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SOME RESULTS ON PROPER BIHARMONIC VECTOR FIELD

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Abstract

In this paper, we establish a necessary and sufficient conditions under which a vector field be biharmonic on Riemannian manifold. We also construct some examples of proper biharmonic vector fields from a resolution of differential equations.

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1 Introduction

Consider a smooth map $\phi : (M^m, g) \to (N^n, h)$ between two Riemannian manifolds, then the energy functional is defined by

$$E(\phi) = \frac{1}{2}\int_M |d\phi|^2 dv_g. \tag{1}$$

(or over any compact subset $K \subset M$).

A map is called harmonic if it is a critical point of the energy functional E (or $E(K)$ for all compact subsets $K \subset M$). For any smooth variation $\{\phi\}_{t\in I}$ of ϕ with $\phi_0 = \phi$ and $V = \frac{d\phi_t}{dt}\Big|_{t=0}$, we have

$$\frac{d}{dt}E(\phi_t)\Big|_{t=0} = -\int_M h(\tau(\phi), V)\, dv_g, \tag{2}$$

where

$$\tau(\phi) = trace_g \nabla d\phi. \tag{3}$$

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is the tension field of ϕ. Then ϕ is harmonic if and only if $\tau(\phi) = 0$.

One can refer to [7], [8], [13] for background on harmonic maps, and [4], [5] for background on generalized harmonic maps.

As a generalization of harmonic maps, biharmonic maps are defined similarily, as follows:
A map φ is said to be biharmonic if it is a critical point of the bi-energy functional

$$E_2(\varphi; D) = \frac{1}{2}\int_D |\tau(\varphi)|^2\, v_g, \tag{4}$$

over any compact domain D. Equivalently, φ is biharmonic if it satisfies the associated Euler-Lagrange equations:

$$\tau_2(\varphi) \equiv -Tr_g R^N(\tau(\varphi), d\varphi)d\varphi - Tr_g\big(\nabla^\varphi\nabla^\varphi\tau(\varphi) - \nabla^\varphi_{\nabla^M}\tau(\varphi)\big) = J_\varphi(\tau(\varphi)) = 0 \tag{5}$$

The operator $\tau_2(\varphi)$ is called the bitension field of φ (see [2], [6], [12]).

The existence and explicit construction of harmonic and biharmonic mappings between two given Riemannian manifolds (M, g) and (N, h) are two of the most fundamental problems of the theory of harmonic mappings. If M is compact and N has nonpositive sectional curvature, then any smooth map from M to N can be deformed into a harmonic map using the heat flow method [Eells and Sampson 1964]. However, there is no general existence theory of harmonic and biharmonic mappings if the target manifold does not satisfy the nonpositivity curvature condition. This fact makes it interesting to find harmonic maps defined by vector fields as a map from Riemannian manifold (M, g) to its tangent bundle TM. Problems of this kind have been studied when TM is endowed with the Riemannian Sasaki metric see ([3] [9] [10] [13]) and with the Riemannian Cheeger-Gromoll metric (see [1]). It is obvious to see that any harmonic map is biharmonic, therefore it is interesting to construct proper biharmonic maps (non-harmonic biharmonic maps).

The main idea in this note consists in the warping of the Sasaki metric. First we introduce a new metric called Mus-Sasaki metric on the tangent bundle TM. This new natural metric will lead us to interesting results (see [16] and [17]). Afterward we establish necessary and sufficient conditions under which a vector field be biharmonic (Theorem 2, Theorem 3, Theorem 4, Theorem 5 and Theorem 6). We also construct some examples of proper biharmonic vector fields from a resolution of partial differential equations (Example 2 to Example 5).

2 Basic notions and definition on TM.

2.1 Horizontal and vertical lifts on TM.

Let (M, g) be an m-dimensional Riemannian manifold and (TM, π, M) be its tangent bundle. A local chart $(U, x^i)_{i=1...n}$ on M induces a local chart $(\pi^{-1}(U), x^i, y^i)_{i=1...n}$ on TM. Denote by Γ_{ij}^k the Christoffel symbols of g and by ∇ the Levi-Civita connection of g.

We have two complementary distributions on TM, the vertical distribution $\mathcal{V}$ and the horizontal distribution $\mathcal{H}$, defined by :

$$\begin{aligned} \mathcal{V}_{(x,u)} &= Ker(d\pi_{(x,u)}) = \{a^i \frac{\partial}{\partial y^i}|_{(x,u)}; \quad a^i \in \mathbb{R}\} \\ \mathcal{H}_{(x,u)} &= \{a^i \frac{\partial}{\partial x^i}|_{(x,u)} - a^i u^j \Gamma_{ij}^k \frac{\partial}{\partial y^k}|_{(x,u)}; \quad a^i \in \mathbb{R}\}, \end{aligned}$$

where $(x, u) \in TM$, such that $T_{(x,u)}TM = \mathcal{H}_{(x,u)} \oplus \mathcal{V}_{(x,u)}$.

Let $X = X^i \frac{\partial}{\partial x^i}$ be a local vector field on M. The vertical and the horizontal lifts of X are defined by

$$X^V = X^i \frac{\partial}{\partial y^i} \tag{6}$$

$$X^H = X^i \frac{\delta}{\delta x^i} = X^i \{\frac{\partial}{\partial x^i} - y^j \Gamma_{ij}^k \frac{\partial}{\partial y^k}\} \tag{7}$$

For consequences, we have $(\frac{\partial}{\partial x^i})^H = \frac{\delta}{\delta x^i}$ and $(\frac{\partial}{\partial x^i})^V = \frac{\partial}{\partial y^i}$, then $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})_{i=1...n}$ is a local adapted frame in TTM.

Remark 1. .

1. *if* $w = w^i \frac{\partial}{\partial x^i} + \overline{w}^j \frac{\partial}{\partial y^j} \in T_{(x,u)}TM$, *then its horizontal and vertical parts are defined by*

$$w^h = w^i \frac{\partial}{\partial x^i} - w^i u^j \Gamma_{ij}^k \frac{\partial}{\partial y^k} \in \mathcal{H}_{(x,u)}$$

$$w^v = \{\overline{w}^k + w^i u^j \Gamma_{ij}^k\} \frac{\partial}{\partial y^k} \in \mathcal{V}_{(x,u)}$$

2. *if* $u = u^i \frac{\partial}{\partial x^i} \in T_x M$ *then its vertical and horizontal lifts are defined by*

$$\begin{aligned} u^V &= u^i \frac{\partial}{\partial y^i} \in \mathcal{V}_{(x,u)} \in \mathcal{H}_{(x,u)} \\ u^H &= u^i \{\frac{\partial}{\partial x^i} - y^j \Gamma_{ij}^k \frac{\partial}{\partial y^k}\}. \end{aligned}$$

Proposition 1 ([15]). *Let (M,g) be a flat Riemannian manifold, then for all vector fields $X,Y\in\Gamma(TM)$ and $p\in T^2M$ we have:*

1. $[X^H,Y^H]_p=[X,Y]^H_p$,
2. $[X^H,Y^V]_p=(\nabla_X Y)^V_p$,
3. $[X^V,Y^V]_p=0$,

where $p=(x,u)$.

Definition 1 ([14]). *Let (M,g) be a Riemannian manifold. The Sasaki metric $\widehat{g}$ is defined on the tangent bundle TM by:*

1. $\widehat{g}(X^H,Y^H)_{(x,u)}=g_x(X,Y)$
2. $\widehat{g}(X^H,Y^V)_{(x,u)}=0$
3. $\widehat{g}(X^V,Y^V)_{(x,u)}=g_x(X,Y)$

where $X,Y\in\Gamma(TM)$ and $(x,u)\in TM$.

Corollary 1. *If (M,g) is flat ($R=0$), then we obtain*

$$\begin{array}{lcl} 1)\ (\widehat{\nabla}_{X^H}Y^H)_{(x,u)} & = & (\nabla_X Y)^H_{(x,u)},\\ 2)\ (\widehat{\nabla}_{X^H}Y^V)_{(x,u)} & = & (\nabla_X Y)^V_{(x,u)},\\ 3)\ (\widehat{\nabla}_{X^V}Y^H)_{(x,u)} & = & 0,\\ 4)\ (\widehat{\nabla}_{X^V}Y^V)_{(x,u)} & = & 0,\\ 5)\ \widehat{R} & = & 0, \end{array}$$

3 Mus-Sasaki metric.

3.1 Mus-Sasaki metric.

Definition 2 ([16]). *Let (M,g) be a Riemannian manifold and $f: M\times\mathbb{R}\to\,]0,+\infty[$. On the tangent bundle TM, we define a Mus-Saski metric noted g_f by*

1. $g_f(X^H,Y^H)_{(x,u)}=g_x(X,Y)$
2. $g_f(X^H,Y^V)_{(x,u)}=0$
3. $g_f(X^V,Y^V)_{(x,u)}=f(x)g_x(X,Y)$

where $X,Y\in\Gamma(TM)$, $(x,u)\in TM$ and $r=g(u,u)$. f is called twisting function.

Note that, if $f = 1$ then g_f is the Sasaki metric [15]. For more detail on geometry of Mus-Sasaki metric see [11], [16].

Lemma 1 ([16]). *Let (M,g) be a Riemannian manifold, then for all $x \in M$ and $u = u^i \frac{\partial}{\partial x^i} \in T_xM$, we have the following*

1. $X^H(g(u,u))_{(x,u)} = 0$
2. $X^H(g(Y,u))_{(x,u)} = g(\nabla_X Y, u)_x$
3. $X^V(g(u,u)_{(x,u)} = 2g(X,u)_x$
4. $X^V(g(Y,u)_{(x,u)} = g(X,Y)_x$

Lemma 2. *Let (M,g) be a Riemannian manifold, then we have the following*

1. $X^V(f)_{(x,u)} = 0$
2. $X^H(f)_{(x,u)} = g_x(grad_M(f), X)$

where $(x,u) \in TM$.

Lemma 3. *Let (M,g) be a flat Riemannian manifold. We have*

$$\begin{array}{rcl}
1)\ (\widetilde{\nabla}_{X^H} Y^H)_{(x,u)} &=& (\nabla_X Y)^H_{(x,u)}, \\
2)\ (\widetilde{\nabla}_{X^H} Y^V)_{(x,u)} &=& (\nabla_X Y)^V_{(x,u)} + \frac{1}{2}X(f)Y^V, \\
3)\ (\widetilde{\nabla}_{X^V} Y^H)_{(x,u)} &=& \frac{1}{2}Y(f)X^V, \\
4)\ (\widetilde{\nabla}_{X^V} Y^V)_{(x,u)} &=& -\frac{1}{2}g(X,Y)\big(grad(f)\big)^H, \\
5)\ \widetilde{R}(X^H,Y^H)Y^H &=& 0, \\
6)\ \widetilde{R}(X^H,Y^V)Y^V &=& -\frac{1}{2}\|Y\|^2\big(\nabla_X grad(f)\big)^H + \frac{1}{4f}\|Y\|^2 X(f)\big(grad(f)\big)^H \\
7)\ \widetilde{R}(X^H,Y^H)Y^V &=& 0 \\
8)\ \widetilde{R}(X^H,Y^V)Y^H &=& \frac{1}{2}\Big[X(Y(f)) + \frac{1}{2}X(f)Y(f) - \big(\nabla_X Y\big)(f)\Big]Y^V \\
9)\ \widetilde{R}(X^V,Y^H)Y^H &=& -\frac{1}{2}\Big[Y^2(f) + \frac{1}{2}\big(Y(f)\big)^2 - \big(\nabla_Y Y\big)(f)\Big]X^V \\
10)\ \widetilde{R}(X^V,Y^V)Y^H &=& 0 \\
11)\ \widetilde{R}(X^V,Y^V)Y^V &=& \frac{f}{4}\|grad(f)\|^2\Big[g(X,Y)Y - \|Y\|^2 X\Big]^V.
\end{array}$$

where $\widetilde{\nabla}$ (resp $\widetilde{R}$) denote the Levi-Civita connection (resp curvature tensor) of $\widetilde{g}$.

The proof of Lemma 3 follows directly from Kozul formula, Lemma 1 and Lemma 2.

Theorem 1. *Let (M, g) be a flat Riemannian manifold and (TM, g_f) be its tangent bundle equipped with the Mus-Sasaki metric. Then the tension field associated with $\pi : (TM, g_f) \to (M, g)$ is given by:*

$$\tau^f(\pi) = \frac{m}{2f} grad(f). \tag{8}$$

So, π is harmonic if and only $f = const$.

Proof. Let $(E_1, ..., E_m$ be a local orthonormal frame on M, then $(E_1^H, ..., E_m^H, \frac{1}{\sqrt{f}}E_1^V, ..., \frac{1}{\sqrt{f}}E_m^V)$ is an orthonormal frame on (TM, g_f). From Equation 3 and Lemma 3 we have

$$\begin{aligned}
\tau^f(\pi)_x &= \sum_i \Big[\nabla_{d\pi(E_i^H)} d\pi(E_i^H) - d\pi(\widetilde{\nabla}_{E_i^H} E_i^H)\Big] \\
&\quad + \sum_i \Big[\nabla_{d\pi(\frac{1}{\sqrt{f}}E_i^V)} d\pi(\frac{1}{\sqrt{f}}E_i^V) - d\pi(\widetilde{\nabla}_{\frac{1}{\sqrt{f}}E_i^V} \frac{1}{\sqrt{f}}E_i^V)\Big] \\
&= -\sum_i \Big[d\pi(\widetilde{\nabla}_{\frac{1}{\sqrt{f}}E_i^V} \frac{1}{\sqrt{f}}E_i^V)\Big] \\
&= \sum_i \frac{1}{2f} g(E_i, E_i) grad(f) \\
&= \frac{m}{2f} grad(f)
\end{aligned}$$

□

From Equation 5 and Theorem 1 we obtain the following theorem

Theorem 2. *Let (M, g) be a flat Riemannian manifold and (TM, g_f) be its tangent bundle equipped with the Mus-Sasaki metric. Then the bitension field associated with $\pi : (TM, g^S) \to (M, g)$ is given by:*

$$\tau_2^f(\pi) = -Tr_g \nabla\nabla\tau^f(\pi) + Tr_g \nabla_\nabla \tau^f(\pi) - \frac{m}{2}\nabla_{grad(f)}\tau^f(\pi). \tag{9}$$

Example 1. *[proper biharmonic map]*

Let $M = \mathbb{R}^m$. If we set $f(x, x_2, ..., x_m) = f(x)$, then π is a proper biharmonic if and only $f' \neq 0$ and f is a solution of the following differential equation

$$[\ln(f)]''' + \frac{m}{2} f'[\ln(f)]'' = 0. \tag{10}$$

If we set $f(x) = k.\exp(a.x)$, $k > 0$ then f is a solution of equation (10) and π is a proper biharmonic.

Lemma 4. *Let (M,g) be a Riemannian manifold . If $X,Y\in\Gamma(TM)$ are a vector fields and $(x,u)\in TM$ such that $X_x=u$, then we have*

$$d_xX(Y_x)=Y^H_{(x,u)}+(\nabla_YX)^V_{(x,u)}.$$

Proof. Let (U,x^i) be a local chart on M in $x\in M$ and $(\pi^{-1}(U),x^i,y^j)$ be the induced chart on TM, if $X_x=X^i(x)\frac{\partial}{\partial x^i}|_x$ and $Y_x=Y^i(x)\frac{\partial}{\partial x^i}|_x$, then

$$d_xX(Y_x)=Y^i(x)\frac{\partial}{\partial x^i}|_{(x,X_x)}+Y^i(x)\frac{\partial X^k}{\partial x^i}(x)\frac{\partial}{\partial y^k}|_{(x,X_x)},$$

thus the horizontal part is given by

$$\begin{aligned}(d_xX(Y_x))^h &= Y^i(x)\frac{\partial}{\partial x^i}|_{(x,X_x)}-Y^i(x)X^j(x)\Gamma^k_{ij}(x)\frac{\partial}{\partial y^k}|_{(x,X_x)}\\ &= Y^H_{(x,X_x)}\end{aligned}$$

and the vertical part is given by

$$\begin{aligned}(d_xX(Y_x))^v &= \{Y^i(x)\frac{\partial X^k}{\partial x^i}(x)+Y^i(x)X^j(x)\Gamma^k_{ij}(x)\}\frac{\partial}{\partial y^k}|_{(x,X_x)}\\ &= (\nabla_YX)^V_{(x,X_x)}.\end{aligned}$$

□

Theorem 3. *Let (M,g) be a flat Riemannian manifold and (TM,g_f) be its tangent bundle equipped with the Mus-Sasaki metric. Then the tension field associated with $X\in\Gamma(T(TM))$ is given by:*

$$\tau^f(X) = \Big[Tr_g\nabla^2X+\nabla_{grad(f)}X\Big]^V-\frac{1}{2}\Big[Tr_gg(\nabla X,\nabla X)grad(f)\Big]^H. \quad (11)$$

Proof. Let $x\in M$ and $\{E_i\}_{i=1}^n$ be a local orthonormal frame on M such that $\nabla_{E_i}E_j=0$ at x and $X_x=u$, then by summing over i, we have

$$\begin{aligned}\tau^f(X)_x &= \Big[\widetilde{\nabla}_{dX(E_i)}dX(E_i)\Big]\\ &= \Big[\widetilde{\nabla}_{E_i^H+(\nabla_{E_i}X)^V}(E_i^H+(\nabla_{E_i}X)^V)\Big]\\ &= \Big[\widetilde{\nabla}_{E_i^H}E_i^H+\widetilde{\nabla}_{E_i^H}(\nabla_{E_i}X)^V+\widetilde{\nabla}_{(\nabla_{E_i}X)^V}E_i^H+\widetilde{\nabla}_{(\nabla_{E_i}X)^V}(\nabla_{E_i}X)^V\Big]\\ &= \big(Tr_g\nabla^2X\big)^V+\big(\nabla_{grad(f)}X\big)^V-\frac{1}{2}Tr_gg(\nabla X,\nabla X)\big(grad(f)\big)^H\end{aligned}$$

by Lemma 4, Theorem 3 follows.

□

Corollary 2. *Let (M,g) be a flat Riemannian manifold and (TM,g_f) be its tangent bundle equipped with the Mus-Sasaki metric. If $X:(M,g)\to(Tm,\hat{g})$ is harmonic, then the tension field associated with $X\in\Gamma(T(TM))$ is given by:*

$$\tau^f(X) \;=\; \Big[\nabla_{grad(f)}X\Big]^V - \frac{1}{2}\Big[Tr_g g(\nabla X,\nabla X)grad(f)\Big]^H. \qquad (12)$$

From Corollary 2 we obtain

Theorem 4. *Let (M,g) be a flat Riemannian manifold and (TM,g_f) be its tangent bundle equipped with the Mus-Sasaki metric. A vector field$X\in\Gamma(TM)$ is harmonic if and only the following conditions are verified*

$$\begin{cases} Tr_g\nabla^2 X + \nabla_{grad(f)}X = 0 \\ \\ Tr_g g(\nabla X,\nabla X)grad(f) = 0 \end{cases}$$

Example 2. *Let $M=\mathbb{R}^m$ and $f=const$. Then $X=(X^1,...,X^m)$ is harmonic if and only if X^k is harmonic for all $k\in\{1,..,m\}$ i.e*

$$\triangle X^k = 0, \quad 1\le k\le m.$$

Example 3. *Let $M=\mathbb{R}^m$ and f be a smooth function non-costant . Then $X=(X^1,...,X^m)$ is harmonic if and only if X^k is constant for all $k\in\{1,..,m\}$.*

Lemma 5. *Let $M=R^m$, $f(x,x_2,...,x_m)=f(x)$ and $X=(ax+b)\partial_1$; $a\neq 0$, $a,x\in\mathbb{R}$. Then we have*

$$\begin{aligned}
\nabla_{\partial_i}X &= (\delta_i^1 a,0,...,0) = \delta_i^1 a\partial_i \\
Tr_g g(\nabla X,\nabla X) &= a^2 \\
Tr_g\nabla^2 X &= 0 \\
\tau^f(X) &= af'\partial_1^V - \frac{1}{2}a^2 f'\partial_1^H \\
dX(\partial_i) &= \partial_i^H + \delta_i^1 a\partial_i^V \\
\tilde{\nabla}_{dX(\partial_i)}\tau^f(X) &= 0, \quad 2\le i\le m. \\
\tilde{\nabla}_{dX(\partial_1)}\tau^f(X) &= F_1(x)\partial_1^V + F_2(x)\partial_1^H.
\end{aligned}$$

where $F_1 = af'' + \frac{a}{2}(f')^2 - \frac{1}{4}a^3(f')^2$ *and* $F_2 = -\frac{a^2}{2}\big[f''+(f')^2\big]$.

Proof.

$$\begin{aligned}
\tilde{\nabla}_{dX(\partial_1)}\tau^f(X) &= \tilde{\nabla}_{\partial_1^H+\left(\nabla_{\partial_1}X\right)^V}\left(af'\partial_1^V-\frac{1}{2}a^2f'\partial_1^H\right)\\
&= \tilde{\nabla}_{\partial_1^H+a\partial_1^V}\left(af'\partial_1^V-\frac{1}{2}a^2f'\partial_1^H\right)\\
&= a\tilde{\nabla}_{\partial_1^H}(f'\partial_1^V)-\frac{1}{2}a^2\tilde{\nabla}_{\partial_1^H}(f'\partial_1^H)+a^2\tilde{\nabla}_{\partial_1^V}\left(f'\partial_1^V\right)\\
&\quad -\frac{1}{2}a^3\tilde{\nabla}_{\partial_1^V}(f'\partial_1^H)\\
&= af''\partial_1^V+\frac{a}{2}(f')^2\partial_1^V-\frac{a^2}{2}f''\partial_1^H-\frac{a^2}{2}(f')^2\partial_1^H\\
&\quad -\frac{1}{4}a^3(f')^2\partial_1^V\\
&= \left[af''+\frac{a}{2}(f')^2-\frac{1}{4}a^3(f')^2\right]\partial_1^V-\frac{a^2}{2}\left[f''+(f')^2\right]\partial_1^H\\
&= F_1(x)\partial_1^V+F_2(x)\partial_1^H.
\end{aligned}$$

□

Lemma 6. *Let* $M=R^m$, $f(x,x_2,...,x_m)=f(x)$ *and* $X=(ax+b)\partial_1;\quad a\neq 0,\quad a,x\in\mathbb{R}$. *Then we have*

$$\begin{aligned}
\tilde{\nabla}_{dX(\partial_i)}\tilde{\nabla}_{dX(\partial_i)}\tau^f(X) &= 0,\quad (2\le i\le m).\\
\tilde{\nabla}_{dX(\partial_1)}\tilde{\nabla}_{dX(\partial_1)}\tau^f(X) &= \left[F_2'-\frac{1}{2}aF_1f'\right]\partial_1^H+\left[F_1'+\frac{1}{2}f'(F_1+aF_2)\right]\partial_1^V\\
\widetilde{R}(\tau^f(X),dX(\partial_i))dX(\partial_i) &= 0,\quad (2\le i\le m).\\
\widetilde{R}(\tau^f(X),dX(\partial_1))dX(\partial_1) &= \frac{a}{2}(1+\frac{a^2}{2})f'\Big\{a\left[f''-\frac{1}{2}\frac{(f')^2}{f}\right]\partial_1^H\\
&\quad -\left[f''+\frac{1}{2}(f')^2\right]\partial_1^V\Big\} \qquad (13)
\end{aligned}$$

Proof. Using Lemma 3, we obtain

$$\widetilde{R}(\tau^f(X),\partial_1^H)\partial_1^H = -\frac{a}{2}f'\left[f''+\frac{1}{2}(f')^2\right]\partial_1^V \qquad (14)$$

$$\widetilde{R}(\tau^f(X),\partial_1^H)\partial_1^V = \frac{a}{2}f'\left[f''-\frac{1}{2}\frac{(f')^2}{f}\right]\partial_1^H \qquad (15)$$

$$\widetilde{R}(\tau^f(X),\partial_1^V)\partial_1^H = -\frac{a^2}{4}f'\left[f''+\frac{1}{2}(f')^2\right]\partial_1^V \qquad (16)$$

$$\widetilde{R}(\tau^f(X),\partial_1^V)\partial_1^V = \frac{a^2}{4}f'\left[f''-\frac{1}{2}\frac{(f')^2}{f}\right]\partial_1^H \qquad (17)$$

$$\begin{aligned}
\widetilde{R}(\tau^f(X),dX(\partial_1))dX(\partial_1) &= \widetilde{R}\left(\tau^f(X),\partial_1^H+a\partial_1^V\right)(\partial_1^H+a\partial_1^V)\\
&= \widetilde{R}\left(\tau^f(X),\partial_1^H\right)\partial_1^H+a\widetilde{R}\left(\tau^f(X),\partial_1^H\right)\partial_1^V\\
&\quad +a\widetilde{R}\left(\tau^f(X),\partial_1^V\right)\partial_1^H+a^2\widetilde{R}\left(\tau^f(X),\partial_1^V\right)\partial_1^V \qquad (18)
\end{aligned}$$

Substituting (14), (15), (16) and (17) in (18) we obtain (13).

□

From Lemma 6, we obtain the following theorem

Theorem 5. *Let* $M = R^m$, $f(x, x_2, ..., x_m) = f(x)$ *and* $X = (ax + b)\partial_1$; $a \neq 0$, $a, x \in \mathbb{R}$. *Then we have*

$$-\tau_2^f(X) = \Big\{F_2' - \frac{1}{2}aF_1f' + \frac{a^2}{2}(1 + \frac{a^2}{2})f'\big[f'' - \frac{1}{2}\frac{(f')^2}{f}\big]\Big\}\partial_1^H$$
$$+\Big\{F_1' + \frac{1}{2}f'(F_1 + aF_2) - \frac{a}{2}(1 + \frac{a^2}{2})f'\big[f'' + \frac{1}{2}(f')^2\big]\Big\}\partial_1^V \qquad (19)$$

So, X is biharmonic if and only if f is solution of the following equations:

$$\begin{cases} F_2' - \frac{1}{2}aF_1f' + \frac{a^2}{2}(1 + \frac{a^2}{2})f'\big[f'' - \frac{1}{2}\frac{(f')^2}{f}\big] = 0 \\ F_1' + \frac{1}{2}f'(F_1 + aF_2) - \frac{a}{2}(1 + \frac{a^2}{2})f'\big[f'' + \frac{1}{2}(f')^2\big] = 0 \end{cases} \qquad (20)$$

where $F_1 = af'' + \frac{a}{2}(f')^2 - \frac{1}{4}a^3(f')^2$ *and* $F_2 = -\frac{a^2}{2}\big[f'' + (f')^2\big]$.

Example 4. *Let* $a = \sqrt{2}$, *then we have:*

$$F_1 = \sqrt{2}f''.$$
$$F_2 = -f'' - (f')^2.$$

and $X = (\sqrt{2}x + b)\partial_1$ *is biharmonic if and only f is solution of the following equations:*

$$\begin{cases} f.f''' + f.f'.f'' + (f')^3 = 0 \\ f''' - f'.f'' - (f')^3 = 0 \end{cases} \qquad (21)$$

Lemma 7. *Let* $M = R^m$, $f(x, y, x_3, ..., x_m) = f(y)$ *and* $X = (ax + b)\partial_1$; $a \neq 0$, $a, x \in \mathbb{R}$. *Then we have*

$$\nabla_{\partial_i}X = \delta_i^1 a\partial_i$$
$$Tr_g g(\nabla X, \nabla X) = a^2$$
$$Tr_g \nabla^2 X = 0$$
$$dX(\partial_i) = \partial_i^H + \delta_i^1 a\partial_i^V$$
$$\tau^f(X) = -\frac{1}{2}a^2 f'\partial_2^H$$
$$\tilde{\nabla}_{dX(\partial_1)}\tau^f(X) = -\frac{a^3}{4}(f')^2\partial_1^V.$$
$$\tilde{\nabla}_{dX(\partial_2)}\tau^f(X) = -\frac{a^2}{2}f''\partial_2^H.$$
$$\tilde{\nabla}_{dX(\partial_i)}\tau^f(X) = 0, \quad 3 \leq i \leq m.$$

Proof. Same as the proof in Lemma 5. Using Lemma 7 we obtain the following lemma □

Lemma 8. *Let $M = R^m$, $f(x, y, x_3, ..., x_m) = f(y)$ and $X = (ax + b)\partial_1$; $a \neq 0$, $a, x, y, x_3, ..., x_m \in \mathbb{R}$. Then we have*

$$\begin{aligned}
\tilde{\nabla}^2_{dX(\partial_1)}\tau^f(X) &= \frac{a^4}{8}(f')^3\partial_2^H. \\
\tilde{\nabla}^2_{dX(\partial_2)}\tau^f(X) &= -\frac{a^2}{2}f'''\partial_2^H. \\
\tilde{\nabla}_{dX(\partial_i)}\tau^f(X) &= 0, \quad 3 \leq i \leq m. \\
\widetilde{R}(\tau^f(X), dX(\partial_2))dX(\partial_2) &= \frac{a^3}{8}(f')^2\big[f'' + \frac{1}{2}(f')^2\big]\partial_1^V \\
\widetilde{R}(\tau^f(X), dX(\partial_i))dX(\partial_i) &= 0, \quad (i \neq 2).
\end{aligned}$$

From Lemma 8 we obtain the following theorem

Theorem 6. *Let $M = R^m$, $f(x, y, x_3, ..., x_m) = f(y)$ and $X = (ax + b)\partial_1$; $a \neq 0$, $a, x, y, x_3, ..., x_m \in \mathbb{R}$. Then we have*

$$-\tau_2^f(X) = \frac{a^3}{8}(f')^2\big[f'' + \frac{1}{2}(f')^2\big]\partial_1^V + \frac{a^2}{2}\big[\frac{a^2}{4}(f')^3 - f'''\big]\partial_2^H \tag{22}$$

So, X is biharmonic if and only if f is solution of the following equations:

$$\begin{cases} f'' + \frac{1}{2}(f')^2 = 0 \\ \frac{a^2}{4}(f')^3 - f''' = 0 \end{cases} \tag{23}$$

Example 5. *Let $a = \sqrt{2}$, then $f(x, y, x_3, ..., x_m) = f(y) = \ln(y^2)$ is a solution of the Equation (23) and $X = X = (ax + b)\partial_1$ is proper biharmonic vector field.*

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