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STATE-SPACE SOLUTION OF SINGULAR LINEAR CONTINUOUS-TIME SYSTEMS USING THE CONFORMABLE DERIVATIVE AND SUMUDU TRANSFORM

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Abstract

The aim of this work is the application of the Sumudu transform for solving singular continuous-time linear systems based on the conformable derivative operator. Thanks to the interesting properties of the conformable Sumudu transform that we have established, a new approach is developed. Through academic and real examples, our method is compared to the existing ones, where the applicability and the accuracy of the developed process are shown.

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Key words: singular systems, conformable derivative operator, conformable fractional Sumudu transform.

1 Introduction

Fractional order systems have generated considerable interests in many fields of applied sciences, engineering, and control theory [21, 22, 28, 31]. However, a new derivative operator, called the conformable derivative operator, has been proposed by *Khalil et al.* [23] and took part on several areas as engineering, finances, biology, medicine, physics and applied mathematics [5, 6, 7, 14, 11, 36]. The most advantages of this derivative is that it preserves the properties of the usual exact derivatives such as: quotient, product, chain rules, Rolle's theorem, and mean-value theorem. More than that, conformable derivative does not contain

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any integral terms, that make it much more easier to apply on the fractional differential equations [1, 23]. In fact, various problems had been solved, certain methods and resolution had been developed and improved, and other definitions of the conformable derivative operator had been exploited in [23]. For example, fractional partial differential equations [36], time-fractional one dimensional cable differential equation [37, 39], fractional Cauchy problem [38], linear/nonlinear differential equations [40], and other applications.

In control theory, for instance, the state of fractional continuous-time systems appeared in [20, 21, 22]. Note that various methods, including integral transformations like Laplace transform, Millen transform, Sumudu transform [2, 3, 4, 33, 12, 15, 18, 24, 25, 34, 35], had been proposed for resolving these systems [9, 16, 17, 22].

The regular linear continuous-time system with conformable derivative in unidimensional (1D) and two dimensional (2D) models has received much attention in the last two years [8, 29, 30]. In this paper, we propose to solve singular linear continuous-time system with conformable derivative using conformable Sumudu transform which has a relation with Sumudu transform and has many interesting and attractive advantages over other integral transforms specifically the unity by providing the convergence when solving differential equations and also the resolvability of problems without resorting to a new frequency domain [1, 35]. The expression of the state of our system has been obtained thanks to some properties and formulas of the conformable fractional Sumudu transform that we have established and proved.

This paper is divided into four Sections. Section 2 gives a brief overview of the definitions and properties, which are used along this paper. In Section 3, the resolution of the singular continuous-time linear systems of order α by conformable Sumudu transform method is introduced and established, furthermore, the solution of the regular continuous-time linear systems is also discussed. Section 4 focuses on the numerical examples where the advantages and the effectiveness of our approach are shown by using a Matlab code. Finally, some conclusions are drawn in the last Section.

2 Preliminaries

In this section, the most important mathematical background used in this work are presented. First, we will start by recalling some definitions and properties of the conformable derivative operator [23]. Then, the definition of the conformable Sumudu transform is presented [1] followed by some of its properties that we have developed. Finally, results on matrix theory are given.

Definition 1. *[23] Given a function $x : [0, +\infty) \to \mathbb{R}$. Then, the conformable derivative of the function x of order α, with $\alpha \in (0, 1]$ is defined by*

$$\mathbf{T}^{\alpha}(x)(t) = \lim_{\epsilon \to 0} \frac{x\left(t + \epsilon t^{1-\alpha}\right) - x(t)}{\epsilon}, \quad \forall\, t > 0.$$

If the conformable derivative of the function x of order α for all $t > 0$ exists, then, we simply say x is α-differentiable.

Theorem 1. *[23] Let $\alpha \in (0,1]$ and x_1, $x_2 : \mathbb{R}_+ \to \mathbb{R}$ be α-differentiable functions. Then, $\forall t > 0$*

(a) $\mathbf{T}^\alpha(ax_1(t) + bx_2(t)) = a\mathbf{T}^\alpha(x_1)(t) + b\mathbf{T}^\alpha(x_2)(t)$, *for all* $a, b \in \mathbb{R}$;

(b) $\mathbf{T}^\alpha(t^p) = pt^{p-\alpha}$, *for all* $p \in \mathbb{R}$;

(c) $\mathbf{T}^\alpha(\lambda) = 0$, *for all constant function* $x_1(t) = \lambda$;

(d) $\mathbf{T}^\alpha\left(x_1(t)x_2(t)\right) = x_1(t)\mathbf{T}^\alpha\left(x_2\right)(t) + x_2(t)\mathbf{T}^\alpha\left(x_1\right)(t)$;

(e) $\mathbf{T}^\alpha\left(\dfrac{x_1(t)}{x_2(t)}\right) = \dfrac{x_2(t)\mathbf{T}^\alpha\left(x_1\right)(t) + x_1(t)\mathbf{T}^\alpha\left(x_2\right)(t)}{x_2^2(t)}$;

(f) *If x_1 is differentiable, then,* $\mathbf{T}^\alpha(x_1)(t) = t^{1-\alpha}\dfrac{\mathrm{d}x_1(t)}{\mathrm{d}t}$.

Definition 2. *[1] Over the following set of function*

$$A_\alpha = \left\{ x(t) : \exists M, \tau_1, \tau_2 > 0, |x(t)| < Me^{\left|\frac{t^\alpha}{\alpha\tau_j}\right|},\ \text{if } t^\alpha \in (-1)^j \times [0,\infty),\ j = 1,2 \right\},$$

then, the conformable Sumudu transform of the function x is defined by

$$\begin{aligned} S_\alpha[x(t)](v) &= X_\alpha(v) \\ &= \frac{1}{v}\int_0^\infty e^{\frac{-t^\alpha}{\alpha v}} x(t)\mathrm{d}t^\alpha, \quad v \in (-\tau_1, \tau_2). \end{aligned} \tag{1}$$

Where $\mathrm{d}t^\alpha = t^{\alpha-1}\mathrm{d}t$ *and* $\alpha \in (0,1]$.

Theorem 2. *[1] Let $x, x_1, x_2 : [0, +\infty) \to \mathbb{R}$ be a given functions, $0 < \alpha \leq 1$, λ, $\mu \in \mathbb{R}$ and $v > 0$. Then, we have the following properties*

1. $S_\alpha[\mathbf{T}^\alpha x(t)](v) = \dfrac{1}{v}\left[S_\alpha[x(t)](v) - x(0)\right], \quad \forall t > 0,$

2. $S_\alpha\left[\dfrac{t^{\alpha n}}{\alpha^n}\right](v) = \Gamma(n+1)v^n, \quad \forall n \in \mathbb{N},$

3. $S_\alpha[\lambda x_1(t) + \mu x_2(t)](v) = \lambda S_\alpha[x_1(t)](v) + \mu S_\alpha[x_2(t)](v).$

Lemma 1. *Let $x_1, x_2 : [0, +\infty) \to \mathbb{R}$ be a given functions. Then, the conformable Sumudu transform of the convolution product of x_1 and x_2 is defined by*

$$S_\alpha\left[\left(x_1 \star x_2\right)(t)\right](v) = vS_\alpha[x_1(t^\alpha)](v)S_\alpha[x_2(t)](v), \quad v > 0,$$

where

$$(x_1 \star x_2)(t) = \int_0^t x_1\left(t^\alpha - \tau^\alpha\right)x_2(\tau)\mathrm{d}\tau^\alpha.$$

Proof. Using the relationship between conformable Sumudu transform and conformable Laplace transform [1], we get

$$\begin{aligned} S_\alpha\left[\left(x_1 \star x_2\right)(t)\right](v) &= \frac{\mathcal{L}_\alpha[(x_1 \star x_2)(t)](s)}{v}, \quad s \to \frac{1}{v}, \\ &= \frac{\left(\mathcal{L}_\alpha[x_1(t^\alpha)]\mathcal{L}_\alpha[x_2(t)]\right)(s)}{v}, \quad s \to \frac{1}{v}, \\ &= vS_\alpha[x_1(t^\alpha)](v)S_\alpha[x_2(t)](v), \end{aligned} \tag{2}$$

where, $\mathcal{L}_\alpha$ is the conformable Laplace transform [14]. □

Theorem 3. *[1] Let $x : [0, +\infty) \to \mathbb{R}$ be an n-differentiable function and α such that, $0 < \alpha \leq 1$. Then,*

$$S_\alpha\left[\mathbf{T}^{n\alpha} x(t)\right](v) = \frac{S_\alpha\left[x(t)\right](v) - x(0)}{v^n}, \quad \forall n \in \mathbb{N} \text{ and } \forall v > 0, \tag{3}$$

and as in [32], $\mathbf{T}^{n\alpha}$ is known as the conformable derivative operator of order n.

Proposition 1. *Let $\alpha \in (0, 1]$ and for all $v > 0$, the conformable Sumudu transform of the conformable derivative of order $(n-1)$ of the function $t^{1-\alpha}\delta(t)$ is given by*

$$S_\alpha\left[\mathbf{T}^{(n-1)\alpha} t^{1-\alpha}\delta(t)\right](v) = \frac{1}{v^{n-1}} S_\alpha\left[t^{1-\alpha}\delta(t)\right](v) = \frac{1}{v^n}, \quad \forall n \in \mathbb{N}^*. \tag{4}$$

Proof. To proof formula (4), we will proceed by induction and we will use the properties of the function δ given in [13].

1. First step: for $n = 1$, we get

$$\begin{aligned} S_\alpha\left[t^{1-\alpha}\delta(t)\right](v) &= \frac{1}{v}\int_0^\infty t^{1-\alpha}\delta(t)e^{-\frac{t^\alpha}{v\alpha}}t^{\alpha-1}\mathrm{d}t \\ &= \frac{1}{v}\int_0^\infty \delta(t)e^{-\frac{t^\alpha}{v\alpha}}\mathrm{d}t, \end{aligned}$$

   using the property of δ function, yields

$$S_\alpha\left[t^{1-\alpha}\delta(t)\right](v) = \frac{1}{v}e^0,$$

   finally,

$$S_\alpha\left[t^{1-\alpha}\delta(t)\right](v) = \frac{1}{v}.$$

2. Second step: we assume that the expression (4) is true up to the order $n-2$ and we proof that it stays true at the order $n-1$.

   For $\alpha \in (0, 1]$ and all $v > 0$, we have

$$S_\alpha\left[\mathbf{T}^{(n-1)\alpha} t^{1-\alpha}\delta(t)\right](v) = \frac{1}{v}\int_0^\infty \mathbf{T}^{(n-1)\alpha}\left[t^{1-\alpha}\delta(t)\right]e^{-\frac{t^\alpha}{v\alpha}}t^{\alpha-1}\mathrm{d}t,$$

applying the definition of $\mathbf{T}^{n\alpha}$, we get

$$S_\alpha\left[\mathbf{T}^{(n-1)\alpha}t^{1-\alpha}\delta(t)\right](v)=\frac{1}{v}\int_0^\infty \mathbf{T}^{(n-2)\alpha}\left[\mathbf{T}^{\alpha}(t^{1-\alpha}\delta(t))\right]e^{-\frac{t^\alpha}{v\alpha}}t^{\alpha-1}\mathrm{d}t,$$

as the formula (4) is true for $n-2$, we obtain

$$S_\alpha\left[\mathbf{T}^{(n-1)\alpha}t^{1-\alpha}\delta(t)\right](v)=\frac{1}{v^{n-1}}\int_0^\infty \mathbf{T}^{\alpha}\left[t^{1-\alpha}\delta(t)\right]e^{-\frac{t^\alpha}{v\alpha}}t^{\alpha-1}\mathrm{d}t,$$

by the use of the definition of $\mathbf{T}^{\alpha}$, we find

$$\begin{aligned}S_\alpha\left[\mathbf{T}^{(n-1)\alpha}t^{1-\alpha}\delta(t)\right](v)&=\frac{1}{v^{n-1}}\int_0^\infty t^{1-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\left[t^{1-\alpha}\delta(t)\right]e^{-\frac{t^\alpha}{v\alpha}}t^{\alpha-1}\mathrm{d}t\\&=\frac{1}{v^{n-1}}\left[\int_0^\infty(1-\alpha)t^{-\alpha}\delta(t)e^{-\frac{t^\alpha}{v\alpha}}\mathrm{d}t\right.\\&\left.+\int_0^\infty t^{1-\alpha}\frac{\mathrm{d}}{\mathrm{d}t}\left[\delta(t)\right]e^{-\frac{t^\alpha}{v\alpha}}\mathrm{d}t\right],\end{aligned}$$

using the property of the function δ, it follows

$$\begin{aligned}S_\alpha\left[\mathbf{T}^{(n-1)\alpha}t^{1-\alpha}\delta(t)\right](v)=&\frac{1}{v^{n-1}}\left[\int_0^\infty(1-\alpha)t^{-\alpha}\delta(t)e^{-\frac{t^\alpha}{v\alpha}}\mathrm{d}t\right.\\&\left.+\frac{1}{v}\int_0^\infty\delta(t)e^{-\frac{t^\alpha}{v\alpha}}\mathrm{d}t-\int_0^\infty(1-\alpha)t^{-\alpha}\delta(t)e^{-\frac{t^\alpha}{v\alpha}}\mathrm{d}t\right],\end{aligned}$$

finally, we obtain

$$S_\alpha\left[\mathbf{T}^{(n-1)\alpha}t^{1-\alpha}\delta(t)\right](v)=\frac{1}{v^{n-1}}S_\alpha\left[t^{1-\alpha}\delta(t)\right](v)=\frac{1}{v^n},\quad\forall n\in\mathbb{N}^*.$$

□

Inspired by [26, 27] and based on [9, 16] we obtain the following results.

Proposition 2. *Let $A,E\in\mathbb{R}^{n_1\times n_1}$ be a real matrices with $\det E=0$, then, we have*

$$\left(\frac{1}{v}E-A\right)^{-1}=\sum_{i=-\mu}^{\infty}\phi_i v^{i+1},\quad v>0,\tag{5}$$

with $\mu=rg(E)-deg\left(\det\left(\frac{1}{v}E-A\right)\right)+1$ represents the index of nilpotency of $\left(\frac{1}{v}E-A\right)$ and ϕ_i are the fundamental matrices, which depend on the regularity of E and satisfy

$$\phi_i=(\phi_0 A)^i\phi_0,\quad\forall i\in\mathbb{N},\tag{6}$$

and

$$\phi_iE-\phi_{i-1}A=\delta_{i0}\mathbb{I}=E\phi_i-A\phi_{i-1},\tag{7}$$

where δ_{i0} is the Kronecker delta.

However, when $\det E \neq 0$, the Laurent series are described by the following proposition

Proposition 3. *Let $A, E \in \mathbb{R}^{n_1 \times n_1}$ be a real matrices with $\det E \neq 0$, then, we have*

$$\left(\frac{1}{v}E - A\right)^{-1} = \sum_{i=0}^{\infty} \phi_i v^{i+1}, \quad v > 0, \tag{8}$$

with ϕ_i are the fundamental matrices, which depend on the regularity of E and satisfy

$$\phi_i = \left(E^{-1}A\right)^i E^{-1}. \tag{9}$$

3 Main Results

This section is devoted to present our main results. For this purpose, we will consider the following continuous-times linear systems

$$E\mathbf{T}^{\alpha}x(t) = Ax(t) + Bu(t), \tag{10}$$

$$y(t) = Cx(t) + Du(t), \tag{11}$$

where $\mathbf{T}^{\alpha}$ presents the conformable derivative operator of order α with $0 < \alpha \leq 1$, $x \in \mathbb{R}^{n_1}$, $u \in \mathbb{R}^{m_1}$ and $y \in \mathbb{R}^{p_1}$ are, respectively, the state, the control, and the output of the system. $E, A \in \mathbb{R}^{n_1 \times n_1}$, $B \in \mathbb{R}^{n_1 \times m_1}$, $C \in \mathbb{R}^{p_1 \times n_1}$ and $D \in \mathbb{R}^{p_1 \times m_1}$ with $\det E = 0$. The boundary condition of the system is given by $x(0) = x_0$.

We take into account the following hypotheses which implies that the solution is impulse free:

(i) $Ex(0)$ and $v^{-i}Ex(0)$ exist for $i = \overline{1, \mu}$ and $v \in (-\tau_1, \tau_2)$,

(ii) $u(t)$ is specified for $t \geq 0$,

(iii) The pencil $\left(\frac{1}{v}E - A\right)$ is regular for all $v \in \mathbb{C}$.

In the following, we denote X_α and U_α the conformable Sumudu transform of x and u respectively.

Applying the conformable Sumudu transform to the equation (10), we obtain

$$S_\alpha\left[E\mathbf{T}^{\alpha}x(t)\right](v) = S_\alpha\left[Ax(t) + Bu(t)\right](v), \quad v > 0.$$

The use of the linearity property of conformable Sumudu transform together with the first property of the theorem 2, yields

$$E\left(\frac{X_\alpha(v) - x(0)}{v}\right) = AX_\alpha(v) + BU_\alpha(v),$$

which is equivalent to

$$\left[\frac{1}{v}E - A\right]X_\alpha(v) = \frac{1}{v}Ex(0) + BU_\alpha(v).$$

As the pencil $(E,\, A)$ is regular, so

$$X_\alpha(v) = \left[\frac{1}{v}E - A\right]^{-1}\left[\frac{1}{v}Ex(0) + BU_\alpha(v)\right]. \tag{12}$$

Thanks to the formula (5), the relation (12) becomes

$$X_\alpha(v) = \sum_{i=-\mu}^{\infty} \phi_i v^i Ex(0) + \sum_{i=-\mu}^{\infty} \phi_i v^{i+1} BU_\alpha(v),$$

by dividing the sum we get

$$\begin{aligned} X_\alpha(v) = &\sum_{i=0}^{\infty} \phi_i v^i Ex(0) + \sum_{i=0}^{\infty} \phi_i v^{i+1} BU_\alpha(v) \\ &+ \sum_{i=1}^{\mu} \phi_{-i} v^{-i} Ex(0) + \sum_{i=1}^{\mu} \phi_{-i} v^{-i+1} BU_\alpha(v). \end{aligned} \tag{13}$$

Finally, by the use of the inverse conformable Sumudu transform and convolution product, we obtain the following theorem which represents the first result of this paper.

Theorem 4. *The solution of the singular dynamical system of order α described by the equation* (10) *is given by*

$$\begin{aligned} x(t) = &\sum_{i=0}^{\infty} \phi_i \left(\frac{t^{\alpha i}}{\alpha^i i!} Ex(0) + \int_0^t \frac{(t^\alpha - \tau^\alpha)^i}{\alpha^i i!} Bu(\tau)\mathrm{d}\tau^\alpha\right) \\ &+ \sum_{i=1}^{\mu} \phi_{-i}\left(B\mathbf{T}^{\alpha(i-1)}u(t) + E\mathbf{T}^{\alpha(i-1)} t^{1-\alpha}\delta(t)x(0)\right), \end{aligned} \tag{14}$$

where $\mu = rg(E) - deg\left(\det\left(\frac{1}{v}E - A\right)\right) + 1$ represents the index of nilpotency of $\left(\frac{1}{v}E - A\right)$, ϕ_i are the fundamental matrices defined in proposition 2, *and δ is the Dirac delta function.*

Theorem 4 can be expressed using the exponential expression and the formula (6) as follow

Corollary 1. *The state of the singular dynamical system of order α described by the equation* (10) *is given by*

$$\begin{aligned} x(t) = &e^{\phi_0 A \frac{t^\alpha}{\alpha}} \phi_0 Ex(0) + \int_0^t e^{\phi_0 A \frac{t^\alpha - \tau^\alpha}{\alpha}} \phi_0 Bu(\tau)\mathrm{d}\tau^\alpha \\ &+ \sum_{i=1}^{\mu} \phi_{-i}\left(B\mathbf{T}^{\alpha(i-1)}u(t) + E\mathbf{T}^{\alpha(i-1)} t^{1-\alpha}\delta(t)x(0)\right), \end{aligned} \tag{15}$$

where $\mu = rg(E) - deg\left(\det\left(\frac{1}{v}E - A\right)\right) + 1$ represents the index of nilpotency of $\left(\frac{1}{v}E - A\right)$, and ϕ_i are the fundamental matrices defined in proposition 2, *and δ is the Dirac delta function.*

Remark 1. *If $\alpha = 1$, we find the state response of the singular dynamical system defined in [10]*

$$\begin{aligned} x(t) =& e^{\phi_0 At}\phi_0 Ex(0) + \int_0^t e^{\phi_0 A(t-\tau)}\phi_0 Bu(\tau)\mathrm{d}\tau \\ &+ \sum_{i=1}^{\mu} \phi_{-i}\left(Bu^{(i-1)}(t) + E\delta^{(i-1)}(t)x(0)\right). \end{aligned} \tag{16}$$

where $\mu = rg(E) - deg\left(\det\left(\frac{1}{v}E - A\right)\right) + 1$ represents the index of nilpotency of $\left(\frac{1}{v}E - A\right)$, and ϕ_i are the fundamental matrices defined in proposition 2, and δ is the Dirac delta function.

Let us, now, discuss the case where E is a regular matrix, i.e., $\det E \neq 0$. For this case, we assume that $[E^{-1}A]^i v^i x(0)$ exist for all $i \in \mathbb{N}$ and $v \in (-\tau_1, \tau_2)$. Hence

Theorem 5. *The solution of the implicit dynamical system of order α given by the equation (10) is*

$$x(t) = \sum_{i=0}^{\infty}\left[E^{-1}A\right]^i \frac{t^{\alpha i}}{\alpha^i i!}x(0) + \int_0^t \sum_{i=0}^{\infty}\left[E^{-1}A\right]^i E^{-1}\frac{(t^\alpha - \tau^\alpha)^i}{\alpha^i i!}Bu(\tau)\mathrm{d}\tau^\alpha. \tag{17}$$

Therefore, by using the exponential expression, we obtain

$$x(t) = e^{\left[E^{-1}A\right]\frac{t^\alpha}{\alpha}}x(0) + \int_0^t e^{\left[E^{-1}A\right]\frac{t^\alpha - \tau^\alpha}{\alpha}}E^{-1}Bu(\tau)\mathrm{d}\tau^\alpha.$$

Proof. Thanks to the formula (8), the relation (12) becomes

$$X(v) = \sum_{i=0}^{\infty}\phi_i v^i Ex(0) + \sum_{i=0}^{\infty}\phi_i v^{i+1}BU_\alpha(v),$$

it follows that

$$X_\alpha(v) = \sum_{i=0}^{\infty}\left[E^{-1}A\right]^i v^i x(0) + \sum_{i=0}^{\infty}\left[E^{-1}A\right]^i E^{-1}Bv^{i+1}U_\alpha(v).$$

Finally by applying the inverse of conformable Sumudu transform and the convolution product, we obtain the solution. □

Remark 2. *If $E = I$, we obtain the standard dynamical system of order α and the state is*

$$x(t) = e^{A\frac{t^\alpha}{\alpha}}x(0) + \int_0^t e^{A\frac{t^\alpha - \tau^\alpha}{\alpha}}Bu(\tau)\mathrm{d}\tau^\alpha.$$

Furthermore, if $\alpha = 1$, the state of the standard dynamical system is

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\mathrm{d}\tau.$$

4 Experimental results

In this section, we present some illustrative academic and real examples in order to show the efficiency and the accuracy of our approach. It must be emphasized that all examples were already discussed in [16] and [22].

Example 1. *Let us consider, for $\alpha \in (0, 1]$, the following system of electrical circuit*

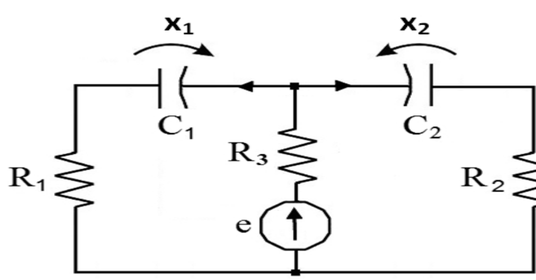

Figure 1: Electrical circuit [22].

R_1, R_2, R_3 represent resistances, C_1, C_2 the capacitances, and e the source voltage (the control $u(t) = e$). Using Kirchhoff's laws, we can write the equations

$$e = R_1C_1\frac{d^\alpha x_1}{dt^\alpha} + x_1 + R_3\left(C_1\frac{d^\alpha x_1}{dt^\alpha} + C_2\frac{d^\alpha x_2}{dt^\alpha}\right), \tag{18}$$

$$e = R_3\left(C_1\frac{d^\alpha x_1}{dt^\alpha} + C_2\frac{d^\alpha x_2}{dt^\alpha}\right) + R_2C_2\frac{d^\alpha x_2}{dt^\alpha} + x_2, \tag{19}$$

which are equivalent to

$$\begin{bmatrix}(R_1+R_3)C_1 & R_3C_2\\ R_3C_1 & (R_2+R_3)C_2\end{bmatrix}\frac{d^\alpha}{dt^\alpha}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} + \begin{bmatrix}1\\1\end{bmatrix}e. \tag{20}$$

The general expression of the system (20) *is*

$$\mathbf{T}^\alpha Ex(t) = Ax(t) + Bu(t), \tag{21}$$

with boundary condition $x_0 = 0_{\mathbb{R}^2}$ and

$$E = \begin{pmatrix}(R_1+R_3)\,C_1 & R_3C_2\\ R_3C_1 & (R_2+R_3)C_2\end{pmatrix},$$

$$A = \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix},\ B = \begin{pmatrix}1\\1\end{pmatrix},$$

as $\det E = [R_1(R_2+R_3) + R_2R_3]C_1C_2 \neq 0$, *then,*

$$E^{-1} = \frac{1}{\det E}\begin{pmatrix}(R_2+R_3)C_2 & -R_3C_2\\ -R_3C_1 & (R_1+R_3)C_1\end{pmatrix},$$

$$E^{-1}A = \frac{1}{\det E}\begin{pmatrix} -(R_2+R_3)C_2 & R_3C2 \\ R_3C_1 & -(R_1+R_3)C_1 \end{pmatrix} \; and\, E^{-1}B = \frac{1}{\det E}\begin{pmatrix} R_2C_2 \\ R_1C_1 \end{pmatrix}.$$

For $e = 1V$, the solution of the electrical circuit is

$$x(t) = \int_0^t e^{E^{-1}A\frac{(t^\alpha - \tau^\alpha)}{\alpha}} E^{-1}B\mathrm{d}\tau^\alpha, \tag{22}$$

which is the same one as in [19].

The solution with Caputo derivative is

$$\tilde{x}(t) = \sum_{k=0}^{\infty}\left(A^k \int_0^t \frac{(t-\tau)^{(k+1)\alpha-1}}{\Gamma[(k+1)\alpha]}\mathrm{d}\tau\right)B, \tag{23}$$

To show the efficiency of our method we will plot, in the following figures, both solutions together with the exact solution for different values of α. We assume that $R_1 = R_2 = 10\Omega$, $R_3 = 20\Omega$, $C_1 = C_2 = 100mF$ ant the input $u(t) = e = 1V$,

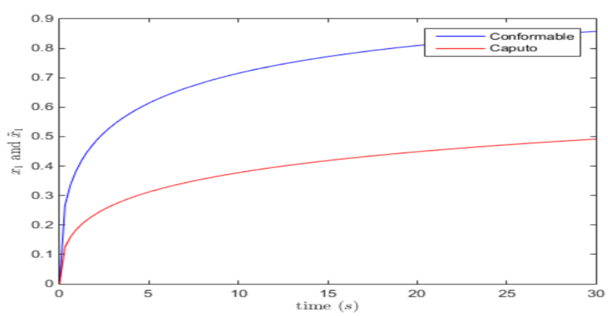

Figure 2: Comparison of the solutions x_1 and $\tilde{x}_1$ for $\alpha = 0.4$.

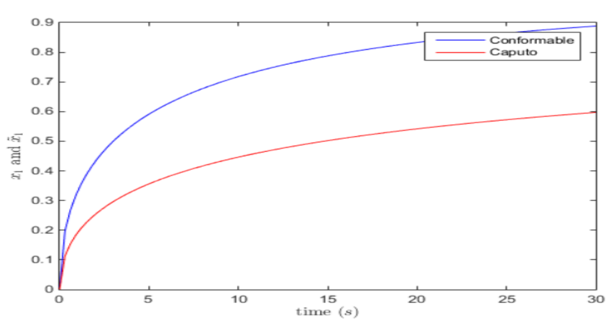

Figure 3: Comparison of the solutions x_1 and $\tilde{x}_1$ for $\alpha = 0.5$.

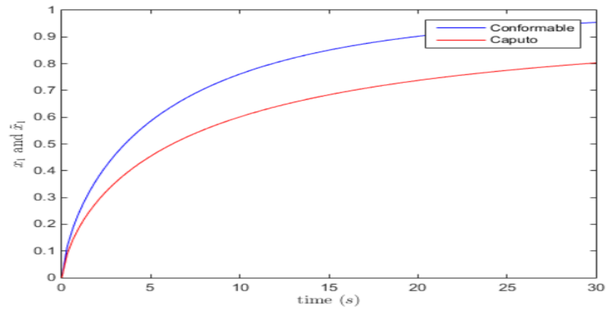

Figure 4: Comparison of the solutions x_1 and $\tilde{x}_1$ for $\alpha = 0.7$.

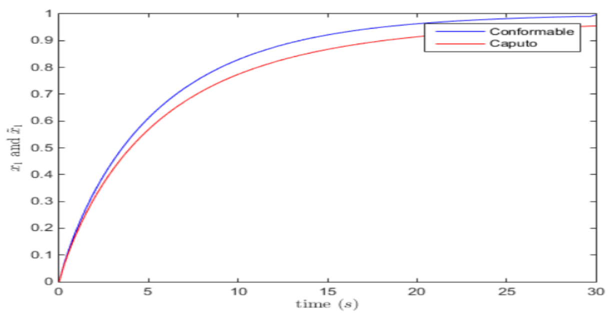

Figure 5: Comparison of the solutions x_1 and $\tilde{x}_1$ for $\alpha = 0.9$.

Example 2. *Let $0 < \alpha \leq 1$ and the following singular system*

$$\mathbf{T}^{\alpha} Ex(t) = Ax(t) + Bu(t), \tag{24}$$

with

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ A = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}, \ B = \begin{pmatrix} 1 \\ 2 \end{pmatrix},$$

and the initial condition

$$x_0 = \begin{pmatrix} x_{0,1} \\ x_{0,2} \end{pmatrix}.$$

Since

$$\det\left(\frac{1}{v}E - A\right) = \frac{2+2v}{v} \neq 0, \quad \forall v > 0,$$

and $\mu = 1$, it follows

$$\phi_{-1} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \phi_{2m} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \phi_{2m+1} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \forall m \in \mathbb{N}.$$

The state of the system (24) *is given by*

$$x(t) = \begin{pmatrix} e^{\frac{-t^{\alpha}}{\alpha}} x_{0,1} + \int_0^t e^{-\frac{t^{\alpha}-\tau^{\alpha}}{\alpha}} u(\tau) \mathrm{d}\tau^{\alpha} \\ u(t) \end{pmatrix}. \tag{25}$$

However, with the Caputo derivative, we find

$$\tilde{x}(t) = \begin{pmatrix} \displaystyle\sum_{i=0}^{\infty}(-1)^i \left[\frac{t^{i\alpha}}{\Gamma(i\alpha+1)}x_{0,1} + \frac{1}{\Gamma((i+1)\alpha)}\int_0^t (t-\tau)^{(i+1)\alpha-1}u(\tau)\mathrm{d}\tau\right] \\ u(t) \end{pmatrix}. \tag{26}$$

For different values of α, $u(t) = 1$, $x_{0,1} = 3$, *and* $x_{0,2} = 0$, *the comparison of the states between conformable derivative* $x(t) = [x_1(t), x_2(t)]^T$, *Caputo derivative* $\tilde{x}(t) = [\tilde{x}_1(t), \tilde{x}_2(t)]^T$ *is plotted in figures* 6, 7, *and* 8.

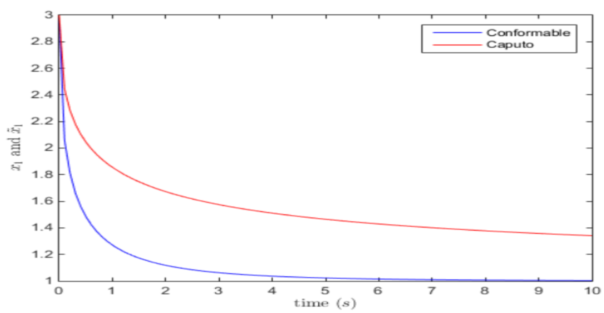

Figure 6: Comparison of the solutions x_1 and $\tilde{x}_1$ for $\alpha = 0.5$.

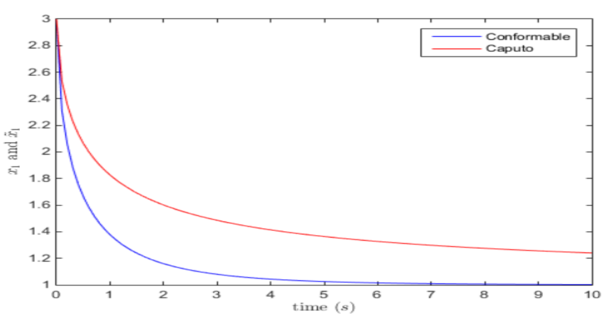

Figure 7: Comparison of the solutions x_1 and $\tilde{x}_1$ for $\alpha = 0.6$.

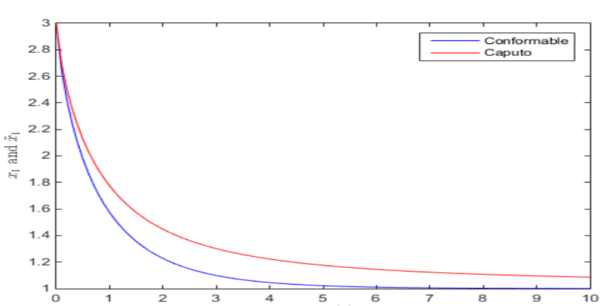

Figure 8: Comparison of the solutions x_1 and $\tilde{x}_1$ for $\alpha = 0.8$.

5 Concluding Remarks

In this paper, the continuous-time linear systems based on the conformable derivatives operator are introduced where another approach to compute there solutions are presented. The main idea behind this approach consists on using the conformable Sumudu transform which is recognized by its important properties. The singular and regular cases are discussed and the method can be used for several practical applications as for instance the electrical circuit. Through the numerical examples presented the final section, it easy to see that the solution of dynamical systems with conformable derivative is consistent to the classical derivative. More then that, it has been shown in [19] that for the conformable derivative, the electrical circuit could be reach its steady state in a shorter time. For our future work, the researches should be undertaken in the conformable Sumudu for other models with conformable derivatives such as; financial models.

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References

[1] Al-Zhour, Z., Alrawajeh, F., Al-Mutairi, N. and Alkhasawneh, R., *New Results on the Conformable Fractional Sumudu Transform: Theories and Applications*, Int. J. Anal. Appl., **17**, (2019), no. 6, 1019–1033.

[2] Asiru, M.A., *Classroom note: application of the Sumudu transform to discrete dynamic systems*, International Journal of Mathematical Education in Science and Technology, **34**, (2003), no. 6, 944–949.

[3] Asiru, M.A., *Further properties of the Sumudu transform and its applications*, International J. of Mathematical Education in Science and Technology, **33**, (2010), no. 3, 441–449.

[4] Asiru, M.A., *Sumudu transform and solution of integral equations of convolution type*, International J. of Mathematical Education in Science and Technology, **32**, (2010), no. 6, 906–910.

[5] Avcı, D., İskender Eroğlu, B.B. and Özdemir, N., *Conformable Fractional Wave-Like Equation on a Radial Symmetric Plate*, In: Babiarz A., Czornik

A., Klamka J., Niezabitowski M. (eds) Theory and Applications of Non-integer Order Systems. Lecture Notes in Electrical Engineering. Springer, Cham, **407**, (2017),137–146.

[6] Avcı, D., İskender Eroğlu, B.B. and Özdemir, N., *Conformable Heat Problem in a Cylinder*, Proceedings, International Conference on Fractional Differentiation and its Applications, Novi Sad, Serbia, 572–580, 2016.

[7] Avcı, D., İskender Eroğlu, B.B. and Özdemir, N., *Conformable Heat Equation on a Radial Symmetric Plate*, Thermal Science, **21**, (2017), no. 2, 819–826.

[8] Benyettou, K., Bouagada, D. and Ghezzar, M.A., *Solution of 2D State Space Continuous-Time Conformable Fractional Linear System Using Laplace and Sumudu Transform*, Computational Mathematics and Modeling, **32**, (2021), 94–109.

[9] Bouagada, D. and Van Dooren, P., *State space solution of implicit fractional continuous-time systems*, Fractional Calculus and Applied Analysis, **15**, (2012), no. 3, 356–361. .

[10] Dai, L., *Singular Control Systems, Lecture Notes in Control and Information Sciences*, 118,Springer-Verlag 19 pages.

[11] Evirgen, F., *Conformable Fractional Gradient Based Dynamic System for Constrained Optimization Problem*, Acta Physica Polonica A, **132**, (2017), no. 3, 1066-1069. .

[12] Gupta, V.G., Shrama, B. and Kilicman, A., *A note on fractional Sumudu transform*, J. of Applied Mathematics, **2010**, (2010), 154189.

[13] Hörmander, L., *The analysis of linear partial differential operators I* (Springer, Berlin, Heidelberg, 2003).

[14] İskender Eroğlu, B.B., Avcı, D., and Özdemir, N., *Optimal Control Problem for a Conformable Fractional Heat Conduction Equation*, Acta Physica Polonica A, **132**, (2017), 658–662. .

[15] Jena, R.M., Chakraverty, S. and Yavuz, M., *Two-hybrid techniques coupled with an integral transform for Caputo time-fractional Navier-Stokes Equations*, Progress in Fractional Differentiation and Applications, **6**, no. 3, (2020), 201–213.

[16] Kaisserli, Z. and Bouagada, D., *Solution of State-Space Singular Continuous-Time Fractional Linear Systems using Sumudu Transform*, Lobachevski J. of Mathematics, **42**, (2021), 110–117.

[17] Kaisserli, Z. and Bouagada, D., *Application of the Sumudu transform to solve regular fractional continuous time linear systems*, Kragujevac J. of mathematics, **45**, (2021), no. 2, 267–274.

[18] Karatas Akgül, E., Akgül, A. and Yavuz, M., *New Illustrative Applications of Integral Transforms to Financial Models with Different Fractional Derivatives*, "Chaos, Solitons &Fractals, **146**, (2021),110877.

[19] Kaczorek, T., *Analysis of positive linear continous-time systems using the conformable derivative*, International J. of Applied Mathematics and Computer Science, **28**, (2018), no. 2, 335–340.

[20] Kaczorek, T., *Realization problem for positive fractional hybrid 2D linear systems*, Fractional Calculus and Applied Analysis, **11**, (2008), no. 3, 353–368.

[21] Kaczorek, T., *Selected Problems of Fractional Systems Theory*, Springer-Verlag, Berlin, 2011.

[22] Kaczorek, T. and Rogowski, K., *Fractional Linear Systems and Electrical Circuits*, Studies in Systems, Decision and Control **13**, Springer International Publishing, Switzerland, 2015.

[23] Khalil, R., Al Horani, M., Yousef, A. and Sababhehb, M., *A new definition of fractional derivative*, J. of Computational and Applied Mathematics, **264**, (2014), 65–70.

[24] Kılıcman, A., Eltayeb, H. and Agarwal, P.R., *On Sumudu transform and system of differential equations*, Abstract and Applied Analysis, **2010**, (2010), 598702.

[25] Kılıcman, A. and Gadain, H., *On the applications of Laplace and Sumudu transforms*, J. of the Franklin Institute, **347**, (2010), no. 5, 848–862.

[26] Lewis, F.L. and Mertzios, B.G., *On the analysis of discrete linear time-invariant singular systems*, IEEE Transactions on Automatic Control , **35**, (1992), no. 5, 399–419.

[27] Mertzios, B.G. and Lewis, F.L., *Fundamental matrix of discrete singular systems circuits*, Circuits, Systems and Signal Processing, **8**, (1989), no. 3, 341–355.

[28] Miller, K.S. and Ross, B., *An Introduction to the Fractional Calculus and Fractional Differential Equations* (John Wiley & Sons Inc, 1994 [in New York].

[29] Piotrowska, E. and Rogowski, K., *Analysis of Fractional Electrical Circuit Using Caputo and Conformable Derivative Definitions*, In: Ostalczyk P., Sankowski D., Nowakowski J. (eds) Non-Integer Order Calculus and its Applications. RRNR 2017. Lecture Notes in Electrical Engineering, **496**, (2019).

[30] Rogowski, K., *General Response Formula for CFD Pseudo-Fractional 2D Continuous Linear Systems Described by the Roesser Model*, J. Symmetry, **12**, (2020), no. 12, 2073–8994.

[31] Sabatier, J., Agrawal, O.P. and Machado, J.A. *Advances in Fractional. Calculus, Theoritical Developments and Applications in Physics and Engineering* Springer, 2007 [in London].

[32] Thabet, A., *On conformable fractional calculus*, J. of Computational and Applied Mathematics, **279**, (2015), 57–66.

[33] Tuluce Demiray, S., Bulut, H. and Belgacem, F.B.M., *Sumudu transform method for analytical solutions of fractional type ordinary differential equations*, Mathematical Problems in Engineering, **2015**, (2015), 131690.

[34] Vashi, J. and Timol, M.G., *Laplace and Sumudu transforms and their application*, International J. of Innovative Science, Engineering and Technology, **3**, (2016), no. 8, 538–542.

[35] Watugala, G.K., *Sumudu transform: a new integral transform to solve differential equations and control engineering problems*, International J. of Mathematical Education in Science and Technology **24**, (1993), no. 1, 35–43.

[36] Yavuz, M., *Novel solution methods for initial boundary value problems of fractional order with conformable differentiation*, An International J. of Optimization and Control: Theories & Applications, **8**, (2018), no. 1, 1–7.

[37] Yaṣkıran, B. and Yavuz, M., *Approximate-analytical solutions of cable equation using conformable fractional operator*, New Trends in Mathematical Sciences, **4**, (2017),209–219.

[38] Yavuz. M. and Özdemir, N., *On the solutions of fractional Cauchy problem featuring conformable derivative*, In ITM Web of Conferences, **22**, p. 01045. EDP Sciences, (2018).

[39] Yavuz, M. and Yaṣkıran, B., *Conformable Derivative Operator in Modelling Neuronal Dynamics*, Applications & Applied Mathematics, An International J., **13**, (2018), no. 2, 803–817.

[40] Yavuz, M. and Yaṣkıran, B., *Dynamical behaviors of separated homotopy method defined by conformable operator*, Konuralp Journal of Mathematics, **7**, (2019), no. 1, 1–6.