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SPATIAL BEHAVIOUR IN THE COUPLED THEORY FOR VISCOELASTIC MATERIALS WITH VOIDS

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Abstract

We analyse viscoelastic porous materials in the coupled linear theory. We consider the coupling between the volume fraction and Darcy's law for a right cylinder. The mathematical model is represented by a system of equations of steady vibrations for the displacement vector, the changes of the volume fraction and the pressure. The spatial behaviour is characterised in terms of some cross-sectional functional.

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1 Introduction

The mechanical and hydraulic coupling was introduced for the first time by Biot in [1]. In this paper, Biot analyses the three-dimensional consolidation theory of solids with voids in the isotropic case by means of Darcy's law. Then, in [2], Biot studies the viscoelastic (time-dependent) effects for elastic solids with voids.

Another important concept in the theory of materials with voids is the volume fraction. Based on this concept, Cowin and Nunziato studied in [12] and [14] a linear and a nonlinear approach, respectively, for coupled elasticity in the case of deformable bodies with voids.

Recently, M. M. Svanadze considered in [18] the coupled linear theory of viscoelasticity for bodies with voids. In this paper, we note the coupling between the concept of volume fraction and Darcy's law. This type of coupling was also studied in [16] and [17]. This mathematical model can be employed to predict the physical properties of some biomaterials (see [13], [15]) and geomaterials, such as soils which are water saturated, rocks which are impregnated with oil or foams that are filled with air (see [7])

Viscoelastic materials were studied in many recent articles, see for example [6], [18], [19], [20], [21], [22]. The spatial behaviour of the solution for some problems

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in thermoelasticity is studied in some recent articles, see for example [3], [4], [5], [8], [11]. Electrical effects were considered in thermoelasticity in the context of the Green-Naghdi theory, see for example [9], [10].

In this article, we analyse the mathematical model introduced by M. M. Svanadze in [18]. More precisely, we study the spatial behaviour of the solution for vibrating viscoelastic porous materials in the coupled theory following the approach from [11].

2 Preliminaries

In this study, we analyse a viscoelastic material with voids which lies in the three-dimensional Euclidean space $\mathbb{R}^3$. For this material, the skeletal portion is represented by a Kelvin-Voigt solid. We consider that $\hat{\mathbf{u}} = (\hat{u}_1, \hat{u}_2, \hat{u}_3)$ is a vector representing the displacement in the body, $\hat{\varphi}$ represents the change of the fraction of volume for pores and $\hat{p}$ is the change of the pressure of the fluid in the voids network.

We consider that $\mathbf{x} = (x_1, x_2, x_3)$ is a point in $\mathbb{R}^3$. Moreover, t is a variable representing the time, with $t \geq 0$. In the sequel, differentiation with respect to t is represented by a superposed dot. Furthermore, partial differentiation in terms of a Cartesian coordinate is represented by the associated subscript preceded by a comma. Note that all functions that have a hat depend both on the time t and on the space coordinate $\mathbf{x}$.

The framework is a system of rectangular Cartesian axes Ox_k, with $k = 1, 2, 3$, which is fixed. Latin subscripts can have the values $1, 2, 3$, while Greek subscripts can have the values $1, 2$. We assume that a regular region B contains an isotropic and homogeneous viscoelastic body with pores.

The mathematical model of the coupled linear theory of viscoelasticity for solids with voids is represented by the governing system of field equations of motion introduced by M. M. Svanadze in [18]. The equations of motion are

$$\begin{aligned} &\hat{t}_{lj,j} = \rho\left(\ddot{\hat{u}}_l - \hat{F}_l'\right), \\ &\hat{\sigma}_{j,j} + \hat{\xi} = \rho_1\ddot{\hat{\varphi}} - \rho\hat{s}_1, l = 1, 2, 3, \end{aligned} \tag{1}$$

with

$$\hat{\xi} = -b\hat{e}_{rr} - \zeta\hat{\varphi} + m\hat{p} - \gamma^*\dot{\hat{e}}_{rr} - \zeta^*\dot{\hat{\varphi}} \tag{2}$$

and

$$\hat{e}_{lj} = \frac{1}{2}\left(\hat{u}_{l,j} + \hat{u}_{j,l}\right). \tag{3}$$

The constitutive equations are [18]

$$\begin{aligned} &\hat{t}_{lj} = 2\mu\hat{e}_{lj} + \lambda\hat{e}_{rr}\delta_{lj} + (b\hat{\varphi} - \beta\hat{p})\delta_{lj} + 2\mu^*\dot{\hat{e}}_{lj} + \lambda^*\dot{\hat{e}}_{rr}\delta_{lj} + b^*\dot{\hat{\varphi}}\delta_{lj}, \\ &\hat{\sigma}_l = \alpha\hat{\varphi}_{,l} + \alpha^*\dot{\hat{\varphi}}_{,l}, \quad l, j = 1, 2, 3. \end{aligned} \tag{4}$$

The equation of fluid mass conservation is [18]

$$\hat{v}_{j,j} + a\dot{\hat{p}} + \beta\dot{\hat{e}}_{rr} + m\dot{\hat{\varphi}} = 0, \tag{5}$$

while Darcy's law is given by

$$\hat{\mathbf{v}} = -\frac{k'}{\mu'}\nabla \hat{p} - \rho_2 \hat{\mathbf{s}}_2. \tag{6}$$

In the equations above, ρ is the mass density in the configuration of reference, with $\rho > 0$, $\hat{\mathbf{F}}' = \left(\hat{F}_1', \hat{F}_2', \hat{F}_3'\right)$ represents the body force per unit mass, $\hat{t}_{lj}$ is the tensor of total stress, $\hat{\sigma}_j$, with $j = 1, 2, 3$ represents the vector of the equilibrated stress corresponding to a single double force system without moment, ρ_1 is the coefficient of the equilibrated inertia, with $\rho_1 > 0$, $\hat{s}_1$ is the extrinsic equilibrated body force, ξ is the intrinsic equilibrated body force and $\hat{e}_{lj}$ is the tensor representing the strain.

In the constitutive equations, λ and μ are the Lame constants, δ_{lj} is Kronecker's delta, β is the effective stress parameter, m is the coefficient of cross-correlation, b, α and ζ are material parameters, while $\lambda^*, \mu^*, b^*, \alpha^*, \zeta^*$ and γ^* are viscoelastic constants.

In the other equations, $\hat{\mathbf{v}} = (\hat{v}_1, \hat{v}_2, \hat{v}_3)$ is the vector representing the fluid flux, a describes the compressibility of the voids, with $a \neq 0$, μ' is the viscosity of the fluid, k' is the macrosopic intrinsic permeability corresponding to the voids network, ρ_2 is the fluid density, $\hat{\mathbf{s}}_2$ is the external force (such as gravity) for the voids phase, while ∇ is the gradient operator.

As in [18], from equations (1) and (5), with (2)-(4) and (6), we deduce a system of equations of motion for the displacement vector $\hat{\mathbf{u}}$, the changes of the volume fraction $\hat{\varphi}$ and the pressure $\hat{p}$

$$\begin{aligned}
&\tilde{\mu}\Delta\hat{\mathbf{u}} + (\tilde{\lambda} + \tilde{\mu})\nabla \mathrm{div}\hat{\mathbf{u}} + \tilde{b}\nabla\hat{\varphi} - \beta\nabla\hat{p} = \rho\left(\ddot{\hat{\mathbf{u}}} - \hat{\mathbf{F}}'\right),\\
&\tilde{\alpha}\Delta\hat{\varphi} - \tilde{\zeta}\hat{\varphi} - \tilde{\gamma}\mathrm{div}\hat{\mathbf{u}} + m\hat{p} = \rho_1\ddot{\hat{\varphi}} - \rho\hat{s}_1,\\
&k\Delta\hat{p} - a\dot{\hat{p}} - \beta \mathrm{div}\dot{\hat{\mathbf{u}}} - m\dot{\hat{\varphi}} = -\rho_2 \mathrm{div}\hat{\mathbf{s}}_2,
\end{aligned} \tag{7}$$

with $k = \frac{k'}{\mu'}$, Δ the Laplacian operator and

$$\begin{aligned}
&\tilde{\lambda} = \lambda + \lambda^*\frac{\partial}{\partial t}, \quad \tilde{\mu} = \mu + \mu^*\frac{\partial}{\partial t}, \quad \tilde{b} = b + b^*\frac{\partial}{\partial t},\\
&\tilde{\alpha} = \alpha + \alpha^*\frac{\partial}{\partial t}, \quad \tilde{\gamma} = b + \gamma^*\frac{\partial}{\partial t}, \quad \tilde{\zeta} = \zeta + \zeta^*\frac{\partial}{\partial t}.
\end{aligned} \tag{8}$$

3 Basic assumptions

Now we consider that B is a right cylinder which has length $L > 0$. We assume that the cross section of B is bounded by one or more curves which are piecewise smooth. The Cartesian coordinates are considered such that the origin appears in the end of the cylinder which is on the lower base. Moreover, the x_3-axis and the generators are parallel. We consider that $D(x_3)$ is the cross section of B associated with the axial distance x_3 and that $\partial D(x_3)$ is the cross-sectional boundary. Moreover, let π be the lateral surface of B, such that $\pi = \partial D \times (0, L)$ and $B = D \times (0, L)$.

Now we consider that $\hat{\mathbf{u}}$, $\hat{\varphi}$, $\hat{p}$, $\hat{\mathbf{F}}$, $\hat{s}_1$ and $\hat{\mathbf{s}}_2$ have a harmonic time variation, i.e.

$$\left\{\hat{\mathbf{u}}, \hat{\varphi}, \hat{p}, \hat{\mathbf{F}}', \hat{s}_1, \hat{\mathbf{s}}_2\right\}(\mathbf{x}, t) = \mathrm{Re}\left[\left\{\mathbf{u}, \varphi, p, \mathbf{F}', s_1, \mathbf{s}_2\right\}(\mathbf{x}) e^{-i\omega t}\right]. \qquad (9)$$

It follows that we have the following differential system of steady vibrations in the coupled linear theory of viscoelasticity for materials with voids when $\mathbf{F}'$, s_1 and $\mathbf{s}_2$ are null [18]

$$\begin{aligned} &\mu_1 u_{s,rr} + \rho\omega^2 u_s + (\lambda_1 + \mu_1) u_{r,rs} + b_1 \varphi_{,s} - \beta p_{,s} = 0 \\ &\alpha_1 \varphi_{,rr} + \zeta_2 \varphi - \gamma_1 u_{m,m} + mp = 0 \\ &kp_{,rr} + a'p + \beta' u_{m,m} + m'\varphi = 0 \end{aligned} \qquad (10)$$

where ω is the frequency of oscillation, with $\omega > 0$ and

$$\begin{aligned} &\lambda_1 = \lambda - i\omega\lambda^*, \quad \mu_1 = \mu - i\omega\mu^*, \quad b_1 = b - i\omega b^*, \\ &\alpha_1 = \alpha - i\omega\alpha^*, \quad \gamma_1 = b - i\omega\gamma^*, \quad \zeta_1 = \zeta - i\omega\zeta^*, \\ &\zeta_2 = \rho_1\omega^2 - \zeta_1, \quad a' = i\omega a, \quad \beta' = i\omega\beta, \quad m' = i\omega m. \end{aligned} \qquad (11)$$

To this system describing the amplitude of the oscillation we add the lateral boundary conditions

$$u_r(\mathbf{x}) = 0, \quad \varphi(\mathbf{x}) = 0, \quad p(\mathbf{x}) = 0 \text{ on } \pi \qquad (12)$$

and the following boundary conditions for the ends of the cylinder

$$\begin{aligned} &u_r(x_1, x_2, 0) = \tilde{u}_r(x_1, x_2), \quad \varphi(x_1, x_2, 0) = \tilde{\varphi}(x_1, x_2), \\ &p(x_1, x_2, 0) = \tilde{p}(x_1, x_2) \text{ for all } (x_1, x_2) \in D(0) \end{aligned} \qquad (13)$$

and

$$\begin{aligned} &u_r(x_1, x_2, L) = 0, \quad \varphi(x_1, x_2, L) = 0, \\ &p(x_1, x_2, L) = 0 \text{ for all } (x_1, x_2) \in D(L), \end{aligned} \qquad (14)$$

where $\tilde{u}_r(x_1, x_2)$, $\tilde{\varphi}(x_1, x_2)$ and $\tilde{p}(x_1, x_2)$ are given functions on $D(0)$.

In the sequel, we will analyse the boundary value problem $\mathcal{P}$ which is characterized by the governing system of equations (10), the boundary conditions (12) on the lateral of the cylinder, the boundary conditions (13) and (14) on the ends of the cylinder. The solution of this boundary value problem $\mathcal{P}$ is the amplitude of oscillation (9).

The quadratic form

$$U_1(\chi_{ij}, \xi) = \lambda^* \chi_{mm}\chi_{nn} + 2\mu^* \chi_{rs}\chi_{rs} + (b^* + \gamma^*)\chi_{mm}\xi + \zeta^*\xi^2, \quad \chi_{rs} = \chi_{sr} \qquad (15)$$

is positive definite if

$$\mu^* > 0, \zeta^* > 0, \frac{1}{4}(b^* + \gamma^*)^2 < \zeta^*(\lambda^* + \frac{2}{3}\mu^*). \qquad (16)$$

The quadratic form

$$U_2(\eta_p, \zeta_q) = \alpha^*\omega^2 \eta_r\eta_r + k\zeta_r\zeta_r \qquad (17)$$

is positive definite if

$$\alpha^* > 0, k > 0. \tag{18}$$

Let us denote by π_2 and π_1 the greatest and the lowest eigenvalues of the quadratic form U_1. Moreover, let ν_2 and ν_1 be the greatest and the lowest eigenvalues of the quadratic form U_2. This implies that [11]

$$\pi_1(\chi_{rs}\chi_{rs} + \xi^2) \leq U_1(\chi_{ij}, \xi) \leq \pi_2(\chi_{rs}\chi_{rs} + \xi^2), \tag{19}$$

$$\nu_1(\eta_p\eta_p + \zeta_q\zeta_q) \leq U_2(\eta_p, \zeta_q) \leq \nu_2(\eta_p\eta_p + \zeta_q\zeta_q). \tag{20}$$

4 Spatial behaviour

In the sequel, our aim is to characterize the spatial growth and decay properties of the amplitude for finite cylinders first and then for semi-infinite cylinders filled with a viscoelastic material with pores.

Let us denote by a superposed bar the complex conjugate. Let us consider the following cross-sectional functional

$$\begin{aligned} J(x_3) = -\int_{D(x_3)} & \{i\omega\left[\mu_1 u_{r,3}\bar{u}_r - \bar{\mu}_1 u_r \bar{u}_{r,3} + (\lambda_1 + \mu_1)\bar{u}_3 u_{r,r} - \right. \\ & \left. - (\bar{\lambda}_1 + \bar{\mu}_1) u_3 \bar{u}_{r,r}\right] + i\omega(b_1\bar{u}_3\varphi - \bar{b}_1 u_3\bar{\varphi}) + i\omega(\beta u_3\bar{p} - \beta\bar{u}_3 p) + \\ & + i\omega(\alpha_1\varphi_{,3}\bar{\varphi} - \bar{\alpha}_1\bar{\varphi}_{,3}\varphi) + (kp_{,3}\bar{p} + k\bar{p}_{,3}p)\}\, da. \end{aligned} \tag{21}$$

Theorem 1. *Let B be a finite cylinder filled with a viscoelastic material with voids and characterized by the boundary value problem $\mathcal{P}$. Then*

$$\begin{aligned} -\frac{dJ}{dx_3}(x_3) = \int_{D(x_3)} & \left\{\omega^2\left[2\mu^* u_{s,r}\bar{u}_{s,r} + 2(\lambda^* + \mu^*)u_{s,s}\bar{u}_{r,r}\right] + \right. \\ & + \omega^2\left[2\alpha^*\varphi_{,r}\bar{\varphi}_{,r} + 2\zeta^*\varphi\bar{\varphi}\right] + 2kp_{,r}\bar{p}_{,r} + \\ & \left. + \omega^2(b^* + \gamma^*)\left(\bar{u}_{s,s}\varphi + u_{s,s}\bar{\varphi}\right)\right\} da. \end{aligned} \tag{22}$$

Proof. We consider the complex conjugate of equation $(10)_1$ and we multiply it with u_s. Then we multiply the equation $(10)_1$ by the complex conjugate $\bar{u}_s$. Therefore, we obtain

$$\begin{aligned} 0 = u_s\left(\rho\omega^2\bar{u}_s\right) - \bar{u}_s\left(\rho\omega^2 u_s\right) = & \left(\mu_1\bar{u}_s u_{s,rr} - \bar{\mu}_1 u_s\bar{u}_{s,rr}\right) + \\ & + \left[(\lambda_1 + \mu_1)\bar{u}_s u_{r,rs} - (\bar{\lambda}_1 + \bar{\mu}_1)u_s\bar{u}_{r,rs}\right] + \\ & + \left(b_1\bar{u}_s\varphi_{,s} - \bar{b}_1 u_s\bar{\varphi}_{,s}\right) + \left(\beta u_s\bar{p}_{,s} - \beta\bar{u}_s p_{,s}\right). \end{aligned} \tag{23}$$

This leads further to

$$\begin{aligned} & \left[(\mu_1 u_{r,s}\bar{u}_r - \bar{\mu}_1 u_r\bar{u}_{r,s}) + (\lambda_1 + \mu_1)\bar{u}_s u_{r,r} - (\bar{\lambda}_1 + \bar{\mu}_1)u_s\bar{u}_{r,r}\right]_{,s} + \\ & + (\bar{\mu}_1 - \mu_1)u_{r,s}\bar{u}_{r,s} + (\bar{\lambda}_1 + \bar{\mu}_1 - \lambda_1 - \mu_1)u_{r,r}\bar{u}_{s,s} + \\ & + \left(b_1\bar{u}_s\varphi_{,s} - \bar{b}_1 u_s\bar{\varphi}_{,s}\right) + \left(\beta u_s\bar{p}_{,s} - \beta\bar{u}_s p_{,s}\right) = 0. \end{aligned} \tag{24}$$

In the sequel, we integrate the identity (24) over $D(x_3)$. We employ the boundary conditions (12) on the lateral of the cylinder and then derive the identity

$$\begin{aligned}&\frac{d}{dx_3}\int_{D(x_3)}[\mu_1 u_{r,3}\bar{u}_r-\bar{\mu}_1 u_r\bar{u}_{r,3}+(\lambda_1+\mu_1)\bar{u}_3 u_{r,r}-\\&-(\bar{\lambda}_1+\bar{\mu}_1)u_3\bar{u}_{r,r}]\,da=\int_{D(x_3)}[(\mu_1-\bar{\mu}_1)u_{s,r}\bar{u}_{s,r}+\\&+(\lambda_1+\mu_1-\bar{\lambda}_1-\bar{\mu}_1)u_{r,r}\bar{u}_{s,s}+(\bar{b}_1 u_s\bar{\varphi}_{,s}-b_1\bar{u}_s\varphi_{,s})+\\&+(\beta\bar{u}_s p_{,s}-\beta u_s\bar{p}_{,s})]\,da.\end{aligned}\tag{25}$$

In a similar way, the equation $(10)_2$ leads to

$$\begin{aligned}&0=(\rho_1\omega^2\bar{\varphi})\varphi-(\rho_1\omega^2\varphi)\bar{\varphi}=\alpha_1\bar{\varphi}\varphi_{,rr}-\bar{\alpha}_1\varphi\bar{\varphi}_{,rr}-\zeta_1\varphi\bar{\varphi}+\bar{\zeta}_1\varphi\bar{\varphi}-\\&-\gamma_1 u_{m,m}\bar{\varphi}+\bar{\gamma}_1\varphi\bar{u}_{m,m}+mp\bar{\varphi}-m\varphi\bar{p}.\end{aligned}\tag{26}$$

This implies that

$$\begin{aligned}&\left(\alpha_1\varphi_{,r}\bar{\varphi}-\bar{\alpha}_1\bar{\varphi}_{,r}\varphi\right)_{,r}=\alpha_1\varphi_{,r}\bar{\varphi}_{,r}-\bar{\alpha}_1\bar{\varphi}_{,r}\varphi_{,r}+(\zeta_1-\bar{\zeta}_1)\varphi\bar{\varphi}+\\&+\gamma_1 u_{m,m}\bar{\varphi}-\bar{\gamma}_1\bar{u}_{m,m}\varphi+m\varphi\bar{p}-mp\bar{\varphi}.\end{aligned}\tag{27}$$

We integrate the identity (27) over $D(x_3)$. Then we use the boundary conditions (12) on the lateral of the cylinder, which leads us to the identity

$$\begin{aligned}&\frac{d}{dx_3}\int_{D(x_3)}(\alpha_1\varphi_{,3}\bar{\varphi}-\bar{\alpha}_1\bar{\varphi}_{,3}\varphi)\,da=\\&=\int_{D(x_3)}\big[\alpha_1\varphi_{,r}\bar{\varphi}_{,r}-\bar{\alpha}_1\bar{\varphi}_{,r}\varphi_{,r}+(\zeta_1-\bar{\zeta}_1)\varphi\bar{\varphi}+\\&+\gamma_1 u_{m,m}\bar{\varphi}-\bar{\gamma}_1\bar{u}_{m,m}\varphi+m\varphi\bar{p}-mp\bar{\varphi}]\,da.\end{aligned}\tag{28}$$

Finally, the equation $(10)_3$ leads to

$$\begin{aligned}&0=(i\omega a\bar{p})p-(i\omega ap)\bar{p}=kp_{,rr}\bar{p}+k\bar{p}_{,rr}p+\\&+i\omega(\beta u_{m,m}\bar{p}-\beta\bar{u}_{m,m}p+m\varphi\bar{p}-m\bar{\varphi}p).\end{aligned}\tag{29}$$

This implies that

$$\left(kp_{,r}\bar{p}+k\bar{p}_{,r}p\right)_{,r}=2kp_{,r}\bar{p}_{,r}-i\omega(\beta u_{m,m}\bar{p}-\beta\bar{u}_{m,m}p+m\varphi\bar{p}-m\bar{\varphi}p).\tag{30}$$

In a similar way, this leads to the following identity

$$\begin{aligned}&\frac{d}{dx_3}\int_{D(x_3)}(kp_{,3}\bar{p}+k\bar{p}_{,3}p)\,da=\int_{D(x_3)}[2kp_{,r}\bar{p}_{,r}-\\&-i\omega(\beta u_{m,m}\bar{p}-\beta\bar{u}_{m,m}p+m\varphi\bar{p}-m\bar{\varphi}p)]\,da.\end{aligned}\tag{31}$$

In the sequel, we compute (25)$\cdot i\omega$+(28)$\cdot i\omega$+(31). Then we rewrite the terms $(\beta u_s\bar{p}-\beta\bar{u}_s p)_{,s}$ and $(b_1\bar{u}_s\varphi-\bar{b}_1 u_s\bar{\varphi})_{,s}$ in order to obtain

$$\begin{aligned}
&\frac{d}{dx_3}\int_{D(x_3)}\{i\omega\left[\mu_1 u_{r,3}\bar{u}_r-\bar{\mu}_1 u_r\bar{u}_{r,3}+(\lambda_1+\mu_1)\bar{u}_3 u_{r,r}-\right.\\
&\left.-(\bar{\lambda}_1+\bar{\mu}_1)u_3\bar{u}_{r,r}\right]+i\omega(b_1\bar{u}_3\varphi-\bar{b}_1 u_3\bar{\varphi})+\\
&+i\omega(\beta u_3\bar{p}-\beta\bar{u}_3 p)+i\omega(\alpha_1\varphi_{,3}\bar{\varphi}-\bar{\alpha}_1\bar{\varphi}_{,3}\varphi)+\\
&+(kp_{,3}\bar{p}+k\bar{p}_{,3}p)\}\,da=\int_{D(x_3)}\{i\omega\left[(\mu_1-\bar{\mu}_1)u_{s,r}\bar{u}_{s,r}+\right.\\
&\left.+\ (\lambda_1+\mu_1-\bar{\lambda}_1-\bar{\mu}_1)u_{s,s}\bar{u}_{r,r}\right]+\\
&+i\omega\left[(\alpha_1-\bar{\alpha}_1)\varphi_{,r}\bar{\varphi}_{,r}+(\zeta_1-\bar{\zeta}_1)\varphi\bar{\varphi}\right]+2kp_{,r}\bar{p}_{,r}+\\
&+i\omega\left[(b_1-\bar{\gamma}_1)\bar{u}_{s,s}\varphi-(\bar{b}_1-\gamma_1)u_{s,s}\bar{\varphi}\right]\}\,da.
\end{aligned}\tag{32}$$

■

Theorem 2. *Let B be a finite cylinder filled with a viscoelastic material with voids characterized by the boundary value problem $\mathcal{P}$. In our setting, the conditions (16), (18), (19) and (20) hold true. Then*

i) $J(x_3)$ is a non-increasing function with respect to x_3 on (0,L) and

$$J(x_3)\geq 0 \text{ for all } x_3\in(0,L). \tag{33}$$

ii) There exists a computable positive constant ν, which depends on the profile of the coupled theory for viscoelastic materials with voids, which allows us to derive the following differential inequality of first-order

$$|J(x_3)|+\frac{1}{\nu}\frac{dJ}{dx_3}(x_3)\leq 0 \text{ for all } x_3\in(0,L). \tag{34}$$

iii) With the same positive constant ν as above, the following exponential decay estimate holds true

$$0\leq J(x_3)\leq J(0)e^{-\nu x_3} \text{ for all } x_3\in[0,L]. \tag{35}$$

Proof. i) We denote by

$$\varepsilon_{rs}=\frac{1}{2}(u_{r,s}+u_{s,r}). \tag{36}$$

By using the conditions (12) on the lateral boundary of the cylinder and integration by parts, we obtain

$$\begin{aligned}
&\int_{D(x_3)}\varepsilon_{rs}\bar{\varepsilon}_{rs}da=\frac{1}{4}\int_{D(x_3)}(u_{r,s}+u_{s,r})(\bar{u}_{r,s}+\bar{u}_{s,r})da=\\
&=\frac{1}{2}\int_{D(x_3)}(u_{r,s}\bar{u}_{r,s}+u_{r,s}\bar{u}_{s,r})da=\\
&=\frac{1}{2}\int_{D(x_3)}(u_{r,s}\bar{u}_{r,s}+u_{r,r}\bar{u}_{s,s})da.
\end{aligned}\tag{37}$$

Then equation (22) can be rewritten as

$$-\frac{dJ}{dx_3}(x_3)=\int_{D(x_3)}\{\omega^2[2\lambda^*\varepsilon_{mm}\bar{\varepsilon}_{nn}+4\mu^*\varepsilon_{rs}\bar{\varepsilon}_{rs}+2\zeta^*\varphi\bar{\varphi}+ \\ +(b^*+\gamma^*)(\bar{\varepsilon}_{ss}\varphi+\varepsilon_{ss}\bar{\varphi})]+2\omega^2\alpha^*\varphi_{,r}\bar{\varphi}_{,r}+2kp_{,r}\bar{p}_{,r}\}da. \tag{38}$$

By considering the assumptions (16) and (18), we can replace the formulas (19) and (20) into (38). This implies that

$$-\frac{dJ}{dx_3}(x_3)\geq\int_{D(x_3)}2\omega^2\pi_1(\varepsilon_{rs}\bar{\varepsilon}_{rs}+\varphi\bar{\varphi})da+ \\ +\int_{D(x_3)}2\nu_1(\varphi_{,r}\bar{\varphi}_{,r}+p_{,r}\bar{p}_{,r})da. \tag{39}$$

Then the identity

$$\varepsilon_{rs}\bar{\varepsilon}_{rs}+\frac{1}{4}(u_{r,s}-u_{s,r})(\bar{u}_{r,s}-\bar{u}_{s,r})=u_{r,s}\bar{u}_{r,s} \tag{40}$$

leads to

$$-\frac{dJ}{dx_3}(x_3)\geq\int_{D(x_3)}2\omega^2\pi_1(u_{r,s}\bar{u}_{r,s}+\varphi\bar{\varphi})da+ \\ +\int_{D(x_3)}2\nu_1(\varphi_{,r}\bar{\varphi}_{,r}+p_{,r}\bar{p}_{,r})da. \tag{41}$$

It follows that $J(x_3)$ is non-increasing with regards to x_3 on (0,L). By the condition on the end boundary, we have $J(L)=0$. These two relations imply that

$$J(x_3)\geq 0 \text{ for all } x_3\in(0,L). \tag{42}$$

ii) In relation (41), we integrate with regards to the variable x_3 upon (x_3,L) and use the formula $J(L)=0$. This implies that

$$J(x_3)\geq\int_{B(x_3)}2\omega^2\pi_1(u_{r,s}\bar{u}_{r,s}+\varphi\bar{\varphi})dv+ \\ +\int_{B(x_3)}2\nu_1(\varphi_{,r}\bar{\varphi}_{,r}+p_{,r}\bar{p}_{,r})dv. \tag{43}$$

In the formula above, we denote by

$$B(x_3)=D\times(x_3,L). \tag{44}$$

The lateral boundary conditions (12) imply that

$$\int_{D(x_3)}u_{r,\alpha}\bar{u}_{r,\alpha}da\geq m_0\int_{D(x_3)}u_r\bar{u}_r da, \tag{45}$$

$$\int_{D(x_3)}\varphi_{,\alpha}\bar{\varphi}_{,\alpha}da\geq m_0\int_{D(x_3)}\varphi\bar{\varphi}da, \tag{46}$$

$$\int_{D(x_3)} p_{,\alpha}\bar{p}_{,\alpha} da \geq m_0 \int_{D(x_3)} p\bar{p} da. \tag{47}$$

In the relations above, m_0 is the lowest eigenvalue which is obtained if we consider clamped membrane problem in the two-dimensional case for the cross section D.

We first apply the arithmetic-geometric mean inequality and then use the inequality of Cauchy-Schwarz. These two imply that we can compute the positive constants n_1, n_2, n_3 and n_4 so that

$$\begin{aligned} |J(x_3)| \leq n_1 \int_{D(x_3)} u_{r,s}\bar{u}_{r,s} da + n_2 \int_{D(x_3)} \varphi\bar{\varphi} da + \\ + n_3 \int_{D(x_3)} \varphi_{,r}\bar{\varphi}_{,r} da + n_4 \int_{D(x_3)} p_{,r}\bar{p}_{,r} da. \end{aligned} \tag{48}$$

Based on the relations (41) and (48) we can derive the differential inequality of first-order from below

$$|J(x_3)| + \frac{1}{\nu}\frac{dJ}{dx_3}(x_3) \leq 0 \text{ for all } x_3 \in (0, L). \tag{49}$$

In the inequality above, we denote by

$$\frac{1}{\nu} = \max\left\{\frac{n_1}{2\omega^2\pi_1}, \frac{n_2}{2\omega^2\pi_1}, \frac{n_3}{2\nu_1}, \frac{n_4}{2\nu_1}\right\}. \tag{50}$$

iii) The estimate can be obtained by integrating (49). ■

In the sequel, the aim is to analyse what happens in a semi-infinite cylinder. To this end, we define the volume energetic function

$$\begin{aligned} \mathcal{E}(x_3) = \int_{B(x_3)} \{\omega^2[2\lambda^*\varepsilon_{mm}\bar{\varepsilon}_{nn} + 4\mu^*\varepsilon_{rs}\bar{\varepsilon}_{rs} + 2\zeta^*\varphi\bar{\varphi} + \\ + (b^* + \gamma^*)(\bar{\varepsilon}_{ss}\varphi + \varepsilon_{ss}\bar{\varphi})] + 2\omega^2\alpha^*\varphi_{,r}\bar{\varphi}_{,r} + 2kp_{,r}\bar{p}_{,r}\} dv. \end{aligned} \tag{51}$$

Theorem 3. *Let B be a semi-infinite cylinder filled with a viscoelastic material with voids characterized by the boundary value problem $\mathcal{P}$. In our setting the conditions (16), (18), (19) and (20) hold true. Then we have an alternative result of Phragmen-Lindelöf type*

i) either we have a a finite volume energetic function $\mathcal{E}(x_3)$ for the amplitude of the steady-state vibration if the cross-sectional measure $J(x_3)$ is equal to $\mathcal{E}(x_3)$ and it has a faster spatial decay than the function $e^{-\nu x_3}$, or

ii) we have an infinite volume energetic function $\mathcal{E}(x_3)$ for the amplitude of the steady-state vibration and then the spatial growth of $-J(x_3)$ is faster than the function $e^{\nu x_3}$.

Proof. Note that the formula (41) holds true for a semi-infinite cylinder. We deduce that $J(x_3)$ is still non-increasing with regards to x_3 on $(0, \infty)$.

As a consequence, there are two possibilities

i) $J(x_3) \geq 0$ for all $x_3 \in (0, \infty)$;

ii) we have the value $x_3^* \in (0, \infty)$ such that $J(x_3) < 0$.

First, we analyse the case (i). In this case, the differential inequality (34) has the form

$$\nu J(x_3) + \frac{dJ}{dx_3}(x_3) \leq 0 \text{ for all } x_3 \in (0, L). \tag{52}$$

This implies that

$$0 \leq J(x_3) \leq J(0)e^{-\nu x_3} \text{ for all } x_3 \in [0, \infty). \tag{53}$$

Therefore, we obtain

$$J(\infty) = \lim_{x_3 \to \infty} J(x_3) = 0. \tag{54}$$

If we integrate (22) over (x_3, ∞), we deduce that

$$J(x_3) = \mathcal{E}(x_3). \tag{55}$$

It follows that $\mathcal{E}(x_3)$ is finite and

$$0 \leq \mathcal{E}(x_3) \leq J(0)e^{-\nu x_3} \text{ for all } x_3 \in [0, \infty). \tag{56}$$

For the case (ii), it follows that

$$J(x_3) < 0 \text{ for all } x_3 \in (x_3^*, \infty). \tag{57}$$

In this case, the differential inequality (34) has the form

$$\frac{dJ}{dx_3}(x_3) - \nu J(x_3) \leq 0, \text{ for all } x_3 \in (x_3^*, \infty). \tag{58}$$

If we integrate (58), then we get the spatial estimate from below

$$-J(x_3) \geq -J(x_3^*)e^{\nu(x_3 - x_3^*)} > 0 \text{ for all } x_3 \in (x_3^*, \infty). \tag{59}$$

Based on the formulas (22) and (51) we can deduce that the volume energetic function $\mathcal{E}(x_3)$ is infinite. ■

5 Conclusions

Although the computations in our study and in S. Chiriţă's paper [11] are quite similar, they refer to different phenomena. While S. Chiriţă analyses the effect of the temperature in a viscoelastic material with voids, we consider that the pores are filled with fluid and describe the effect of the pressure in the voids. Moreover, we neglect the thermal effects.

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