x))^n \\ &= \frac{\alpha^*(x) \alpha^{**}(x) \alpha^{2n}(x) - \beta^*(x) \beta^{**}(x) \beta^{2n}(x)}{\alpha(x) - \beta(x)} = \hat{M}_{2n}(x). \end{aligned}$$

The other case can be done similarly. □

Theorem 7. The sum of the first m terms of the sequence $\{\hat{M}_m(x)\}_{m=0}^{\infty}$ is given by

$$\begin{aligned} & \sum_{k=0}^m \hat{M}_k(x) \\ &= \frac{\hat{M}_0(x) + 2\hat{M}_m(x) - \hat{M}_{m+1}(x) - \alpha^*(x)\alpha^{**}(x)\beta(x) - \beta^*(x)\beta^{**}(x)\alpha(x)}{(1-\alpha(x))(1-\beta(x))}. \end{aligned}$$

Proof. From the Binet's formula of Mersenne hybrinomial quaternion, we have

$$\begin{aligned} & \sum_{k=0}^m \hat{M}_k(x) \\ &= \sum_{k=0}^m \frac{\alpha^*(x)\alpha^{**}(x)\alpha^k(x) - \beta^*(x)\beta^{**}(x)\beta^k(x)}{\alpha(x) - \beta(x)} \\ &= \frac{\alpha^*(x)\alpha^{**}(x)}{\alpha(x) - \beta(x)} \sum_{m=0}^n \alpha^k(x) - \frac{\beta^*(x)\beta^{**}(x)}{\alpha(x) - \beta(x)} \sum_{m=0}^n \beta^k(x) \\ &= \frac{\alpha^*(x)\alpha^{**}(x)}{\alpha(x) - \beta(x)} \frac{1 - \alpha^{m+1}(x)}{1 - \alpha(x)} - \frac{\beta^*(x)\beta^{**}(x)}{\alpha(x) - \beta(x)} \frac{1 - \beta^{m+1}(x)}{1 - \beta(x)} \\ &= \frac{\alpha^*(x)\alpha^{**}(x) - \alpha^*(x)\alpha^{**}(x)\beta(x) - \alpha^*(x)\alpha^{**}(x)\alpha^{m+1}(x)}{(1-\alpha(x))(1-\beta(x))(\alpha(x) - \beta(x))} \\ &+ \frac{\alpha^*(x)\alpha^{**}(x)\alpha^m(x)\alpha(x)\beta(x)}{(1-\alpha(x))(1-\beta(x))(\alpha(x) - \beta(x))} \\ &- \frac{\beta^*(x)\beta^{**}(x) - \beta^*(x)\beta^{**}(x)\alpha(x) - \beta^*(x)\beta^{**}(x)\beta^{m+1}(x)}{(1-\alpha(x))(1-\beta(x))(\alpha(x) - \beta(x))} \\ &+ \frac{+\beta^*(x)\beta^{**}(x)\beta^m(x)\beta(x)\alpha(x)}{(1-\alpha(x))(1-\beta(x))(\alpha(x) - \beta(x))} \\ &= \frac{\hat{M}_0(x) + 2\hat{M}_m(x) - \hat{M}_{m+1}(x) - \alpha^*(x)\alpha^{**}(x)\beta(x) - \beta^*(x)\beta^{**}(x)\alpha(x)}{(1-\alpha(x))(1-\beta(x))}. \end{aligned}$$

□

Theorem 8. The exponential generating function for the $\hat{M}_m(x)$ is

$$\sum_{k=0}^{\infty} \frac{\hat{M}_k(x)}{k!} \ell^k = \frac{\alpha^*(x)\alpha^{**}(x)e^{\alpha(x)\ell} - \beta^*(x)\beta^{**}(x)e^{\beta(x)\ell}}{\alpha(x) - \beta(x)}.$$

Proof. From the Binet's formula for Mersenne hybrinomial quaternion, we have

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{\hat{M}_k(x)}{k!} \ell^k &= \sum_{k=0}^{\infty} \left(\frac{\alpha^*(x)\alpha^{**}(x)\alpha^k(x) - \beta^*(x)\beta^{**}(x)\beta^k(x)}{\alpha(x) - \beta(x)} \right) \frac{\ell^k}{k!} \\ &= \frac{\alpha^*(x)\alpha^{**}(x)}{\alpha(x) - \beta(x)} \sum_{k=0}^{\infty} \frac{(\alpha(x)\ell)^k}{k!} - \frac{\beta^*(x)\beta^{**}(x)}{\alpha(x) - \beta(x)} \sum_{k=0}^{\infty} \frac{(\beta(x)\ell)^k}{k!} \end{aligned}$$

$$= \frac{\alpha^*(x) \alpha^{**}(x) e^{\alpha(x)\ell} - \beta^*(x) \beta^{**}(x) e^{\beta(x)\ell}}{\alpha(x) - \beta(x)}.$$

□

Theorem 9. For $r \geq s \geq 1$, Catalan identity for the Mersenne hybrid quaternion is

$$\begin{aligned} \hat{M}_{r+s}(x) \hat{M}_{r-s}(x) - \left(\hat{M}_r(x)\right)^2 \\ = (-2)^{r-s} \alpha^*(x) \alpha^{**}(x) \beta^*(x) \beta^{**}(x) \left(\frac{\alpha^s(x) - \beta^s(x)}{\alpha(x) - \beta(x)}\right)^2. \end{aligned}$$

Proof. From the Binet's formula for Mersenne hybrid quaternion, we have

$$\begin{aligned} \hat{M}_{r+s}(x) \hat{M}_{r-s}(x) - \left(\hat{M}_r(x)\right)^2 \\ = \frac{-\alpha^*(x) \alpha^{**}(x) \beta^*(x) \beta^{**}(x)}{(\alpha(x) - \beta(x))^2} \left[\alpha^{r+s}(x) \beta^{r-s}(x) + \alpha^{r-s}(x) \beta^{r+s}(x) \right. \\ \left. - 2\alpha^r(x) \beta^r(x) \right] \\ = \frac{-\alpha^*(x) \alpha^{**}(x) \beta^*(x) \beta^{**}(x)}{(\alpha(x) - \beta(x))^2} \times \\ \times \frac{((\alpha(x) \beta(x))^r \left(\frac{\alpha(x)}{\beta(x)}\right)^s + (\alpha(x) \beta(x))^r \left(\frac{\beta(x)}{\alpha(x)}\right)^s - 2(\alpha(x) \beta(x))^r)}{(\alpha(x) - \beta(x))^2} \\ = \alpha^*(x) \alpha^{**}(x) \beta^*(x) \beta^{**}(x) (-2)^{r-s} \left(\frac{\alpha^s(x) - \beta^s(x)}{\alpha(x) - \beta(x)}\right)^2. \end{aligned}$$

□

Theorem 10. For $r \in \mathbb{N}$, Cassini's identity for the Mersenne hybrid quaternion is

$$\hat{M}_{r+1}(x) \hat{M}_{r-1}(x) - \left(\hat{M}_r(x)\right)^2 = (-2)^{r-1} \alpha^*(x) \alpha^{**}(x) \beta^*(x) \beta^{**}(x).$$

Proof. Taking $s = 1$ in Catalan identity, the proof is completed. □

Theorem 11. For $s > r$, d'Ocagne's identity for the Mersenne hybrid quaternion is

$$\begin{aligned} \hat{M}_s(x) \hat{M}_{r+1}(x) - \hat{M}_{s+1}(x) \hat{M}_r(x) \\ = \alpha^*(x) \alpha^{**}(x) \beta^*(x) \beta^{**}(x) 2^r \left(\frac{\alpha^{s-r}(x) - \beta^{s-r}(x)}{\alpha(x) - \beta(x)}\right). \end{aligned}$$

Proof. From the Binet's formula for Mersenne hybrinomial quaternion, we have

$$\begin{aligned}
 & \hat{M}_s(x) \hat{M}_{r+1}(x) - \hat{M}_{s+1}(x) \hat{M}_r(x) \\
 &= \frac{-\alpha^*(x) \alpha^{**}(x) \beta^*(x) \beta^{**}(x)}{(\alpha(x) - \beta(x))^2} \\
 &= \frac{((\alpha^s(x) \beta^r(x) (\beta(x) - \alpha(x)) - \alpha^r(x) \beta^s(x) (\beta(x) - \alpha(x))))}{(\alpha(x) - \beta(x))^2} \\
 &= \alpha^*(x) \alpha^{**}(x) \beta^*(x) \beta^{**}(x) 2^r \left(\frac{\alpha^{s-r}(x) - \beta^{s-r}(x)}{\alpha(x) - \beta(x)} \right)
 \end{aligned}$$

□

3 Mersenne-Lucas hybrinomial quaternions

Now, we define Mersenne-Lucas hybrinomial quaternions by using Mersenne-Lucas polynomials and give some of their properties.

Definition 5. We denote the Mersenne-Lucas hybrinomial quaternions by $\hat{m}_n(x)$ as follows

$$\hat{m}_n(x) = \check{m}_n(x) + i\check{m}_{n+1}(x) + j\check{m}_{n+2}(x) + k\check{m}_{n+3}(x).$$

Furthermore, every Mersenne-Lucas hybrinomial quaternions can be written as

$$\begin{aligned}
 \hat{m}_n(x) &= (m_n(x) + im_{n+1}(x) + \varepsilon m_{n+2}(x) + hm_{n+3}(x)) + \\
 & i(m_{n+1}(x) + im_{n+2}(x) + \varepsilon m_{n+3}(x) + hm_{n+4}(x)) + \\
 & j(m_{n+2}(x) + im_{n+3}(x) + \varepsilon m_{n+4}(x) + hm_{n+5}(x)) + \\
 & k(m_{n+3}(x) + im_{n+4}(x) + \varepsilon m_{n+5}(x) + hm_{n+6}(x)) \\
 &= \check{m}_n(x) + i\check{m}_{n+1}(x) + \varepsilon\check{m}_{n+2}(x) + h\check{m}_{n+3}(x).
 \end{aligned}$$

Theorem 12. The Binet's formulas for these sequences are given as follows:

$$\hat{m}_n(x) = \alpha^{**}(x) \alpha^*(x) \alpha^n(x) + \beta^{**}(x) \beta^*(x) \beta^n(x)$$

Proof.

$$\begin{aligned}
\hat{m}_n(x) &= \tilde{m}_n(x) + i\tilde{m}_{n+1}(x) + \varepsilon\tilde{m}_{n+2}(x) + h\tilde{m}_{n+3}(x) \\
&= \alpha^{**}(x)\alpha^n(x) + \beta^{**}(x)\beta^n(x) + i(\alpha^{**}(x)\alpha^{n+1}(x) + \beta^{**}(x)\beta^{n+1}(x)) \\
&\quad + \varepsilon(\alpha^{**}(x)\alpha^{n+2}(x) + \beta^{**}(x)\beta^{n+2}(x)) \\
&\quad + h(\alpha^{**}(x)\alpha^{n+3}(x) + \beta^{**}(x)\beta^{n+3}(x)) \\
&= \alpha^{**}(x)\alpha^n(x) + i\alpha^{**}(x)\alpha^{n+1}(x) + \varepsilon\alpha^{**}(x)\alpha^{n+2}(x) \\
&\quad + h\alpha^{**}(x)\alpha^{n+3}(x) + \beta^{**}(x)\beta^n(x) \\
&\quad + i\beta^{**}(x)\beta^{n+1}(x) + \varepsilon\beta^{**}(x)\beta^{n+2}(x) + h\beta^{**}(x)\beta^{n+3}(x) \\
&= \alpha^{**}(x)\alpha^n(x)((1 + i\alpha(x) + \varepsilon\alpha^2(x) + h\alpha^3(x)) \\
&\quad + \beta^{**}(x)\beta^n(x)((1 + i\beta(x) + \varepsilon\beta^2(x) + h\beta^3(x)) \\
&= \alpha^{**}(x)\alpha^*(x)\alpha^n(x) + \beta^{**}(x)\beta^*(x)\beta^n(x)
\end{aligned}$$

□

Theorem 13. For $n \geq 0$, $\hat{m}_n(x)$ have the following binomial sum formula for odd and even terms,

$$\begin{aligned}
\hat{m}_{2n}(x) &= \sum_{m=0}^n \binom{n}{m} (-2)^{n-m} (3x)^m \hat{m}_m(x), \\
\hat{m}_{2n+1}(x) &= \sum_{m=0}^n \binom{n}{m} (-2)^{n-m} (3x)^m \hat{m}_{m+1}(x).
\end{aligned}$$

Proof. The proof is done similarly to Theorem 6. □

Theorem 14. The sum of the first m terms of the sequence $\{\hat{m}_m(x)\}_{m=0}^{\infty}$ is given by

$$\begin{aligned}
\sum_{k=0}^m \hat{m}_k(x) &= \\
&= \frac{\hat{m}_0(x) + 2\hat{m}_m(x) - \hat{m}_{m+1}(x) - \alpha^{**}(x)\alpha^*(x)\beta(x) - \beta^{**}(x)\beta^*(x)\alpha(x)}{(1 - \alpha(x))(1 - \beta(x))}.
\end{aligned}$$

Proof. The proof is done similarly to Theorem 2. □

Theorem 15. The exponential generating function for the $\hat{m}_m(x)$ is

$$\sum_{k=0}^{\infty} \frac{\hat{m}_k(x)}{k!} t^k = \alpha^{**}(x)\alpha^*(x)e^{\alpha(x)t} + \beta^{**}(x)\beta^*(x)e^{\beta(x)t}.$$

Proof. The proof is done similarly to Theorem 7. □

Theorem 16. For $r \geq s \geq 1$, Catalan identity for the Mersenne-Lucas hybrinomial quaternion is

$$\hat{m}_{r+s}(x) \hat{m}_{r-s}(x) - \hat{m}_r^2(x) = 2^{r-s} \alpha^{**}(x) \alpha^*(x) \beta^{**}(x) \beta^*(x) (\alpha^s - \beta^s)^2.$$

Proof. The proof is done similarly to Theorem 8. \square

Theorem 17. For $r \in \mathbb{N}$, Cassini's identity for the Mersenne-Lucas hybrinomial quaternion is

$$\hat{m}_{r-1}(x) \hat{m}_{r+1}(x) - (\hat{m}_r(x))^2 = 2^{r-1} \alpha^{**}(x) \alpha^*(x) \beta^{**}(x) \beta^*(x) (\alpha(x) - \beta(x))^2.$$

Proof. Taking $s = 1$ in Catalan identity, the proof is completed. \square

Theorem 18. For $s > r$, d'Ocagne's identity for the Mersenne-Lucas hybrinomial quaternion is

$$\begin{aligned} \hat{m}_s(x) \hat{m}_{r+1}(x) - \hat{m}_{s+1}(x) \hat{m}_r(x) \\ = \alpha^*(x) \alpha^{**}(x) \beta^*(x) \beta^{**}(x) (2^s - 2^r) (\alpha^{r-s+1} + \beta^{r-s+1}) \end{aligned}$$

Proof. The proof is done similarly to Theorem 10. \square

4 Conclusions

In this work, we gave Mersenne and Mersenne-Lucas hybrinomial quaternions with their properties. Furthermore, we obtained some important identities such as the Binet formulas, Catalan, Cassini, d'Ocagne identity and generating and exponential generating function of these hybrinomial quaternions.

This work can be moved to other number sequences and the relationships between them can be examined.

References

- [1] Chelgham, M. and Boussayoud, A. *On the k -Mersenne-Lucas numbers.*, Notes on Number Theory and Discrete Mathematics **27** (2021), no. 1, 7-13.
- [2] Dağdeviren, A. and Kürüz, F. *On The Horadam hybrid quaternions*, arXiv:2012.08277 (2020).
- [3] Erkan, E. and Dağdeviren, A. *k -Fibonacci and k -Lucas hybrid numbers*, Tamap. Journal of Mathematics and Statistics DOI:10.29371/2021.16.125. (2021).
- [4] Hamilton, W.R., *Elements of quaternions*, London: Longmans, Green, & Company, 1866.
- [5] Horzum, T. and Kocer, E.G. *On some properties of Horadam polynomials*, Int. Math. Forum **4** (2009), no. 25-28, 1243-1252.

- [6] Kızılateş, C. *A new generalization of Fibonacci hybrid and Lucas hybrid numbers*, Chaos, Solitons and Fractals **130**, 109449, (2020).
- [7] Kızılateş C. *A Note on Horadam Hybrinomials*, FUJMA **5** (2022), no. 1, 1-9.
- [8] Kılıc, N. *The $h(x)$ -Lucas quaternion polynomials*, Annales Mathematicae et Informaticae **47** (2017), 119-128.
- [9] Kumari, M., Tanti, J. and Prasad, K. *On some new families of k -Mersenne and generalized k -Gaussian Mersenne numbers and their polynomials*, arXiv preprint arXiv:2111.09592, (2021).
- [10] Kürüz, F., Dağdeviren, A. and Catarino, P. *On Leonardo Pisano Hybrinomials.*, Mathematics **9** (2021), no. 22, 2923.
- [11] Özdemir, M. *Introduction to hybrid numbers.*, Advances in Applied Clifford Algebras, **28** (2018), no. 1, 1-32.
- [12] Özkan E. and Uysal, M., *Mersenne-Lucas hybrid numbers*, Mathematica Montisnigri **52** (2021), no. 2, 17-29.
- [13] Özkan, E., Uysal, M. and Godase, A.D. *Hyperbolic k -Jacobsthal and k -Jacobsthal-Lucas quaternions*, Indian J. Pure Appl. Math. **53** (2022), no. 4, 956-967.
- [14] Özkan E. and Uysal, M., *On quaternions with higher order Jacobsthal numbers components*, Gazi University Journal of Science, **36** (2023), no. 1, 336-347.
- [15] Özkoç, A. and Porsuk, A. *A note for the (p, q) -Fibonacci and Lucas quaternion polynomials*, Konuralp Journal of Mathematics **5** (2017), no. 2, 36-46.
- [16] Patel, B.K. and Ray, P.K. *On the properties of (p, q) -Fibonacci and (p, q) -Lucas quaternions*, Math. Rep., Buchar. **21(71)** (2019), 15-25.
- [17] Polatlı, E. *A note on ratios of Fibonacci hybrid and Lucas hybrid numbers*, Notes Number Theory Discrete Math **27** (2021), no. 3, 73-78.
- [18] Ramírez, J.L., *Some combinatorial properties of the k -Fibonacci and the k -Lucas quaternions*, An. Ştiinţ. Univ. "Ovidius" Constanţa, Ser. Mat. **23** (2015). no. 2, 201-212.
- [19] Suvarnamani, A., and Tatong, M. *Some properties of (p, q) -Fibonacci numbers*, Progress in Applied Science and Technology **5** (2015), no. 2, 17-21.
- [20] Szynal-Liana, A., *The Horadam hybrid numbers*, Discuss. Math., Gen. Algebra Appl. **38** (2018), no. 1, 91-98.
- [21] Szynal-Liana, A. and Włoch, I. *On Jacobsthal and Jacobsthal-Lucas hybrid numbers*, Ann. Math. Sil. **33** (2019), 276-283.

- [22] Szynal-Liana, A. and Włoch, I., *On Pell and Pell-Lucas hybrid numbers*, Commentat. Math **58** (2018), no. 1-2, 11-17.
- [23] Szynal-Liana, A. and Włoch, I., *Introduction to Fibonacci and Lucas hybrid nomials*, Complex Var. Elliptic Equ. **65** (2020), no. 10, 1736-1747.
- [24] Uysal, M. and Özkan, E., *Padovan hybrid quaternions and some properties*, Journal of Science and Arts **22** (2022), no. 1, 121-132.
- [25] Uysal, M. and Özkan, E., *Higher-order Jacobsthal-Lucas quaternions*, Axioms (accepted for publication) (2022).

