Bulletin of the *Transilvania* University of Braşov
Series III: Mathematics and Computer Science, Vol. 3(65), No. 1 - 2023, 93-104
https://doi.org/10.31926/but.mif.2023.3.65.1.7

A STUDY ON THE k-STEP GENERALIZED BALANCING SEQUENCES

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Abstract

In this paper, firstly, we define the k-step generalized Balancing sequences and study the Binet formula of these sequences. Also, we find families of super-diagonal matrices such that the permanents of these matrices are the elements of the k-step generalized Balancing sequences. Finally, we examine the periods of the k-step Balancing sequences in the semi-direct product presented by $G = \left\langle x, y \mid x^{2^{m-1}} = y^2 = 1,\ yxy = x^{-1} \right\rangle$ for the generating pair (x, y).

2010 *Mathematics Subject Classification:* 11B50, 11K31, 20F05.
Key words: balancing sequence, groups, period, presentation.

1 Introduction

The study of number sequences has been a source of attraction to the mathematicians since ancient times. Since then many of them are focusing their interest on the study of the fascinating triangular numbers. In 1999, Behera and Panda [2] introduced the notion of Balancing numbers $(B_n)_{n\in\mathbb{N}}$ as solutions to a certain Diophantine equation. Then, the recurrence relation of this number is $B_{n+1} = 6B_n - B_{n-1}$ for $n \geqslant 1$, where $B_0 = 0,\ B_1 = 1$. A study on the Lucas-Balancing numbers $C_n = \sqrt{8B_n^2 + 1}$ was published in 2006 by Panda [18]. The recurrence relation of this number is $C_{n+1} = 6C_n - C_{n-1}$ for $n \geqslant 1$, where $C_0 = 1,\ C_1 = 3$. Also, the authors examined the periodicity of these numbers in [19, 20].

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Kalman [11] mentioned that these sequences are special cases of a sequence which is defined recursively as a linear combination of the preceding k-step terms:

$$a_{n+k} = c_0 a_n + c_1 a_{n+1} + c_2 a_{n+2} + \cdots + c_{k-2} a_{n+k-2} + c_{k-1} a_{n+k-1},$$

where $c_0, c_1, c_2, \cdots, c_{k-1}$ are real constants. In [11], Kalman derived a number of closed-form formulas for the generalized sequence by companion matrix method as follows:

$$A_k = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ c_0 & c_1 & c_2 & c_3 & \cdots & c_{k-3} & c_{k-2} & c_{k-1} \end{bmatrix}_{k \times k}. \tag{1}$$

By inductive argument it is obtained

$$A_k^n \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{k-2} \\ a_{k-1} \end{bmatrix} = \begin{bmatrix} a_n \\ a_{n+1} \\ a_{n+2} \\ \vdots \\ a_{n+k-2} \\ a_{n+k-1} \end{bmatrix} \tag{2}$$

for $n \geq 0$.

In [5], the authors introduced the k-step Balancing sequences as follows:

$$B_k(n+k) = 6B_k(n+k-1) - B_k(n+k-2) + B_k(n+k-3) + \ldots + B_k(n), \tag{3}$$

where $n \geq 0$, $B_k(u) = 0, (0 \leq u < k-1)$ and $B_k(k-1) = 1$. Also, the authors found the following generating matrix for the k-step Balancing sequences:

$$C = [c_{ij}]_{k \times k} = \begin{bmatrix} 6 & -1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{bmatrix}. \tag{4}$$

It is well knowledge that a sequence as periodic if, at a given point, it solely consists of repeated instances of a specified subsequence. The period of the sequence is equal to the number of elements in the repeating subsequence. In [21], the investigation of Fibonacci sequences in cyclic groups served as the foundation for the research of linear recurrence sequences in groups. Many writers have recently examined various unique in groups of linear recurrence sequences; for instance, [3, 4, 7, 8, 9, 10, 12, 14, 22].

On the other hand, the concept of a semi-direct product is a generalization of a direct product in group theory. There are many studies on this subject in the different areas in mathematics. The reader is referred to [1, 15] for studies on semi-direct product of groups. For example, in [6], Deveci investigated the periods of the k-nacci sequences and the generalized order-k Pell sequences in the semi-direct product of groups.

Lemma 1. *Suppose that $\wp_A = \langle X \mid R\rangle$ and $\wp_B = \langle Y \mid S\rangle$ are presentations for the groups A and B, respectively under the maps $\varphi : A \to Aut(B)$, $y \mapsto k_y \in B$, and $x \mapsto a_x \in A$. Then we have a presentation $\langle X, Y \mid S, R, T\rangle$ for semi-direct product $G = B \rtimes_\varphi A$, B by A where $T = \{yx\lambda_{yx}^{-1}x^{-1}r \mid x \in X, y \in Y\}$ and λ_{yx} is a word on y representing the element $\varphi_{a_x}(k_y)$ of B where $a \in A$, $k \in B$, $x \in X$, $y \in Y$.*

Let A be a cyclic group of order $2^{m-1}(m \geq 4)$ with a presentation $\langle x \mid x^{2^{m-1}} = 1\rangle$, and let B be finite cyclic group of order 2 presented by $\langle y \mid y^2 = 1\rangle$. Then, by Lemma 1, a presentation for $G = B \rtimes_\varphi A$ is given by

$$G = \langle x, y \mid x^{2^{m-1}} = y^2 = 1,\ yxy = x^{-1}\rangle. \tag{5}$$

In Section 2 of this paper, we find the Binet formula, permanental representation of the k-step generalized Balancing sequences. In Section 3, we examine the periods of the k-step Balancing sequences in the semi-direct product given in (5) for the generating pair (x, y).

2 The properties of the k-step generalized Balancing sequences

The object of this section is to investigate the k-step generalized Balancing sequences and obtain the Binet formula of these sequences. Then, we get families of super-diagonal matrices such that the permanents of these matrices are the k-step generalized Balancing sequences.

Definition 1. *We defined the k-step generalized Balancing sequences as follows:*

$$B_n^i = 6B_{n-1}^i - B_{n-2}^i + B_{n-3}^i + \ldots + B_{n-k}^i, \tag{6}$$

where $n > 0$, $1 \leq i \leq k$ and with initial conditions

$$B_n^i = \begin{cases} 1 & \text{if } i = 1 - n \\ 0 & \text{otherwise} \end{cases} \quad \text{for } 1 - k \leq n \leq 0\ .$$

These sequences are also referred to be order-k Balancing sequences.

- By taking $k = 2$, $i = 1$ in the equation (6), these sequences reduce to the usual Balancing sequence $\{B_2^1\}$ in OEIS $A001109$.
- By taking $i = k$ in the equation (6), these sequences reduce to the k-step Balancing sequences in [5].

Let us to define a k-square matrix $E_n = [e_{ij}]$ to deal with the k sequences of the k-step generalized Balancing sequences, as following:

$$E_n = \begin{bmatrix} B_n^1 & B_n^2 & \cdots & B_n^k \\ B_{n-1}^1 & B_{n-1}^2 & \cdots & B_{n-1}^k \\ B_{n-2}^1 & B_{n-2}^2 & \cdots & B_{n-2}^k \\ \vdots & \vdots & \ddots & \vdots \\ B_{n-k+1}^1 & B_{n-k+1}^2 & \cdots & B_{n-k+1}^k \end{bmatrix}. \tag{7}$$

Then we get the following matrix relation:

$$E_n = C \cdot E_{n-1} \tag{8}$$

where the matrix C is defined as the equation (4).

Lemma 2. *Let C and E_n be as in* (4) *and* (7), *respectively. Then for all integers $n \geq 0$*

$$E_n = C^n.$$

Proof. By equation (8), we have $E_n = C \cdot E_{n-1}$. Then, by an inductive argument, we can write

$$E_n = C^{n-1} \cdot E_1.$$

By definition of the k-step generalized Balancing sequences, $E_1 = C$, hence we get $E_n = C^n$. $\square$

Now we concentrate on finding the Binet formula for the k-step generalized Balancing sequences.

Lemma 3. *The characteristic equation of the k-step generalized Balancing sequences $x^k - 6x^{k-1} + x^{k-2} - x^{k-3} - \cdots - x - 1 = 0$ does not have multiple roots for $k \geq 3$.*

Proof. Let $f(x) = x^k - 6x^{k-1} + x^{k-2} - x^{k-3} - \cdots - x - 1$. It is clear that $f(0) \neq 0$ and $f(1) \neq 0$ for all $k \geq 3$. Suppose that $h(x) = (x-1) f(x) = x^{k+1} - 7x^k + 7x^{k-1} - 2x^{k-2} + 1$. Let α be a multiple root of $h(x)$, then $\alpha \notin \{0, 1\}$. If possible, α is a multiple root of $h(x)$ in which case $h(\alpha) = 0$ and $h'(\alpha) = 0$. Now $h'(\alpha) = 0$ and $\alpha \neq 0$, we give

$$\begin{aligned} h'(\alpha) &= (k+1)\alpha^k - 7k\alpha^{k-1} + 7(k-1)\alpha^{k-2} - 2(k-2)\alpha^{k-3} \\ &= \alpha^{k-3}\left((k+1)\alpha^3 - 7k\alpha^2 + 7(k-1)\alpha - 2(k-2)\right) = 0. \end{aligned}$$

Thus we obtain

$$\alpha_1 = \frac{\eta^{\frac{1}{3}}}{6(k+1)} + \frac{14(4k^2+3)}{3(k+1)\eta^{\frac{1}{3}}} + \frac{7k}{3(k+1)},$$

$$\alpha_{2,3} = -\frac{\eta^{\frac{1}{3}}}{12(k+1)} - \frac{7(4k^2+3)}{3(k+1)\eta^{\frac{1}{3}}} + \frac{7k}{3(k+1)} \pm \frac{1}{2} i\sqrt{3}\left[\frac{\eta^{\frac{1}{3}}}{6(k+1)} - \frac{14(4k^2+3)}{3(k+1)\eta^{\frac{1}{3}}}\right],$$

where $\eta = 1196k^3 + 1116k - 432 + 12\sqrt{3}\xi k + 12\sqrt{3}\xi$ and $\xi = \sqrt{59k^4 - 118k^3 - 961k^2 - 352k - 940}$. It is easy to see that α_i are distinct from each other. Hence

$$\begin{aligned} 0 = -h(\alpha_i) &= \alpha_i^{k-2}\left[-\alpha_i^3 + 7\alpha_i^2 - 7\alpha_i + 2\right] - 1, \\ &= u_{k,i} - 1, \end{aligned}$$

where $u_{k,i} = \alpha_i^{k-2}\left[-\alpha_i^3 + 7\alpha_i^2 - 7\alpha_i + 2\right]$. If we take $k = 3$ and $1 \leq i \leq 3$, it can be obtained

$$\begin{aligned} 0 = -h(\alpha_1) &= \alpha_1\left[-\alpha_1^3 + 7\alpha_1^2 - 7\alpha_1 + 2\right] - 1, \\ &= u_{3,1} - 1 = 0. \end{aligned}$$

Since $u_{3,1} = 95.06296046 + 0.1348895085 \cdot 10^{-8} i \neq 1$, there is a contradiction. Similarly for α_2 and α_3, we obtain $u_{3,2} = 0.1744052781 + 0.2168473807 \cdot 10^{-10} i \neq 1$ and $u_{3,3} = 0.05560336719 + 0.5686014344 \cdot 10^{-10} i \neq 1$. Thus $k \geq 3$, $h(\alpha_i) \neq 0$, which is a contraction. So, the equation $f(x) = 0$ does not have multiple roots. □

Let $f(x)$ be the characteristic polynomial of the matrix C. Then we have $f(x) = x^k - 6x^{k-1} + x^{k-2} - x^{k-3} - \cdots - x - 1$, which is a well-known fact from the companion matrices. If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are roots of the equation $x^k - 6x^{k-1} + x^{k-2} - x^{k-3} - \cdots - x - 1 = 0$, then by Lemma 3, it is known that $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct. Define the $k \times k$ Vandermonde matrix V as follows:

$$V = \begin{bmatrix} \lambda_1^{k-1} & \lambda_1^{k-2} & \ldots & \lambda_1 & 1 \\ \lambda_2^{k-1} & \lambda_2^{k-2} & \ldots & \lambda_2 & 1 \\ \vdots & \vdots & & \vdots & \\ \lambda_k^{k-1} & \lambda_k^{k-2} & \ldots & \lambda_k & 1 \end{bmatrix}.$$

Assume that $V_j^{(i)}$ is a $k \times k$ matrix obtained from the Vandermonde matrix V by replacing the j^{th} column of V by W_k^i, where, W_k^i is a $k \times 1$ matrix as follows:

$$W_k^i = \begin{bmatrix} \lambda_1^{n+k-i} \\ \lambda_2^{n+k-i} \\ \vdots \\ \lambda_k^{n+k-i} \end{bmatrix}.$$

Then we can give the generalized Binet formula for the k-step generalized Balancing sequences with the following Theorem.

Theorem 1. *Let B_n^i be the n^{th} term of the i^{th} sequence for $1 \leq i \leq k$, $n \geq 1$ and $k \geq 3$, then*

$$B_{n-i+1}^j = \frac{\det(V_j^{(i)})}{\det(V)}.$$

Proof. Since the equation $x^k - 6x^{k-1} + x^{k-2} - x^{k-3} - \cdots - x - 1 = 0$ does not have multiple roots for $k \geq 3$, the eigenvalues of the k-step generalized Balancing matrix C are distinct. Then, it is clear that C is diagonalizable. Since D is invertible, $D^{-1} \cdot C \cdot D = diag(\lambda_1, \lambda_1, \cdots, \lambda_k) = \Lambda$. Thus C is similar to Λ. So we get $C^n \cdot D = D \cdot \Lambda^n$. It is known that $E_n = C^n$ from Lemma 2. Then we have the following linear system of equations:

$$\begin{cases} e_{i1}\lambda_1^{k-1} + e_{i2}\lambda_1^{k-2} + \cdots + e_{ik} = \lambda_1^{n+k-i} \\ e_{i1}\lambda_2^{k-1} + e_{i2}\lambda_2^{k-2} + \cdots + e_{ik} = \lambda_2^{n+k-i} \\ \vdots \\ e_{i1}\lambda_k^{k-1} + e_{i2}\lambda_k^{k-2} + \cdots + e_{ik} = \lambda_k^{n+k-i}, \end{cases}$$

where $E_n = [e_{ij}]_{k \times k}$. Then we conclude that

$$e_{ij} = \frac{\det(V_j^{(i)})}{\det(V)}$$

for each $j = 1, 2, \ldots, k$. Note that $e_{ij} = B_{n-i+1}^{j}$. So we complete the proof. $\square$

We now construct an n-square matrix whose permanents are the k-step generalized Balancing sequences.

In [16, 17], the permanent of an $n \times n$ matrix $A_n = (a_{ij})$ is defined as

$$per(A_n) = \sum_{\sigma \in S_n} \prod_{i=1}^{n} a_{i\sigma(i)}.$$

The sum here extends over all elements σ of the symmetric group S_n; i.e. over all permutations of the numbers $1, 2, \ldots, n$.

Let A_n be $n \times n$ lower Hessenberg matrix for all $n \geq 1$ and $per(A_0) = 1$. Then,

$$\begin{aligned} &per(A_1) = a_{11} \text{ and for } n \geq 2, \\ &per(A_n) = a_{n,n} per(A_{n-1}) + \sum_{r=1}^{n-1} \left(a_{n,r} \prod_{j=1}^{n-1} a_{j,j+1} per(A_{r-1}) \right). \end{aligned} \tag{9}$$

We define an $n \times n$ k^{st} super-diagonal $(0, 1, 6, -1)$-matrix $F_n^k = f_{ij}$, $k \leq n+1$, with $f_{i+1,i} = 1$ for $1 \leq i \leq n-1$, $f_{ii} = 6$ for $1 \leq i \leq n$, $f_{i,i+1} = -1$ for $1 \leq i \leq n-1$

and $f_{ij} = 1$ for $i+2 \le j \le i+k-2$. That is, we write

$$F_n^k = \begin{bmatrix} 6 & -1 & 1 & 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 6 & -1 & 1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 6 & -1 & \cdots & 1 & 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 6 & \cdots & 1 & 1 & 1 & 1 & \cdots & 0 & 0 \\ & & \vdots & & & & & & & \vdots & & \\ 0 & & \cdots & & 0 & 1 & 6 & -1 & 1 & \cdots & 1 & 0 \\ 0 & & \cdots & & 0 & 0 & 1 & 6 & -1 & \cdots & 1 & 1 \\ & & \vdots & & & & & & & \vdots & & \\ 0 & & \cdots & & 0 & & \cdots & & 0 & 1 & 6 & -1 \\ 0 & & \cdots & & 0 & & \cdots & & 0 & 0 & 1 & 6 \end{bmatrix}.$$

Theorem 2. *We have*

(i). *$per(F_n^3) = B_n^1$, for $n \ge 2$,*

(ii). *$per(F_n^k) = B_{n+1}^{k-1}$, for $k-1 \le n$ and $k \ge 4$,*

where B_n^k is the n^{th} term of the k-step generalized Balancing sequences.

Proof. (i). Firstly, we will prove it with induction on n. For $n = 2$,

$$per(F_2^3) = 35 = B_2^1.$$

Now assume that it is true for n. That is, $per(F_n^3) = B_n^1$. Then, by considering (9) and the our assumption, we have

$$\begin{aligned} per(F_{n+1}^3) &= 6per(F_n^3) - per(F_{n-1}^3) \\ &= 6B_n^1 - B_{n-1}^1. \end{aligned}$$

From the Definition 1, we obtain

$$per(F_{n+1}^3) = B_{n+1}^1,$$

which ends up the proof.

(ii). We consider two cases as $k-1 = n$ and $k-1 < n$.
1. Case: Let $k-1 = n$. We will prove it with induction method on t. For $3 \le t \le k-1$, we will show that $per(F_t^k) = B_{t+1}^{k-1}$. For $t = 3$, we have

$$per(F_3^4) = 205 = B_4^3.$$

Now assume that it is true $per(F_{t-1}^k) = B_t^{k-1}$. Then, by considering (9), the our assumption and Definition 1, we have

$$\begin{aligned} per(F_t^k) &= 6per(F_{t-1}^k) - per(F_{t-2}^k) + per(F_{t-3}^k) + \cdots + per(F_{t-k+1}^k) \\ &= 6B_t^{k-1} - B_{t-1}^{k-1} + B_{t-2}^{k-1} + \cdots + B_{t-k+2}^{k-1} \\ &= B_{t+1}^{k-1}. \end{aligned}$$

which ends up the case.
2. Case: Let $k-1<n$. Again, we will prove it with induction method on t. For $k \leq t \leq n$, we will show that $per(F_t^k) = B_{t+1}^{k-1}$. If $t=k$, then we have

$$\begin{aligned} per(F_k^k) &= 6per(F_{k-1}^k) - per(F_{k-2}^k) + per(F_{k-3}^k) + \cdots + per(F_1^k) \\ &= 6B_k^{k-1} - B_{k-1}^{k-1} + B_{k-2}^{k-1} + \cdots + B_2^{k-1}. \end{aligned}$$

From Definition 1, we obtain $per(F_k^k) = B_{k+1}^{k-1}$. We assume that the equation holds for t and $k \leq t \leq n$, then we have $per(F_t^k) = B_{t+1}^{k-1}$.
Now we show that the equation holds for $t+1$. Computing $per(F_{t+1}^k)$ by the Laplace expansion of the permanent with respect to the first row, we obtain for $k \leq t \leq n$

$$\begin{aligned} per(F_{t+1}^k) &= 6per(F_t^k) - per(F_{t-1}^k) + per(F_{t-2}^k) + \cdots + per(F_{t-k+2}^k) \\ &= 6B_{t+1}^{k-1} - B_t^{k-1} + B_{t-1}^{k-1} + \cdots + B_{t-k+3}^{k-1}. \end{aligned}$$

From Definition 1, we obtain $per(F_{t+1}^k) = B_{t+2}^{k-1}$. So the proof is complete. □

3 The lengths of the periods of the k-step Balancing sequences in the semi-direct of finite cyclic groups

In this section, for the generating pair (x,y), we calculate the periods of the k-step Balancing sequences in the semi-direct product of finite cyclic groups with the presentation $G = \langle x, y \mid x^{2^{m-1}} = y^2 = 1,\ yxy = x^{-1} \rangle$.

Definition 2. *[5] Let G be finite group and let (x,y) be a generating pair for G. For a generating pair $(x,y) \in G$, the Balancing orbit*

$$\{a_0 = x, \quad a_1 = y, \quad a_{i+2} = a_i^{-1} a_{i+1}^6, \ \ i \geq 0\},$$

denoted by $B_{x,y}(G) = \{a_i\}$.

Definition 3. *[5] A k-step Balancing sequence in a finite group is a sequence of group elements $a_0, a_1, \cdots, a_n, \cdots$, for which, given an initial set $a_0 = x_0$, $a_1 = x_1$, $a_2 = x_2, \cdots, a_{j-1} = x_{j-1}$, $a_j = x_j$, each element is defined by*

$$a_n = \begin{cases} a_0 a_1 a_2 \cdots a_{n-3} a_{n-2}^{-1} a_{n-3}^6 & j < n < k \\ a_{n-k} a_{n-k+1} \cdots a_{n-3} a_{n-2}^{-1} a_{n-3}^6 & n \geq k \end{cases}.$$

It is require that the initial elements of the sequence $x_0,\ x_1,\ x_2, \cdots, x_{j-1}$ generate the group, thus, forcing the k-step Balancing sequences to reflect the structure of the group. We denoted by $B(G, x_0, x_1, \cdots, x_{j-1})$ the k-step Balancing sequences in a group G generated by $x_0,\ x_1,\ x_2, \cdots, x_{j-1}$.

Now we give the following result for the periodic of k-step Balancing sequences in a finite group.

Theorem 3. *[5] A k-step Balancing sequence in a finite group is periodic.*

We note that by the definition it is clear that the period of the k-step Balancing sequences in a finite group depend on the chosen generating set and the order in which the assignments of $x_0,\ x_1,\ x_2, \cdots, x_{j-1}$. We shall address the lengths of the periods of the k-step Balancing sequences in the semi-direct product G by $LB_k(G, x, y)$ for the generating pair (x, y). We have the following main result of this section.

Theorem 4. *For $m \geq 4$, lengths of the periods of the k-step Balancing sequences in the semi-direct product G are as follows:*

(*i*). $LB_2(G, x, y) = 4$,

(*ii*). $LB_k(G, x, y) = 7.2^{k-3}(m-1)$ *for* $k \geq 3$.

Proof. (*i*). We prove this by direct calculation. First, we note that group by $\langle x, y \mid x^{2^{m-1}} = y^2 = 1,\ yxy = x^{-1} \rangle$, $xy = yx^{-1}$ and $yx = x^{-1}y$. We have the sequence

$$a_0 = x,\ a_1 = y,\ a_2 = x^{-1},\ a_3 = x^6 y,\ a_4 = x,\ a_5 = y,\ a_6 = x^{-1}, \ldots,$$

which has period 4.

(*ii*). If $k = 3$, we have the sequence

$$x,\ y,\ x^{-1},\ x^7 y,\ x^{-1} y,\ x^6 y,\ x^8,\ x^{41},\ x^{-232} y,\ x^{-33},\ x^7 y,\ x^{-265} y,\ x^{-26} y,$$
$$x^{272},\ x^{1393},\ x^{-8112} y,\ x^{-1121},\ x^7 y,\ x^{-9233} y,\ x^{-1114} y,\ x^{9240},\ \cdots.$$

Using the above, the sequence becomes:

$$\begin{aligned}
&a_0 = x,\ a_1 = y,\ a_2 = x^{-1},\ a_3 = x^7 y,\ a_4 = x^{-1} y,\ a_5 = x^6 y,\ a_6 = x^8,\\
&a_7 = x^{41}, a_8 = x^{-232} y,\ a_9 = x^{-33},\ a_{10} = x^7 y,\ a_{11} = x^{-265} y,\\
&a_{12} = x^{-26} y,\ a_{13} = x^{272},\ a_{14} = x^{1393}, a_{15} = x^{-8112} y,\ a_{16} = x^{-1121},\\
&a_{17} = x^7 y,\ a_{18} = x^{-9233} y,\ a_{19} = x^{-1114} y,\ a_{20} = x^{9240},\\
&\cdots,\\
&a_{7i} = x^{2^i \lambda_{1_i} + 1}, a_{7i+1} = x^{-2^i \lambda_{2_i}} y,\ a_{7i+2} = x^{-2^i \lambda_{3_i} - 1},\ a_{7i+3} = x^7 y,\\
&a_{7i+4} = x^{-2^i \lambda_{5_i} - 1} y,\ a_{7i+5} = x^{-2^i \lambda_{6_i} - 2} y,\ a_{7i+6} = x^{2^i \lambda_{7_i}},
\end{aligned}$$

where i is an nonnegative integer, $\lambda_{j_0} = 0$ $(1 \leq j \leq 7)$ and linebreak $gcd(\lambda_{1_i}, \lambda_{2_i}, \cdots, \lambda_{7_i}) = 1$ $(i > 0)$. So we need the smallest integer i such that $2^i \mid 2^{m-1}$ for $m \geq 4$. If we choose $i = m-1$, then we obtain $a_{7(m-1)} = x$ and $a_{7m-6} = y$. Since the elements succeding $a_{7(m-1)}$ and a_{7m-6}, depend on x and y for their values, the cycle begins again with the $(7(m-1))^{\text{nd}}$. So we have $LB_3(G, x, y) = 7(m-1)$.

Let $k \geq 4$. We have the sequence

$$\begin{aligned}
&a_0 = x,\ a_1 = y,\ a_2 = x^{-1},\ a_3 = x^7 y,\ a_4 = y,\ a_5 = x^{-5},\ a_6 = x^{25} y,\\
&a_7 = x^{-10} y,\ \cdots, a_{7.2^{k-3}-k+2} = 1,\ a_{7.2^{k-3}-k+1} = 1, \cdots, a_{7.2^{k-3}-1} = 1,\\
&a_{7.2^{k-3}} = x^{2^n \beta_1 + 1},\ a_{7.2^{k-3}+1} = x^{-2^n \beta_2} y,\ a_{7.2^{k-3}+2} = x^{-2^n \beta_3}, \cdots,\\
&a_{(7.2^{k-3})i-k+2} = 1,\ a_{(7.2^{k-3})i-k+1} = 1, \cdots, a_{7.2^{k-3}i-1} = 1,\\
&a_{(7.2^{k-3})i} = x^{2^i \lambda_{1_i} + 1},\ a_{(7.2^{k-3})i+1} = x^{-2^i \lambda_{2_i}} y,\ a_{(7.2^{k-3})i+2} = x^{-2^i \lambda_{3_i} - 1}, \cdots,
\end{aligned}$$

where i, n and β_v $(1 \leq v \leq 7.2^{k-3})$ are an nonnegative integer, $\lambda_{j_0} = 0$ $(1 \leq j \leq 7.2^{k-3})$ and $gcd(\lambda_{1_i}, \lambda_{2_i}, \cdots, \lambda_{(7.2^{k-3})_i}) = 1$ $(i > 0)$. So we need the smallest integer i such that $2^i \mid 2^{m-1}$ for $m \geq 4$. If we choose $i = m-1$, then we obtain $a_{7.2^{k-3}(m-1)-k+2} = 1$, $a_{7.2^{k-3}(m-1)-k+1} = 1, \cdots$, $a_{7.2^{k-3}(m-1)-1} = 1$, $a_{7.2^{k-3}(m-1)} = x$ and $a_{7.2^{k-3}(m-1)+1} = y$. Thus, the cycle begins again with the $(a_{7.2^{k-3}(m-1)})^{nd}$ element. So we obtain $LB_k(G, x, y) = 7.2^{k-3}(m-1)$ for $k \geq 3$.

□

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