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ON THE CIRCULAR NUMERICAL RANGE OF 5-BY-5 PARTIAL ISOMETRIES

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Abstract

We prove, in some cases in term of kippenhahn curve, that if 5-by-5 partial isometry whose numerical range is a circular disc then its center is must be the origin. This gives a partial affirmative answer of the Conjecture 5.1. of [H. l. Gau et al., Linear and Multilinear Algebra, 64 (1) 2016, 14–35.], for the five dimensional case.

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1 Introduction

Let A be an $n \times n$ complex matrix, its numerical range $W(A)$ is, by definition, the set of complex numbers

$$W(A) = \{\langle Ax, x\rangle : \quad x \in \mathbb{C}^n, \quad \|x\| = 1\}.$$

It is well known that $W(A)$ is a nonempty compact convex subset of $\mathbb{C}$, also contains all the eigenvalues of A and therefore its convex hull, see for instance [6]. The matrix A is said to be a partial isometry if it is isometric on the orthogonal complement of the kernel of A, $Ker(A)$. Assume that A is a partial isometry whose numerical range $W(A)$ is a circular disc. The question is whether the center of $W(A)$ must be the origin. Gau et al. ,[5], gave an affirmative answer if the dimension is at most 4, as follows,

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Theorem 1. *[5, Theorem 2.1] If A is an n-by-n ($n \leq 4$) partial isometry with $W(A) = \{z \in \mathbb{C} : |z - a| \leq r\}$, ($r > 0$), then $a = 0$.*

Also have conjectured that theorem remains valid if A is an n-by-n partial isometry, see [5, Conjecture 5.1.]. By the same procedure of [5], we give an affirmative answer for this conjecture for 5-by-5 partial isometry in some cases in terms of kippenhahn curve, to be more specific, our main theorem reads as follows,

Theorem 2. *Let A be a 5×5 partial isometry matrix $W(A) = \{z \in \mathbb{C} : |z - a| \leq r\}$, ($r > 0$). If the kippenhahn curve $C_R(A)$ has one of the following shapes,*

(i) $C_R(A)$ consists of three point and an ellipse.

(ii) $C_R(A)$ consists of two ellipses and a point.

(iii) $C_R(A)$ consists of a curve of degree 4 with double tangent and an ellipse.

Then $a = 0$.

Two successful approaches to establishing this result are a canonical decomposition of $n \times n$ partial isometry matrix and the Kippenhahn's result for the numerical range of $n \times n$ matrix.

It well known that the numerical range of an $n \times n$ matrix A is completely determined by its Kippenhahn polynomial $P_A(x, y, z) = \det(x\mathrm{Re}(A) + y\mathrm{Im}(A) + zI_n)$, where $\mathrm{Re}(A) = (A + A^*)/2$ and $\mathrm{Im}(A) = (A - A^*)/2\mathbf{i}$ are the real and the imaginary part of A, respectively. I_n denotes the $n \times n$ identity matrix and $\mathbf{i}$ is the complex number $\mathbf{i}^2 = -1$. Let $C(A)$ be the dual of the algebraic curve defined to be the zero set of $P_A(x, y, z) = 0$, on the complex projective plane $\mathbb{CP}^2$, which consists of all equivalence classes of points in $\mathbb{C}^3 \setminus (0, 0, 0)$ under the equivalence relation $\sim$, this relation is defined by $(x, y, z) \sim (x', y', z')$ if and only if there is a nonzero $\lambda \in \mathbb{C}$ such that $(x, y, z) = \lambda(x', y', z')$. Kippenhahn showed that $W(A)$ is the convex hull of the real points of $C(A)$, see [8] and its English translation [9] for a detailed discussion of the connections between the polynomial P_A, and the numerical range of A. This characterization is used by many authors to answer the question when the numerical range of a matrix is an elliptic disc. For 2×2 matrices a complete description of the numerical range is well known, that is $W(A)$ elliptic disk (with possibly degenerate interior), see [6]. In [8] Kippenhahn showed that there are four classes of shapes which the numerical range of matrices of order three. This was improved in [7] by expressing the conditions in terms of the eigenvalues and entries of A, which are easier to apply. By the same procedure, these results are generalized for 4×4 matrices. Let us mention here, that numerous results are known in this direction only for some special classes of matrices, for partial isomertry, nilpotent, doubly stochastic matrices (etc...), see [11], [10]. But no unifying and general theory is not yet available.

In this paper, firstly, with a similar approach used in [2], we will give necessary and sufficient conditions for 5-by-5 matrix A to have an ellipse in the associated real Kippenhahn curve $C_R(A)$. We also express those conditions in terms of eigenvalues and entries of A, the main difficult is the heavy computations. All these conditions will be useful for construct a 5×5 matrix with an elliptic numerical range (Section 2.). Secondly, we establish the result of Theorem 2 in the special case of S_5-matrices (A is S_n-matrix if A is a contraction, the eigenvalues of A are all in the open unit disc $\mathbb{D}$ and $rank(I_n - A^*A) = 1$) (Section 3.). Finally, using the results of two preceding sections, we give the proof of Theorem 2 (Section 4.).

2 Necessary and sufficient conditions for 5×5 matrix to have an ellipse in its real Kippenhahn curve

Let A be a 5×5 complex matrix. Here we give necessary and sufficient conditions for which the associated curve $C_R(A)$ contains an ellipse or a circle. It is well known that, by Schur's theorem, every square matrix is unitarily equivalent to an upper triangular matrix. So, without loss of generality, we can assume that

$$A = \begin{bmatrix} \lambda_1 & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & \lambda_2 & a_{23} & a_{24} & a_{25} \\ 0 & 0 & \lambda_3 & a_{34} & a_{35} \\ 0 & 0 & 0 & \lambda_4 & a_{45} \\ 0 & 0 & 0 & 0 & \lambda_5 \end{bmatrix}, \tag{1}$$

where $\lambda_j = \alpha_j + \mathbf{i}\beta_j$, with α_j and β_j are real for $j = 1, 2, 3, 4, 5$.

Then, we have

$$P_A(x, y, z) = \det(x\mathrm{Re}(A) + y\mathrm{Im}(A) + zI_5)$$
$$= \begin{vmatrix} \psi_1(x,y) & \frac{a_{12}}{2}(x-\mathbf{i}y) & \frac{a_{13}}{2}(x-\mathbf{i}y) & \frac{a_{14}}{2}(x-\mathbf{i}y) & \frac{a_{15}}{2}(x-\mathbf{i}y) \\ (x+\mathbf{i}y)\frac{\overline{a_{12}}}{2} & \psi_2(x,y) & \frac{a_{23}}{2}(x-\mathbf{i}y) & \frac{a_{24}}{2}(x-\mathbf{i}y) & \frac{a_{25}}{2}(x-\mathbf{i}y) \\ (x+\mathbf{i}y)\frac{\overline{a_{13}}}{2} & (x+\mathbf{i}y)\frac{\overline{a_{23}}}{2} & \psi_3(x,y) & \frac{a_{34}}{2}(x-\mathbf{i}y) & \frac{a_{35}}{2}(x-\mathbf{i}y) \\ (x+\mathbf{i}y)\frac{\overline{a_{14}}}{2} & (x+\mathbf{i}y)\frac{\overline{a_{24}}}{2} & (x+\mathbf{i}y)\frac{\overline{a_{34}}}{2} & \psi_4(x,y) & \frac{a_{45}}{2}(x-\mathbf{i}y) \\ (x+\mathbf{i}y)\frac{\overline{a_{15}}}{2} & (x+\mathbf{i}y)\frac{\overline{a_{25}}}{2} & (x+\mathbf{i}y)\frac{\overline{a_{35}}}{2} & (x+\mathbf{i}y)\frac{\overline{a_{45}}}{2} & \psi_5(x,y) \end{vmatrix},$$

where $\psi_j(x, y) = \alpha_j x + \beta_j y + z$, for $j = 1, 2, 3, 4, 5$. By straightforward calculus, we obtain

$$P_A(x, y, z) = \prod_{i=1}^{5}(\alpha_i x + \beta_i y + z) - \frac{x^2 + y^2}{4} Q(x, y, z), \tag{2}$$

where

$$
\begin{aligned}
Q(x,y,z) &= \sum_{S_{ijk}S_{lm}} |a_{lm}|^2(\alpha_i x+\beta_i y+z)(\alpha_j x+\beta_j y+z)(\alpha_k x+\beta_k y+z) \\
&- \sum_{S_{ij}S_{klm}} [x\mathrm{Re}(a_{kl}a_{lm}\overline{a_{km}})+y\mathrm{Im}(a_{kl}a_{lm}\overline{a_{km}})](\alpha_i x+\beta_i y+z)(\alpha_j x+\beta_j y+z) \\
&- \frac{x^2+y^2}{4}\sum_{i=1}^{5}(\alpha_i x+\beta_i y+z)P_i + \sum_{S_iS_{jklm}}(\alpha_i x+\beta_i y+z)[\frac{x^2-y^2}{2}\mathrm{Re}(a_{jk}a_{kl}a_{lm}\overline{a_{jm}}) \\
&+ xy\mathrm{Im}(a_{jk}a_{kl}a_{lm}\overline{a_{jm}})] + \frac{x^2+y^2}{4}\sum_{S_{ij}S_{klm}}[x\mathrm{Re}(a_{kl}a_{lm}\overline{a_{km}})+y\mathrm{Im}(a_{kl}a_{lm}\overline{a_{km}})]|a_{ij}|^2 \\
&- \frac{x^2+y^2}{4}\sum_{S_{ijkl}S_{iml}}[x\mathrm{Re}(a_{ij}a_{jk}a_{kl}\overline{a_{im}a_{ml}})+y\mathrm{Im}(a_{ij}a_{jk}a_{kl}\overline{a_{im}a_{ml}})] \\
&- \frac{x^2+y^2}{4}\sum_{S_{ijk}S_{lm}S_{im}S_{lk}}[x\mathrm{Re}(a_{ij}a_{jk}a_{lm}\overline{a_{im}a_{lk}})+y\mathrm{Im}(a_{ij}a_{jk}a_{lm}\overline{a_{im}a_{lk}})] \\
&- \frac{1}{4}[(x^3-3xy^2)\mathrm{Re}(a_{12}a_{23}a_{34}a_{45}\overline{a_{15}})+(-y^3+3yx^2)\mathrm{Im}(a_{12}a_{23}a_{34}a_{45}\overline{a_{15}})].
\end{aligned}
$$

With $S_{i_1i_2\ldots i_n}$, for some $n\leq 5$, denotes the collection of all n-tuples $(i_1,i_2,\ldots,i_n)$ of natural numbers such that $1\leq i_1<i_2<\ldots\leq i_5$ and

$$P_i = \sum_{S_{jkm}S_{lm}}|a_{jk}|^2|a_{lm}|^2 - \sum_{S_{jkl}S_{jml}}\mathrm{Re}(a_{jk}a_{kl}\overline{a_{jm}a_{ml}}) - \sum_{S_{jk}S_{lkm}}\mathrm{Re}(a_{jk}a_{lm}\overline{a_{jm}a_{lk}})$$

for every $i=1,..,5$. The sums are taken with two by two different indexes which means that $i\neq j\neq k\neq l\neq m$.

We begin by the following Lemma.

Lemma 1. *Let A be a 5×5 matrix. Then the Kippenhahn curve $C_R(A)$ consists of two ellipses, one with foci λ_1, λ_2 and minor axis of length r, the other with foci λ_3, λ_4 and minor axis of length s, and λ_5 if and only if*

$P_A(x,y,z) = [(\alpha_1x+\beta_1y+z)(\alpha_2x+\beta_2y+z)-\frac{r^2}{4}(x^2+y^2)][(\alpha_3x+\beta_3y+z)(\alpha_4x+\beta_4y+z)-\frac{s^2}{4}(x^2+y^2)](\alpha_5x+\beta_5y+z)$,

where $\lambda_j=\alpha_j+\mathbf{i}\beta_j, j=1,2,3,4,5$ and the $\alpha_j's$ and $\beta_j's$ are real.

Proof. Let $B=\begin{bmatrix}\lambda_1 & r\\ 0 & \lambda_2\end{bmatrix}\oplus\begin{bmatrix}\lambda_3 & s\\ 0 & \lambda_4\end{bmatrix}\oplus\lambda_5$. As $C_R(A)=C_R(B)$, by duality the polynomials P_A and P_B are the same, therefore

$P_A(x,y,z)=[(\alpha_1x+\beta_1y+z)(\alpha_2x+\beta_2y+z)-\frac{r^2(x^2+y^2)}{4}][(\alpha_3x+\beta_3y+z)(\alpha_4x+\beta_4y+z)-\frac{s^2(x^2+y^2)}{4}](\alpha_5x+\beta_5y+z)$.

The converse is clear. $\square$

Using the above lemma, we can prove the following theorem.

Theorem 3. *Let A be in upper-triangular form* (1)*. Then the Kippenhahn curve $C_R(A)$ consists of two ellipses, one with foci λ_p, λ_q and minor axis of length r, the other with foci λ_t, λ_v and minor axis of length s, and a point λ_w if and only if*

(a) $r^2 + s^2 = \sum_{S_{lm}} |a_{lm}|^2.$

(b) $$r^2(\lambda_w + \lambda_t + \lambda_v) + s^2(\lambda_w + \lambda_q + \lambda_p) = \sum_{S_{ijk}S_{lm}} |a_{lm}|^2(\lambda_i + \lambda_j + \lambda_k) - \sum_{S_{klm}} a_{kl}a_{lm}\overline{a_{km}}.$$

(c) $$\begin{aligned} &r^2(\lambda_w\lambda_v + \lambda_w\lambda_t + \lambda_t\lambda_v) + s^2(\lambda_w\lambda_p + \lambda_w\lambda_q + \lambda_p\lambda_q)\\ &= \sum_{S_{ijk}S_{lm}} |a_{lm}|^2(\lambda_i\lambda_j + \lambda_i\lambda_k + \lambda_j\lambda_k) - \sum_{S_{ij}S_{klm}} (\lambda_i + \lambda_j)a_{kl}a_{lm}\overline{a_{km}}\\ &+ \sum_{S_{jklm}} a_{jk}a_{kl}a_{lm}\overline{a_{jm}}. \end{aligned}$$

(d) $$\begin{aligned} &r^2\lambda_w\lambda_t\lambda_v + s^2\lambda_w\lambda_p\lambda_q = \sum_{S_{ijk}S_{lm}} |a_{lm}|^2\lambda_i\lambda_j\lambda_k - \sum_{S_{ij}S_{klm}} \lambda_i\lambda_j a_{kl}a_{lm}\overline{a_{km}}\\ &+ \sum_{S_iS_{jklm}} a_{jk}a_{kl}a_{lm}\overline{a_{jm}}\lambda_i - a_{12}a_{23}a_{34}a_{45}\overline{a_{15}}. \end{aligned}$$

(e) $$\begin{aligned} &r^2\alpha_w\alpha_t\alpha_v + s^2\alpha_w\alpha_p\alpha_q - \frac{r^2s^2}{4}\alpha_w = \sum_{S_{ijk}S_{lm}} |a_{lm}|^2\alpha_i\alpha_j\alpha_k\\ &- \sum_{S_{ij}S_{klm}} \mathrm{Re}(a_{kl}a_{lm}\overline{a_{km}})\alpha_i\alpha_j - \frac{1}{4}\sum_{i=1}^{5} P_i\alpha_i + \frac{1}{2}\sum_{S_iS_{jklm}} \mathrm{Re}(a_{jk}a_{kl}a_{lm}\overline{a_{jm}})\alpha_i\\ &+ \frac{1}{4}\sum_{S_{ij}S_{klm}} \mathrm{Re}(a_{kl}a_{lm}\overline{a_{km}})|a_{ij}|^2 - \frac{1}{4}\sum_{S_{ijkl}S_{iml}} \mathrm{Re}(a_{ij}a_{jk}a_{kl}\overline{a_{im}a_{ml}})\\ &- \frac{1}{4}\sum_{S_{ijk}S_{lm}S_{im}S_{lk}} \mathrm{Re}(a_{ij}a_{jk}a_{lm}\overline{a_{im}a_{lk}}) - \frac{1}{4}\mathrm{Re}(a_{12}a_{23}a_{34}a_{45}\overline{a_{15}}). \end{aligned}$$

(f) $$\begin{aligned} &r^2\beta_w\beta_t\beta_v + s^2\beta_w\beta_p\beta_q - \frac{r^2s^2}{4}\beta_w = \sum_{S_{ijk}S_{lm}} |a_{lm}|^2\beta_i\beta_j\beta_k\\ &- \sum_{S_{ij}S_{klm}} \mathrm{Im}(a_{kl}a_{lm}\overline{a_{km}})\beta_i\beta_j - \frac{1}{4}\sum_{i=1}^{5} P_i\beta_i - \frac{1}{2}\sum_{S_iS_{jklm}} \mathrm{Re}(a_{jk}a_{kl}a_{lm}\overline{a_{jm}})\beta_i\\ &+ \frac{1}{4}\sum_{S_{ij}S_{klm}} \mathrm{Im}(a_{kl}a_{lm}\overline{a_{km}})|a_{ij}|^2 - \frac{1}{4}\sum_{S_{ijkl}S_{iml}} \mathrm{Im}(a_{ij}a_{jk}a_{kl}\overline{a_{im}a_{ml}})\\ &- \frac{1}{4}\sum_{S_{ijk}S_{lm}S_{im}S_{lk}} \mathrm{Im}(a_{ij}a_{jk}a_{lm}\overline{a_{im}a_{lk}}) + \frac{1}{4}\mathrm{Im}(a_{12}a_{23}a_{34}a_{45}\overline{a_{15}}). \end{aligned}$$

(g) $$\begin{aligned} &r^2(\alpha_w\alpha_t + \alpha_w\alpha_v + \alpha_v\alpha_t) + s^2(\alpha_w\alpha_p + \alpha_w\alpha_q + \alpha_p\alpha_q) - \frac{r^2s^2}{4}\\ &= \sum_{S_{ijk}S_{lm}} |a_{lm}|^2(\alpha_i\alpha_j + \alpha_i\alpha_k + \alpha_j\alpha_k) - \sum_{S_{ij}S_{klm}} \mathrm{Re}(a_{kl}a_{lm}\overline{a_{km}})(\alpha_i + \alpha_j)\\ &- \frac{1}{4}\sum_{i=1}^{5} P_i + \frac{1}{2}\sum_{S_{jklm}} \mathrm{Re}(a_{jk}a_{kl}a_{lm}\overline{a_{jm}}), \end{aligned}$$

where $P_i = \sum\limits_{S_{jkm}S_{lm}} |a_{jk}|^2|a_{lm}|^2 - \sum\limits_{S_{jkl}S_{jml}} \mathrm{Re}(a_{jk}a_{kl}\overline{a_{jm}a_{ml}})$
$- \sum\limits_{S_{jk}S_{lkm}} \mathrm{Re}(a_{jk}a_{lm}\overline{a_{jm}a_{lk}})$, *for every* $i = 1, .., 5$.

Proof. By lemma 1, we have,

$$\begin{aligned} P_A(x,y,z) = &(\alpha_w x + \beta_w y + z)(\alpha_p x + \beta_p y + z)(\alpha_q x + \beta_q y + z) \\ &(\alpha_t x + \beta_t y + z)(\alpha_v x + \beta_v y + z) \\ &- \frac{x^2+y^2}{4}[r^2(\alpha_w x + \beta_w y + z)(\alpha_t x + \beta_t y + z)(\alpha_v x + \beta_v y + z) \\ &+ s^2(\alpha_w x + \beta_w y + z)(\alpha_p x + \beta_p y + z)(\alpha_q x + \beta_q y + z) \\ &- \frac{x^2+y^2}{4} r^2 s^2 (\alpha_w x + \beta_w y + z)]. \end{aligned}$$

Comparing the previous formula of $P_A(x,y,z)$ with (2), we obtain,

$$\begin{aligned} Q(x,y,z) = &r^2(\alpha_w x + \beta_w y + z)(\alpha_t x + \beta_t y + z)(\alpha_v x + \beta_v y + z) \\ &+ s^2(\alpha_w x + \beta_w y + z)(\alpha_p x + \beta_p y + z)(\alpha_q x + \beta_q y + z) \\ &- \frac{x^2+y^2}{4} r^2 s^2 (\alpha_w x + \beta_w y + z). \end{aligned}$$

Computing the coefficients of $x^3, y^3, z^3, x^2y, xy^2, x^2z, xz^2, y^2z, yz^2, xyz$ by identification, we find, respectively

1. $r^2\alpha_w\alpha_t\alpha_v + s^2\alpha_w\alpha_p\alpha_q - \frac{r^2s^2}{4}\alpha_w = \sum\limits_{S_{ijk}S_{lm}} |a_{lm}|^2\alpha_i\alpha_j\alpha_k$
$- \sum\limits_{S_{ij}S_{klm}} \mathrm{Re}(a_{kl}a_{lm}\overline{a_{km}})\alpha_i\alpha_j - \frac{1}{4}\sum\limits_{i=1}^{5} P_i\alpha_i + \frac{1}{2}\sum\limits_{S_iS_{jklm}} \mathrm{Re}(a_{jk}a_{kl}a_{lm}\overline{a_{jm}})\alpha_i$
$+ \frac{1}{4}\sum\limits_{S_{ij}S_{klm}} \mathrm{Re}(a_{kl}a_{lm}\overline{a_{km}})|a_{ij}|^2 - \frac{1}{4}\sum\limits_{S_{ijkl}S_{iml}} \mathrm{Re}(a_{ij}a_{jk}a_{kl}\overline{a_{im}a_{ml}})$
$- \frac{1}{4}\sum\limits_{S_{ijk}S_{lm}S_{im}S_{lk}} \mathrm{Re}(a_{ij}a_{jk}a_{lm}\overline{a_{im}a_{lk}}) - \frac{1}{4}\mathrm{Re}(a_{12}a_{23}a_{34}a_{45}\overline{a_{15}})$.

2. $r^2\beta_w\beta_t\beta_v + s^2\beta_w\beta_p\beta_q - \frac{r^2s^2}{4}\beta_w = \sum\limits_{S_{ijk}S_{lm}} |a_{lm}|^2\beta_i\beta_j\beta_k$
$- \sum\limits_{S_{ij}S_{klm}} \mathrm{Im}(a_{kl}a_{lm}\overline{a_{km}})\beta_i\beta_j - \frac{1}{4}\sum\limits_{i=1}^{5} P_i\beta_i - \frac{1}{2}\sum\limits_{S_iS_{jklm}} \mathrm{Re}(a_{jk}a_{kl}a_{lm}\overline{a_{jm}})\beta_i$
$+ \frac{1}{4}\sum\limits_{S_{ij}S_{klm}} \mathrm{Im}(a_{kl}a_{lm}\overline{a_{km}})|a_{ij}|^2 - \frac{1}{4}\sum\limits_{S_{ijkl}S_{iml}} \mathrm{Im}(a_{ij}a_{jk}a_{kl}\overline{a_{im}a_{ml}})$
$- \frac{1}{4}\sum\limits_{S_{ijk}S_{lm}S_{im}S_{lk}} \mathrm{Im}(a_{ij}a_{jk}a_{lm}\overline{a_{im}a_{lk}}) + \frac{1}{4}\mathrm{Im}(a_{12}a_{23}a_{34}a_{45}\overline{a_{15}})$.

3. $r^2 + s^2 = \sum\limits_{S_{lm}} |a_{lm}|^2$.

4. $r^2(\beta_w\alpha_t\alpha_v+\beta_t\alpha_w\alpha_v+\beta_v\alpha_w\alpha_t)+s^2(\beta_w\alpha_p\alpha_q+\beta_p\alpha_w\alpha_q+\beta_q\alpha_w\alpha_p)-\dfrac{r^2s^2}{4}\beta_w$
$= \sum\limits_{S_{ijk}S_{lm}} |a_{lm}|^2(\alpha_i\alpha_j\beta_k+\alpha_i\alpha_k\beta_j+\alpha_j\alpha_k\beta_i) - \sum\limits_{S_{ij}S_{klm}} \mathrm{Im}(a_{kl}a_{lm}\overline{a_{km}})\alpha_i\alpha_j$
$- \sum\limits_{S_{ij}S_{klm}} \mathrm{Re}(a_{kl}a_{lm}\overline{a_{km}})(\alpha_i\beta_j+\alpha_j\beta_i) + \dfrac{1}{2}\sum\limits_{S_iS_{jklm}} \mathrm{Re}(a_{jk}a_{kl}a_{lm}\overline{a_{jm}})\beta_i$
$+ \sum\limits_{S_iS_{jklm}} \mathrm{Im}(a_{jk}a_{kl}a_{lm}\overline{a_{jm}})\alpha_i + \dfrac{1}{4}\sum\limits_{S_{ij}S_{klm}} \mathrm{Im}(a_{kl}a_{lm}\overline{a_{km}})|a_{ij}|^2 - \dfrac{1}{4}\sum\limits_{i=1}^{5} P_i\beta_i$
$- \dfrac{1}{4}\sum\limits_{S_{ijkl}S_{iml}} \mathrm{Im}(a_{ij}a_{jk}a_{kl}\overline{a_{im}a_{ml}}) - \dfrac{1}{4}\sum\limits_{S_{ijk}S_{lm}S_{im}S_{lk}} \mathrm{Im}(a_{ij}a_{jk}a_{lm}\overline{a_{im}a_{lk}})$
$- \dfrac{3}{4}\mathrm{Im}(a_{12}a_{23}a_{34}a_{45}\overline{a_{15}})$.

5. $r^2(\alpha_w\beta_t\beta_v+\alpha_t\beta_w\beta_v+\alpha_v\beta_w\beta_t)+s^2(\alpha_w\beta_p\beta_q+\alpha_p\beta_w\beta_q+\alpha_q\beta_w\beta_p)-\dfrac{r^2s^2}{4}\alpha_w$
$= \sum\limits_{S_{ijk}S_{lm}} |a_{lm}|^2(\alpha_i\beta_j\beta_k+\alpha_j\beta_i\beta_k+\alpha_k\beta_i\beta_j) - \sum\limits_{S_{ij}S_{klm}} \mathrm{Re}(a_{kl}a_{lm}\overline{a_{km}})\beta_i\beta_j$
$- \sum\limits_{S_{ij}S_{klm}} \mathrm{Im}(a_{kl}a_{lm}\overline{a_{km}})(\alpha_i\beta_j+\alpha_j\beta_i) - \dfrac{1}{4}\sum\limits_{i=1}^{5} P_i\alpha_i$
$- \dfrac{1}{2}\sum\limits_{S_iS_{jklm}} \mathrm{Re}(a_{jk}a_{kl}a_{lm}\overline{a_{jm}})\alpha_i + \sum\limits_{S_iS_{jklm}} \mathrm{Im}(a_{jk}a_{kl}a_{lm}\overline{a_{jm}})\beta_i$
$+ \dfrac{1}{4}\sum\limits_{S_{ij}S_{klm}} \mathrm{Re}(a_{kl}a_{lm}\overline{a_{km}})|a_{ij}|^2 - \dfrac{1}{4}\sum\limits_{S_{ijkl}S_{iml}} \mathrm{Re}(a_{ij}a_{jk}a_{kl}\overline{a_{im}a_{ml}})$
$- \dfrac{1}{4}\sum\limits_{S_{ijk}S_{lm}S_{im}S_{lk}} \mathrm{Re}(a_{ij}a_{jk}a_{lm}\overline{a_{im}a_{lk}}) + \dfrac{3}{4}\mathrm{Re}(a_{12}a_{23}a_{34}a_{45}\overline{a_{15}})$.

6. $r^2(\alpha_w\alpha_t+\alpha_w\alpha_v+\alpha_v\alpha_t)+s^2(\alpha_w\alpha_p+\alpha_w\alpha_q+\alpha_p\alpha_q)-\dfrac{r^2s^2}{4}$
$= \sum\limits_{S_{ijk}S_{lm}} |a_{lm}|^2(\alpha_i\alpha_j+\alpha_i\alpha_k+\alpha_j\alpha_k) - \sum\limits_{S_{ij}S_{klm}} \mathrm{Re}(a_{kl}a_{lm}\overline{a_{km}})(\alpha_i+\alpha_j)$
$- \dfrac{1}{4}\sum\limits_{i=1}^{5} P_i + \dfrac{1}{2}\sum\limits_{S_{jklm}} \mathrm{Re}(a_{jk}a_{kl}a_{lm}\overline{a_{jm}})$.

7. $r^2(\alpha_w+\alpha_t+\alpha_v)+s^2(\alpha_w+\alpha_p+\alpha_q)$
$= \sum\limits_{S_{ijk}S_{lm}} |a_{lm}|^2(\alpha_i+\alpha_j+\alpha_k) - \sum\limits_{S_{klm}} \mathrm{Re}(a_{kl}a_{lm}\overline{a_{km}})$.

8. $r^2(\beta_w\beta_t+\beta_w\beta_v+\beta_v\beta_t)+s^2(\beta_w\beta_p+\beta_w\beta_q+\beta_p\beta_q)-\dfrac{r^2s^2}{4}$
$= \sum\limits_{S_{ijk}S_{lm}} |a_{lm}|^2(\beta_i\beta_j+\beta_i\beta_k+\beta_j\beta_k) - \sum\limits_{S_{ij}S_{klm}} \mathrm{Im}(a_{kl}a_{lm}\overline{a_{km}})(\beta_i+\beta_j)$
$- \dfrac{1}{4}\sum\limits_{i=1}^{5} P_i - \dfrac{1}{2}\sum\limits_{S_{jklm}} \mathrm{Re}(a_{jk}a_{kl}a_{lm}\overline{a_{jm}})$.

9. $r^2(\beta_w+\beta_t+\beta_v)+s^2(\beta_w+\beta_p+\beta_q) = \sum\limits_{S_{ijk}S_{lm}} |a_{lm}|^2(\beta_i+\beta_j+\beta_k)$
$- \sum\limits_{S_{klm}} \mathrm{Im}(a_{kl}a_{lm}\overline{a_{km}})$.

10. $r^2(\alpha_w\beta_t + \alpha_w\beta_v + \alpha_t\beta_w + \alpha_t\beta_v + \alpha_v\beta_w + \alpha_v\beta_t) + s^2(\alpha_w\beta_p + \alpha_w\beta_q + \alpha_p\beta_w + \alpha_p\beta_q + \alpha_q\beta_w + \alpha_q\beta_p) = \sum_{S_{ijk}S_{lm}} |a_{lm}|^2(\alpha_i\beta_j + \alpha_j\beta_i + \alpha_i\beta_k + \alpha_k\beta_i + \alpha_j\beta_k + \alpha_k\beta_j) - \sum_{S_{ij}S_{klm}} \mathrm{Re}(a_{kl}a_{lm}\overline{a_{km}})(\beta_i + \beta_j) - \sum_{S_{ij}S_{klm}} \mathrm{Im}(a_{kl}a_{lm}\overline{a_{km}})(\alpha_i + \alpha_j) + \sum_{S_{jklm}} \mathrm{Im}(a_{jk}a_{kl}a_{lm}\overline{a_{jm}})$.

Note that the combination of (1), (2), (4) and (5) is equivalent to (d), (e) and (f), since $(1) - (5) - \mathbf{i}(2) + \mathbf{i}(4)$ yields (d).

(6), (8) and (10) is equivalent to (c) and (g) as $(6) - (8) + \mathbf{i}(10)$ yields (c).

(7) and (9) is equivalent to (b), it follows from $(7) + \mathbf{i}(9)$.

This completes the proof. □

Remark 1.
- *The last theorem was obtained in [1, Theorem 2.2], but, there exists a gap in this Theorem, we shall point out that in its proof the errors is exactly on the left side of equation $(4')$ and $(5')$ and we should also add the condition (e), (f), and (g) to ensure the converse implication.*
- *If we take $s = 0$ in the previous theorem, then $C_R(A)$ consists of ellipse and three points.*

In the following, we can see that $C_R(A)$ contains an ellipse and a curve of degree 4 with a double tangent. Using the same conditions derived in [13], to determine whether the numerical range boundary has a flatness for a 3×3 irreducible matrix A, these conditions are given in term of geometrical properties of flatness.

Let L be the supporting line of the convex set $W(A)$ containing the flatness and perpendicular to the line which pass through the origin and forms angle θ from the positive $x-$axis, and let μ be the (signed) distance from the origin to L. It is seen that μ is the largest eigenvalue of $\mathrm{Re}(e^{-\mathbf{i}\theta}A)$ (cf.[13], [8]).

Lemma 2. *Let A be a 5×5 matrix. Then the Kippenhahn curve $C_R(A)$ consists of one ellipse with foci λ_1,λ_2 and minor axis of length r, and a curve of degree 4 with a double tangent and foci at λ_3 , λ_4 and λ_5 if and only if the following conditions hold*

(i) there exist $\theta \in [0, 2\pi[$ and a real μ such that

$$\begin{aligned}
P_A(x,y,z) &= \left[(\alpha_1x + \beta_1y + z)(\alpha_2x + \beta_2y + z) - \frac{r^2}{4}(x^2 + y^2)\right] \\
&\times [(\alpha_3x + \beta_3y + z)(\alpha_4x + \beta_4y + z)(\alpha_5x + \beta_5y + z) \\
&- (\alpha_3x + \beta_3y + z)(x^2 + y^2)(\mathrm{Re}(e^{-\mathbf{i}\theta}\lambda_4) + \mu)(\mathrm{Re}(e^{-\mathbf{i}\theta}\lambda_5) + \mu) \\
&- (\alpha_4x + \beta_4y + z)(x^2 + y^2)(\mathrm{Re}(e^{-\mathbf{i}\theta}\lambda_3) + \mu)(\mathrm{Re}(e^{-\mathbf{i}\theta}\lambda_5) + \mu) \\
&- (\alpha_5x + \beta_5y + z)(x^2 + y^2)(\mathrm{Re}(e^{-\mathbf{i}\theta}\lambda_3) + \mu)(\mathrm{Re}(e^{-\mathbf{i}\theta}\lambda_4) + \mu) \\
&+ (x - iy)(x^2 + y^2)(\mathrm{Re}(e^{-\mathbf{i}\theta}\lambda_3) + \mu)(\mathrm{Re}(e^{-\mathbf{i}\theta}\lambda_4) + \mu)(\mathrm{Re}(e^{-\mathbf{i}\theta}\lambda_5) + \mu)e^{\mathbf{i}\theta} \\
&+ (x + iy)(x^2 + y^2)(\mathrm{Re}(e^{-\mathbf{i}\theta}\lambda_3) + \mu)(\mathrm{Re}(e^{-\mathbf{i}\theta}\lambda_4) + \mu)(\mathrm{Re}(e^{-\mathbf{i}\theta}\lambda_5) + \mu)e^{-\mathbf{i}\theta}].
\end{aligned}$$

(ii) $(\mathrm{Re}(e^{-\mathbf{i}\theta}\lambda_3)+\mu)(\mathrm{Re}(e^{-\mathbf{i}\theta}\lambda_4)+\mu)(\mathrm{Re}(e^{-\mathbf{i}\theta}\lambda_5)+\mu)\neq 0.$

(iii) $\lambda_j(\mathrm{Re}(e^{-\mathbf{i}\theta}\lambda_k)+\mu)+\lambda_k(\mathrm{Re}(e^{-\mathbf{i}\theta}\lambda_j)+\mu)-2(\mathrm{Re}(e^{-\mathbf{i}\theta}\lambda_j)+\mu)(\mathrm{Re}(e^{-\mathbf{i}\theta}\lambda_k)+\mu)e^{\mathbf{i}\theta}\neq\lambda_i((\mathrm{Re}(e^{-\mathbf{i}\theta}\lambda_k)+\mu)+(\mathrm{Re}(e^{-\mathbf{i}\theta}\lambda_j)+\mu))$, *for every* $3\leq i\neq j\neq k\leq 5$.

Proof. Let $B=\begin{bmatrix}\lambda_1 & r\\ 0 & \lambda_2\end{bmatrix}\oplus C$ where C is 3-by-3 irreducible matrix whose Kippenhahn curve is of degree 4 with a double tangent, and eigenvalues at λ_3 , λ_4 and λ_5.

Since $C_R(A)=C_R(B)$, the polynomials P_A and P_B are the same.

$$P_B(x,y,z)=\left[(\alpha_1x+\beta_1y+z)(\alpha_2x+\beta_2y+z)-\frac{r^2}{4}(x^2+y^2)\right]P_C(x,y,z)$$

Put C in an upper triangular form

$$\begin{bmatrix}\lambda_3 & a & b\\ 0 & \lambda_4 & c\\ 0 & 0 & \lambda_5\end{bmatrix}.$$

By assumption $\partial W(C)$ has a flat of portion (containing in the supporting line L), let $\theta\in[0,2\pi[$ be the angular between x-axis and the line which pass through the origin and perpendicular to L, so $e^{-\mathbf{i}\theta}C$ has a vertical flatness. According to Kippenhahn's classification, $\mathrm{Re}(e^{-\mathbf{i}\theta}C)$ must have a multiple eigenvalue, so there exist a real μ such that

$$\mathrm{Re}(e^{-\mathbf{i}\theta}C)+I\mu=\begin{bmatrix}\mathrm{Re}(e^{-\mathbf{i}\theta}\lambda_3)+\mu & e^{-\mathbf{i}\theta}a/2 & e^{-\mathbf{i}\theta}b/2\\ e^{\mathbf{i}\theta}\overline{a}/2 & \mathrm{Re}(e^{-\mathbf{i}\theta}\lambda_4)+\mu & e^{-\mathbf{i}\theta}c/2\\ e^{\mathbf{i}\theta}\overline{b}/2 & e^{\mathbf{i}\theta}\overline{c}/2 & \mathrm{Re}(e^{-\mathbf{i}\theta}\lambda_5)+\mu\end{bmatrix}$$

has rank one, because if otherwise $\mathrm{Re}(e^{-\mathbf{i}\theta}C)+I\mu$ has zero rank, then $\mathrm{Re}(e^{-\mathbf{i}\theta}C)$ and $\mathrm{Im}(e^{-\mathbf{i}\theta}C)$ commutes, and C is therefore reducible, while due to the latter all 2×2 minors of $\mathrm{Re}(e^{-\mathbf{i}\theta}C)+I\mu$ are equal to zero. Consequently,

$$\begin{aligned}|a|^2&=4(\mathrm{Re}(e^{-\mathbf{i}\theta}\lambda_3)+\mu)(\mathrm{Re}(e^{-\mathbf{i}\theta}\lambda_4)+\mu).\\ |b|^2&=4(\mathrm{Re}(e^{-\mathbf{i}\theta}\lambda_3)+\mu)(\mathrm{Re}(e^{-\mathbf{i}\theta}\lambda_5)+\mu).\\ |c|^2&=4(\mathrm{Re}(e^{-\mathbf{i}\theta}\lambda_4)+\mu)(\mathrm{Re}(e^{-\mathbf{i}\theta}\lambda_5)+\mu).\\ a\overline{b}&=2(\mathrm{Re}(e^{-\mathbf{i}\theta}\lambda_3)+\mu)\overline{c}e^{\mathbf{i}\theta}.\end{aligned}\tag{3}$$

It is easy to see from equations above that if one of off-diagonal a,b or c is zero, then at least two of them are equal to zero, this contradict the irreducibility of C, so $abc\neq 0$.

On the other hand,

$$P_C(x,y,z) = \det \begin{bmatrix} (\alpha_3 x + \beta_3 y + z) & a/2(x - iy) & b/2(x - \mathbf{i}y) \\ \overline{a}/2(x + \mathbf{i}y) & (\alpha_4 x + \beta_4 y + z) & c/2(x - \mathbf{i}y) \\ \overline{b}/2(x + \mathbf{i}y) & \overline{c}/2(x + \mathbf{i}y) & (\alpha_5 x + \beta_5 y + z) \end{bmatrix}$$

$$= (\alpha_3 x + \beta_3 y + z)(\alpha_4 x + \beta_4 y + z)(\alpha_5 x + \beta_5 y + z)$$

$$- \frac{1}{4}(x^2 + y^2)[(\alpha_3 x + \beta_3 y + z)|c|^2 + (\alpha_4 x + \beta_4 y + z)|b|^2 + (\alpha_5 x + \beta_5 y + z)|a|^2]$$

$$+ \frac{1}{8}(x^2 + y^2)(x - \mathbf{i}y)a\overline{b}c + \frac{1}{8}(x^2 + y^2)(x + \mathbf{i}y)\overline{ac}b.$$

Combining this relation with (3) and $a\overline{b}c \neq 0$ we get successively (i) and (ii). Moreover the polynomial P_C (and therefore the matrix C) is irreducible. Thus, P_C cannot be factored into three linear factors, or into a quadratic factor and a linear one. We note that linear factors in the left-hand side of the equation $P_C(x,y,z) = 0$ are corresponding always to eigenvalues of the matrix C (see [9], [8] and [13]).

Assume that $P_C(x,y,z)$ has a linear factor $(\alpha_i x + \beta_i y + z)$ which corresponds to $\lambda_i, i = 3,4,5$. Also,

$$\begin{aligned} P_C(x,y,z) = {} & (\alpha_3 x + \beta_3 y + z)(\alpha_4 x + \beta_4 y + z)(\alpha_5 x + \beta_5 y + z) \\ & - (\alpha_3 x + \beta_3 y + z)(x^2 + y^2)\mu_4\mu_5 \\ & - (\alpha_4 x + \beta_4 y + z)(x^2 + y^2)\mu_3\mu_5 \\ & - (\alpha_5 x + \beta_5 y + z)(x^2 + y^2)\mu_3\mu_4 \\ & + (x - \mathbf{i}y)(x^2 + y^2)\mu_3\mu_4\mu_5 e^{\mathbf{i}\theta} \\ & + (x + \mathbf{i}y)(x^2 + y^2)\mu_3\mu_4\mu_5 e^{-\mathbf{i}\theta}, \end{aligned}$$

where $\mu_i = \mathrm{Re}(e^{-\mathbf{i}\theta}\lambda_i) + \mu$, we can see that for two by two equal index $i,j,k \in \{1,2,3\}$, $(\alpha_i x + \beta_i y + z)$ gives,

$$\begin{aligned} & (\alpha_j x + \beta_j y + z)(x^2 + y^2)\mu_i\mu_k \\ & + (\alpha_k x + \beta_k y + z)(x^2 + y^2)\mu_i\mu_j \\ & - (x - \mathbf{i}y)(x^2 + y^2)\mu_i\mu_j\mu_k e^{\mathbf{i}\theta} \\ & - (x + \mathbf{i}y)(x^2 + y^2)\mu_i\mu_j\mu_k e^{-\mathbf{i}\theta}. \end{aligned}$$

This means that the coefficients of $z, \alpha_i x, \beta_i y$ in the last polynomial are equals, which gives

$$\beta_i[\alpha_j\mu_i\mu_k + \alpha_k\mu_i\mu_j - 2\mu_i\mu_j\mu_k\cos(\theta)] = \alpha_i[\beta_j\mu_i\mu_k + \beta_k\mu_i\mu_j - 2\mu_i\mu_j\mu_k\sin(\theta)] \quad (4)$$

$$\alpha_j\mu_i\mu_k + \alpha_k\mu_i\mu_j - 2\mu_i\mu_j\mu_k\cos(\theta) = \alpha_i[\mu_i\mu_k + \mu_i\mu_j] \quad (5)$$

and

$$\beta_j\mu_i\mu_k + \beta_k\mu_i\mu_j - 2\mu_i\mu_j\mu_k\sin(\theta) = \beta_i[\mu_i\mu_k + \mu_i\mu_j]. \quad (6)$$

One can see that (4), (5) and (6) are equivalent to $\lambda_j\mu_k + \lambda_k\mu_j - 2\mu_j\mu_k e^{\mathbf{i}\theta} = \lambda_i(\mu_k + \mu_j)$ and hence (iii).

The converse is obvious. □

In what follows, set $\mu_j = \mathrm{Re}(e^{-\mathbf{i}\theta}\lambda_j) + \mu$ for every $j = 1, ..., 5$.

Theorem 4. *Let A be in upper-triangular form (1). Then $C_R(A)$ consists of one ellipse with foci λ_q,λ_p and minor axis of length r, and a curve of degree 4 with a double tangent and foci at λ_w , λ_t and λ_v if and only if there exist $\theta \in [0, 2\pi[$ and a real μ such that*

(a) $r^2 + 4(\mu_t\mu_v + \mu_w\mu_v + \mu_w\mu_t) = \sum_{S_{lm}} |a_{lm}|^2.$

(b)
$$\begin{aligned} & r^2(\lambda_w + \lambda_t + \lambda_v) + 4[(\lambda_w\mu_t\mu_v + \lambda_t\mu_w\mu_v + \lambda_v\mu_t\mu_w) - 2\mu_w\mu_t\mu_v e^{\mathbf{i}\theta} \\ & + (\lambda_p + \lambda_q)(\mu_t\mu_v + \mu_w\mu_v + \mu_w\mu_t)] \\ & = \sum_{S_{ijk}S_{lm}} |a_{lm}|^2(\lambda_i + \lambda_j + \lambda_k) - \sum_{S_{klm}} a_{kl}a_{lm}\overline{a_{km}}. \end{aligned}$$

(c)
$$\begin{aligned} & r^2(\lambda_w\lambda_t + \lambda_t\lambda_v + \lambda_w\lambda_v) + 4[\lambda_p\lambda_q(\mu_t\mu_v + \mu_w\mu_v + \mu_t\mu_w) \\ & + (\lambda_p + \lambda_q)(\lambda_w\mu_t\mu_v + \lambda_t\mu_w\mu_v + \lambda_v\mu_t\mu_w) - (\lambda_p + \lambda_q)(2\mu_w\mu_t\mu_v e^{\mathbf{i}\theta})] \\ & = \sum_{S_{ijk}S_{lm}} |a_{lm}|^2(\lambda_i\lambda_j + \lambda_i\lambda_k + \lambda_j\lambda_k) - \sum_{S_{ij}S_{klm}} (\lambda_i + \lambda_j)a_{kl}a_{lm}\overline{a_{km}} \\ & + \sum_{S_{jklm}} a_{jk}a_{kl}a_{lm}\overline{a_{jm}}. \end{aligned}$$

(d)
$$\begin{aligned} & r^2(\lambda_w\lambda_t\lambda_v) + 4[\lambda_p\lambda_q((\lambda_w\mu_t\mu_v + \lambda_t\mu_w\mu_v + \lambda_v\mu_t\mu_w) - 2\mu_w\mu_t\mu_v e^{\mathbf{i}\theta})] \\ & = \sum_{S_{ijk}S_{lm}} |a_{lm}|^2\lambda_i\lambda_j\lambda_k - \sum_{S_{ij}S_{klm}} \lambda_i\lambda_j a_{kl}a_{lm}\overline{a_{km}} + \sum_{S_iS_{jklm}} a_{jk}a_{kl}a_{lm}\overline{a_{jm}}\lambda_i \\ & - a_{12}a_{23}a_{34}a_{45}\overline{a_{15}}. \end{aligned}$$

(e)
$$\begin{aligned} & r^2[\alpha_w\alpha_t\alpha_v - (\alpha_w\mu_t\mu_v + \alpha_t\mu_w\mu_v + \alpha_v\mu_w\mu_t) + 2\mu_w\mu_t\mu_v\cos(\theta)] \\ & + 4\alpha_p\alpha_q[(\alpha_w\mu_t\mu_v + \alpha_t\mu_w\mu_v + \alpha_v\mu_w\mu_t) - 2\mu_w\mu_t\mu_v\cos(\theta)] \\ & = \sum_{S_{ijk}S_{lm}} |a_{lm}|^2\alpha_i\alpha_j\alpha_k - \sum_{S_{ij}S_{klm}} \mathrm{Re}(a_{kl}a_{lm}\overline{a_{km}})\alpha_i\alpha_j - \frac{1}{4}\sum_{i=1}^{5} P_i\alpha_i \\ & + \frac{1}{2}\sum_{S_iS_{jklm}} \mathrm{Re}(a_{jk}a_{kl}a_{lm}\overline{a_{jm}})\alpha_i \\ & + \frac{1}{4}\sum_{S_{ij}S_{klm}} \mathrm{Re}(a_{kl}a_{lm}\overline{a_{km}})|a_{ij}|^2 - \frac{1}{4}\sum_{S_{ijkl}S_{iml}} \mathrm{Re}(a_{ij}a_{jk}a_{kl}\overline{a_{im}}\overline{a_{ml}}) \\ & - \frac{1}{4}\sum_{S_{ijk}S_{lm}S_{im}S_{lk}} \mathrm{Re}(a_{ij}a_{jk}a_{lm}\overline{a_{im}}\overline{a_{lk}}) - \frac{1}{4}\mathrm{Re}(a_{12}a_{23}a_{34}a_{45}\overline{a_{15}}). \end{aligned}$$

(f)
$$\begin{aligned} & r^2[\beta_w\beta_t\beta_v - (\beta_w\mu_t\mu_v + \beta_t\mu_w\mu_v + \beta_v\mu_w\mu_t) + 2\mu_w\mu_t\mu_v\sin(\theta)] \\ & + 4\beta_p\beta_q[(\beta_w\mu_t\mu_v + \beta_t\mu_w\mu_v + \beta_v\mu_w\mu_t) - 2\mu_w\mu_t\mu_v\sin(\theta)] \\ & = \sum_{S_{ijk}S_{lm}} |a_{lm}|^2\beta_i\beta_j\beta_k - \sum_{S_{ij}S_{klm}} \mathrm{Im}(a_{kl}a_{lm}\overline{a_{km}})\beta_i\beta_j - \frac{1}{4}\sum_{i=1}^{5} P_i\beta_i \\ & - \frac{1}{2}\sum_{S_iS_{jklm}} \mathrm{Re}(a_{jk}a_{kl}a_{lm}\overline{a_{jm}})\beta_i \\ & + \frac{1}{4}\sum_{S_{ij}S_{klm}} \mathrm{Im}(a_{kl}a_{lm}\overline{a_{km}})|a_{ij}|^2 - \frac{1}{4}\sum_{S_{ijkl}S_{iml}} \mathrm{Im}(a_{ij}a_{jk}a_{kl}\overline{a_{im}}\overline{a_{ml}}) \\ & - \frac{1}{4}\sum_{S_{ijk}S_{lm}S_{im}S_{lk}} \mathrm{Im}(a_{ij}a_{jk}a_{lm}\overline{a_{im}}\overline{a_{lk}}) + \frac{1}{4}\mathrm{Im}(a_{12}a_{23}a_{34}a_{45}\overline{a_{15}}). \end{aligned}$$

(g) $r^2[(\alpha_t\alpha_v + \alpha_w\alpha_v + \alpha_w\alpha_t) - (\mu_t\mu_v + \mu_w\mu_v + \mu_w\mu_t)]$
$+ 4\alpha_p\alpha_q(\mu_t\mu_v + \mu_w\mu_v + \mu_w\mu_t)$
$+ 4(\alpha_p + \alpha_q)[(\alpha_w\mu_t\mu_v + \alpha_t\mu_w\mu_v + \alpha_v\mu_w\mu_t) - 2\mu_w\mu_t\mu_v\cos(\theta)]$
$= \sum_{S_{ijk}S_{lm}} |a_{lm}|^2(\alpha_i\alpha_j + \alpha_i\alpha_k + \alpha_j\alpha_k) - \sum_{S_{ij}S_{klm}} \mathrm{Re}(a_{kl}a_{lm}\overline{a_{km}})(\alpha_i + \alpha_j)$
$- \frac{1}{4}\sum_{i=1}^{5} P_i + \frac{1}{2}\sum_{S_{jklm}} \mathrm{Re}(a_{jk}a_{kl}a_{lm}\overline{a_{jm}})$.

(h) $\mu_w\mu_t\mu_v \neq 0$.

(i) $\lambda_j\mu_k + \lambda_k\mu_j - 2\mu_j\mu_k e^{\mathbf{i}\theta} \neq \lambda_i(\mu_k + \mu_j)$ *for every* $i, j, k \in \{w, t, v\} : i \neq j \neq k$,

where $P_i = \sum_{S_{jkm}S_{lm}} |a_{jk}|^2|a_{lm}|^2 - \sum_{S_{jkl}S_{jml}} \mathrm{Re}(a_{jk}a_{kl}\overline{a_{jm}a_{ml}}) - \sum_{S_{jk}S_{lkm}} \mathrm{Re}(a_{jk}a_{lm}\overline{a_{jm}a_{lk}})$. *for every* $i = 1, .., 5$.

Proof. Taking conditions from lemma 2, we have,

$$\begin{aligned}
P_A(x,y,z) = &\left[(\alpha_p x + \beta_p y + z)(\alpha_q x + \beta_q y + z) - \frac{r^2}{4}(x^2+y^2)\right] \\
&\times [(\alpha_w x + \beta_w y + z)(\alpha_t x + \beta_t y + z)(\alpha_v x + \beta_v y + z) \\
&- (\alpha_w x + \beta_w y + z)(x^2+y^2)\mu_t\mu_v \\
&- (\alpha_t x + \beta_t y + z)(x^2+y^2)\mu_w\mu_v \\
&- (\alpha_v x + \beta_v y + z)(x^2+y^2)\mu_w\mu_t \\
&+ (x - \mathbf{i}y)(x^2+y^2)\mu_w\mu_t\mu_v e^{\mathbf{i}\theta} \\
&+ (x + \mathbf{i}y)(x^2+y^2)\mu_w\mu_t\mu_v e^{-\mathbf{i}\theta}],
\end{aligned}$$

$\mu_w\mu_t\mu_v \neq 0$, and $\lambda_j\mu_k + \lambda_k\mu_j - 2\mu_j\mu_k e^{\mathbf{i}\theta} \neq \lambda_i(\mu_k + \mu_j)$ for every $i, j, k \in \{w, t, v\}$: $i \neq j \neq k$. Comparing the previous formula of $P_A(x, y, z)$ with polynomial (2). We obtain

$$\begin{aligned}
&Q(x,y,z) = r^2[(\alpha_w x + \beta_w y + z)(\alpha_t x + \beta_t y + z)(\alpha_v x + \beta_v y + z) \\
&- (x^2+y^2)\left((\alpha_w x + \beta_w y + z)\mu_t\mu_v + (\alpha_t x + \beta_t y + z)\mu_w\mu_v + (\alpha_v x + \beta_v y + z)\mu_w\mu_t\right) \\
&+ 2(x^2+y^2)(x\mu_w\mu_t\mu_v\cos(\theta) + y\mu_w\mu_t\mu_v\sin(\theta))] \\
&+ 4(\alpha_p x + \beta_p y + z)(\alpha_q x + \beta_q y + z) \\
&[((\alpha_w x + \beta_w y + z)\mu_t\mu_v + (\alpha_t x + \beta_t y + z)\mu_w\mu_v + (\alpha_v x + \beta_v y + z)\mu_w\mu_t) \\
&- 2(x\mu_w\mu_t\mu_v\cos(\theta) + y\mu_w\mu_t\mu_v\sin(\theta))].
\end{aligned}$$

Computing the coefficients of $x^3, y^3, z^3, x^2y, xy^2, x^2z, xz^2, y^2z, yz^2, xyz$. By identification, we get, respectively,

1. $r^2[\alpha_w\alpha_t\alpha_v - (\alpha_w\mu_t\mu_v + \alpha_t\mu_w\mu_v + \alpha_v\mu_w\mu_t) + 2\mu_w\mu_t\mu_v\cos(\theta)]$
$+ 4\alpha_p\alpha_q[(\alpha_w\mu_t\mu_v + \alpha_t\mu_w\mu_v + \alpha_v\mu_w\mu_t) - 2\mu_w\mu_t\mu_v\cos(\theta)]$
$= \sum_{S_{ijk}S_{lm}} |a_{lm}|^2\alpha_i\alpha_j\alpha_k - \sum_{S_{ij}S_{klm}} \mathrm{Re}(a_{kl}a_{lm}\overline{a_{km}})\alpha_i\alpha_j - \frac{1}{4}\sum_{i=1}^{5} P_i\alpha_i$

$$+\frac{1}{2}\sum_{S_iS_{jklm}}\mathrm{Re}(a_{jk}a_{kl}a_{lm}\overline{a_{jm}})\alpha_i$$
$$+\frac{1}{4}\sum_{S_{ij}S_{klm}}\mathrm{Re}(a_{kl}a_{lm}\overline{a_{km}})|a_{ij}|^2-\frac{1}{4}\sum_{S_{ijkl}S_{iml}}\mathrm{Re}(a_{ij}a_{jk}a_{kl}\overline{a_{im}}\overline{a_{ml}})$$
$$-\frac{1}{4}\sum_{S_{ijk}S_{lm}S_{im}S_{lk}}\mathrm{Re}(a_{ij}a_{jk}a_{lm}\overline{a_{im}}\overline{a_{lk}})-\frac{1}{4}\mathrm{Re}(a_{12}a_{23}a_{34}a_{45}\overline{a_{15}}).$$

2. $r^2[\beta_w\beta_t\beta_v-(\beta_w\mu_t\mu_v+\beta_t\mu_w\mu_v+\beta_v\mu_w\mu_t)+2\mu_w\mu_t\mu_v\sin(\theta)]$
$+4\beta_p\beta_q[(\beta_w\mu_t\mu_v+\beta_t\mu_w\mu_v+\beta_v\mu_w\mu_t)-2\mu_w\mu_t\mu_v\sin(\theta)]$
$$=\sum_{S_{ijk}S_{lm}}|a_{lm}|^2\beta_i\beta_j\beta_k-\sum_{S_{ij}S_{klm}}\mathrm{Im}(a_{kl}a_{lm}\overline{a_{km}})\beta_i\beta_j-\frac{1}{4}\sum_{i=1}^{5}P_i\beta_i$$
$$-\frac{1}{2}\sum_{S_iS_{jklm}}\mathrm{Re}(a_{jk}a_{kl}a_{lm}\overline{a_{jm}})\beta_i$$
$$+\frac{1}{4}\sum_{S_{ij}S_{klm}}\mathrm{Im}(a_{kl}a_{lm}\overline{a_{km}})|a_{ij}|^2-\frac{1}{4}\sum_{S_{ijkl}S_{iml}}\mathrm{Im}(a_{ij}a_{jk}a_{kl}\overline{a_{im}}\overline{a_{ml}})$$
$$-\frac{1}{4}\sum_{S_{ijk}S_{lm}S_{im}S_{lk}}\mathrm{Im}(a_{ij}a_{jk}a_{lm}\overline{a_{im}}\overline{a_{lk}})+\frac{1}{4}\mathrm{Im}(a_{12}a_{23}a_{34}a_{45}\overline{a_{15}}).$$

3. $r^2+4(\mu_t\mu_v+\mu_w\mu_v+\mu_w\mu_t)=\sum_{S_{lm}}|a_{lm}|^2.$

4. $r^2[(\beta_w\alpha_t\alpha_v+\beta_t\alpha_w\alpha_v+\beta_v\alpha_w\alpha_t)+2\mu_w\mu_t\mu_v\sin(\theta)$
$-(\beta_w\mu_t\mu_v+\beta_t\mu_w\mu_v+\beta_v\mu_w\mu_t)]+4\alpha_p\alpha_q[(\beta_w\mu_t\mu_v+\beta_t\mu_w\mu_v+\beta_v\mu_w\mu_t)-2\mu_w\mu_t\mu_v\sin(\theta)]$
$+4(\alpha_p\beta_q+\alpha_q\beta_p)[(\alpha_w\mu_t\mu_v+\alpha_t\mu_w\mu_v+\alpha_v\mu_w\mu_t)-2\mu_w\mu_t\mu_v\cos(\theta)]$
$$=\sum_{S_{ijk}S_{lm}}|a_{lm}|^2(\alpha_i\alpha_j\beta_k+\alpha_i\alpha_k\beta_j+\alpha_j\alpha_k\beta_i)-\sum_{S_{ij}S_{klm}}\mathrm{Im}(a_{kl}a_{lm}\overline{a_{km}})\alpha_i\alpha_j$$
$$-\sum_{S_{ij}S_{klm}}\mathrm{Re}(a_{kl}a_{lm}\overline{a_{km}})(\alpha_i\beta_j+\alpha_j\beta_i)+\frac{1}{2}\sum_{S_iS_{jklm}}\mathrm{Re}(a_{jk}a_{kl}a_{lm}\overline{a_{jm}})\beta_i$$
$$-\frac{1}{4}\sum_{i=1}^{5}P_i\beta_i+\sum_{S_iS_{jklm}}\mathrm{Im}(a_{jk}a_{kl}a_{lm}\overline{a_{jm}})\alpha_i+\frac{1}{4}\sum_{S_{ij}S_{klm}}\mathrm{Im}(a_{kl}a_{lm}\overline{a_{km}})|a_{ij}|^2$$
$$-\frac{1}{4}\sum_{S_{ijkl}S_{iml}}\mathrm{Im}(a_{ij}a_{jk}a_{kl}\overline{a_{im}}\overline{a_{ml}})-\frac{1}{4}\sum_{S_{ijk}S_{lm}S_{im}S_{lk}}\mathrm{Im}(a_{ij}a_{jk}a_{lm}\overline{a_{im}}\overline{a_{lk}})$$
$$-\frac{3}{4}\mathrm{Im}(a_{12}a_{23}a_{34}a_{45}\overline{a_{15}}).$$

5. $r^2[(\alpha_w\beta_t\beta_v+\alpha_t\beta_w\beta_v+\alpha_v\beta_w\beta_t)+2\mu_w\mu_t\mu_v\cos(\theta)-(\alpha_w\mu_t\mu_v+\alpha_t\mu_w\mu_v+\alpha_v\mu_w\mu_t)]$
$+4\beta_p\beta_q[(\alpha_w\mu_t\mu_v+\alpha_t\mu_w\mu_v+\alpha_v\mu_w\mu_t)-2\mu_w\mu_t\mu_v\cos(\theta)]$
$+4(\alpha_p\beta_q+\alpha_q\beta_p)[(\beta_w\mu_t\mu_v+\beta_t\mu_w\mu_v+\beta_v\mu_w\mu_t)-2\mu_w\mu_t\mu_v\sin(\theta)]$
$$=\sum_{S_{ijk}S_{lm}}|a_{lm}|^2(\alpha_i\beta_j\beta_k+\alpha_j\beta_i\beta_k+\alpha_k\beta_i\beta_j)-\sum_{S_{ij}S_{klm}}\mathrm{Re}(a_{kl}a_{lm}\overline{a_{km}})\beta_i\beta_j$$
$$-\sum_{S_{ij}S_{klm}}\mathrm{Im}(a_{kl}a_{lm}\overline{a_{km}})(\alpha_i\beta_j+\alpha_j\beta_i)-\frac{1}{4}\sum_{i=1}^{5}P_i\alpha_i$$
$$-\frac{1}{2}\sum_{S_iS_{jklm}}\mathrm{Re}(a_{jk}a_{kl}a_{lm}\overline{a_{jm}})\alpha_i$$

$$+\sum_{S_iS_{jklm}}\mathrm{Im}(a_{jk}a_{kl}a_{lm}\overline{a_{jm}})\beta_i+\frac{1}{4}\sum_{S_{ij}S_{klm}}\mathrm{Re}(a_{kl}a_{lm}\overline{a_{km}})|a_{ij}|^2$$
$$-\frac{1}{4}\sum_{S_{ijkl}S_{iml}}\mathrm{Re}(a_{ij}a_{jk}a_{kl}\overline{a_{im}a_{ml}})-\frac{1}{4}\sum_{S_{ijk}S_{lm}S_{im}S_{lk}}\mathrm{Re}(a_{ij}a_{jk}a_{lm}\overline{a_{im}a_{lk}})$$
$$+\frac{3}{4}\mathrm{Re}(a_{12}a_{23}a_{34}a_{45}\overline{a_{15}}).$$

6. $$r^2[(\alpha_t\alpha_v+\alpha_w\alpha_v+\alpha_w\alpha_t)-(\mu_t\mu_v+\mu_w\mu_v+\mu_w\mu_t)]+4\alpha_p\alpha_q(\mu_t\mu_v+\mu_w\mu_v+\mu_w\mu_t)$$
$$+4(\alpha_p+\alpha_q)[(\alpha_w\mu_t\mu_v+\alpha_t\mu_w\mu_v+\alpha_v\mu_w\mu_t)-2\mu_w\mu_t\mu_v\cos(\theta)]$$
$$=\sum_{S_{ijk}S_{lm}}|a_{lm}|^2(\alpha_i\alpha_j+\alpha_i\alpha_k+\alpha_j\alpha_k)-\sum_{S_{ij}S_{klm}}\mathrm{Re}(a_{kl}a_{lm}\overline{a_{km}})(\alpha_i+\alpha_j)$$
$$-\frac{1}{4}\sum_{i=1}^{5}P_i+\frac{1}{2}\sum_{S_{jklm}}\mathrm{Re}(a_{jk}a_{kl}a_{lm}\overline{a_{jm}}).$$

7. $$r^2(\alpha_w+\alpha_t+\alpha_v)+4(\alpha_w\mu_t\mu_v+\alpha_t\mu_w\mu_v+\alpha_v\mu_w\mu_t)-8\mu_w\mu_t\mu_v\cos(\theta)$$
$$+4(\alpha_p+\alpha_q)(\mu_t\mu_v+\mu_w\mu_v+\mu_w\mu_t)$$
$$=\sum_{S_{ijk}S_{lm}}|a_{lm}|^2(\alpha_i+\alpha_j+\alpha_k)-\sum_{S_{klm}}\mathrm{Re}(a_{kl}a_{lm}\overline{a_{km}}).$$

8. $$r^2[(\beta_t\beta_v+\beta_w\beta_v+\beta_w\beta_t)-(\mu_t\mu_v+\mu_w\mu_v+\mu_w\mu_t)]+4\beta_p\beta_q(\mu_t\mu_v+\mu_w\mu_v+\mu_w\mu_t)$$
$$+4(\beta_p+\beta_q)[(\beta_w\mu_t\mu_v+\beta_t\mu_w\mu_v+\beta_v\mu_w\mu_t)-2\mu_w\mu_t\mu_v\sin(\theta)]$$
$$=\sum_{S_{ijk}S_{lm}}|a_{lm}|^2(\beta_i\beta_j+\beta_i\beta_k+\beta_j\beta_k)-\sum_{S_{ij}S_{klm}}\mathrm{Im}(a_{kl}a_{lm}\overline{a_{km}})(\beta_i+\beta_j)$$
$$-\frac{1}{4}\sum_{i=1}^{5}P_i-\frac{1}{2}\sum_{S_{jklm}}\mathrm{Re}(a_{jk}a_{kl}a_{lm}\overline{a_{jm}}).$$

9. $$r^2(\beta_w+\beta_t+\beta_v)+4(\beta_w\mu_t\mu_v+\beta_t\mu_w\mu_v+\beta_v\mu_w\mu_t)-8\mu_w\mu_t\mu_v\sin(\theta)$$
$$+4(\beta_p+\beta_q)(\mu_t\mu_v+\mu_w\mu_v+\mu_w\mu_t)$$
$$=\sum_{S_{ijk}S_{lm}}|a_{lm}|^2(\beta_i+\beta_j+\beta_k)-\sum_{S_{klm}}\mathrm{Im}(a_{kl}a_{lm}\overline{a_{km}}).$$

10. $$r^2(\alpha_w\beta_t+\alpha_w\beta_v+\alpha_t\beta_w+\alpha_t\beta_v+\alpha_v\beta_w+\alpha_v\beta_t)+4(\alpha_p\beta_q+\alpha_q\beta_p)(\mu_t\mu_v+\mu_w\mu_v+\mu_w\mu_t)$$
$$+4(\alpha_p+\alpha_q)[(\beta_w\mu_t\mu_v+\beta_t\mu_w\mu_v+\beta_v\mu_w\mu_t)-2\mu_w\mu_t\mu_v\sin(\theta)]$$
$$+4(\beta_p+\beta_q)[(\alpha_w\mu_t\mu_v+\alpha_t\mu_w\mu_v+\alpha_v\mu_w\mu_t)-2\mu_w\mu_t\mu_v\cos(\theta)]$$
$$=\sum_{S_{ijk}S_{lm}}|a_{lm}|^2(\alpha_i\beta_j+\alpha_j\beta_i+\alpha_i\beta_k+\alpha_k\beta_i+\alpha_j\beta_k+\alpha_k\beta_j)$$
$$-\sum_{S_{ij}S_{klm}}\mathrm{Re}(a_{kl}a_{lm}\overline{a_{km}})(\beta_i+\beta_j)$$
$$-\sum_{S_{ij}S_{klm}}\mathrm{Im}(a_{kl}a_{lm}\overline{a_{km}})(\alpha_i+\alpha_j)+\sum_{S_{jklm}}\mathrm{Im}(a_{jk}a_{kl}a_{lm}\overline{a_{jm}}).$$

Note that the combination of (1), (2), (4) and (5) is equivalent to (d), (e) and (f), because $(1)-(5)-\mathbf{i}(2)+\mathbf{i}(4)$ yields (d).

(6), (8) and (10) is equivalent to (c) and (g), since $(6)-(8)+\mathbf{i}(10)$ yields (c).

(7) and (9) is equivalent to (b), it follows from $(7)+\mathbf{i}(9)$.

This completes the proof. □

3 On the circular numerical range of S_5 matrices

Recall that an n-by-n matrix A is said to be of class S_n if A is a contraction, the eigenvalues of A are all in the open unit disc $\mathbb{D}$ and $rank(I_n - A^*A) = 1$. Two unitary equivalent S_n-matrices have the following useful characterization.

Lemma 3. *[4, Theorem 4.1] Let A_1 and A_2 be two n-dimensional operators with A_2 in S_n. Then A_1 is unitary equivalent to A_2 if and only if A_1 is a contraction and $W(A_1) = W(A_2)$.*

Now, we establish the result of Theorem 2 in the special case of S_5-matrices.

Theorem 5. *Let A be a non-invertible S_5 matrix with $W(A) = \{z \in \mathbb{C} : |z - a| \le r\}$, $(r > 0)$. If kippenhahn curve $C_R(A)$ has one of the following shapes,*

(i) $C_R(A)$ consists of three points and an ellipse.

(ii) $C_R(A)$ consists of two ellipses and a point.

Then $a = 0$.

Proof. Without loss of generality, we assume that A is an upper triangular matrix. The assumption on the numerical range of A implies that the origin a is an eigenvalue with algebraic multiplicity at least 2. So, by [3, Corollary 1.3] A takes the following form

$$A = \begin{bmatrix} a & 1-a^2 & -a\sqrt{1-a^2} & 0 & 0 \\ 0 & a & \sqrt{1-a^2} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{1-|b|^2} & -\bar{b}\sqrt{1-|c|^2} \\ 0 & 0 & 0 & b & \sqrt{1-|b|^2}\sqrt{1-|c|^2} \\ 0 & 0 & 0 & 0 & c \end{bmatrix}.$$

Moreover, we can take a positive by a suitable rotation, thus $W(A)$ is symmetric with respect to the real axis, which means that $W(A) = W(A^*)$, (A^* is the adjoint matrix of A), as we mentioned below A is of class S_5 and therefore by Lemma 3 A and A^* are unitary equivalent, moreover one can see that the eigenvalues b and c of A must be real or complex conjugates. Let

$$B = A - aI_5 = \begin{bmatrix} 0 & 1-a^2 & -a\sqrt{1-a^2} & 0 & 0 \\ 0 & 0 & \sqrt{1-a^2} & 0 & 0 \\ 0 & 0 & -a & \sqrt{1-|b|^2} & -\bar{b}\sqrt{1-|c|^2} \\ 0 & 0 & 0 & b-a & \sqrt{1-|b|^2}\sqrt{1-|c|^2} \\ 0 & 0 & 0 & 0 & c-a \end{bmatrix}.$$

Consider the homogeneous Kippenhahn polynomial $P_B(x,y,z) = \det(x\mathrm{Re}B + y\mathrm{Im}B + zI_5)$ of degree 5 on the complex projective plane $\mathbb{CP}^2$. Since by hypotheses $W(B)$ is a circular disc with center 0 and radius r, then $C_R(B)$ has one of two possibles shapes,

(i) A circle with center 0 and radius r together with three points $-a, b-a, c-a$ inside it.

(ii) A circle with center 0 and radius r together with a point($-a, b-a$ or $c-a$) and an ellipse with (minor axis length $s \le 2r$) and the two remaining points as the foci, all inside the circle.

Applying condition (d) of Theorem 3 to the upper-triangular matrix B yields to

$$\begin{aligned}
&4r^2(-a(b-a)(c-a))\\
&\quad = (1-a^2)^2(-a)(b-a)(c-a) - (1-a^2)(-a)\sqrt{1-a^2}\sqrt{1-a^2}(b-a)(c-a)\\
&\quad = 0.
\end{aligned}$$

Then either $a=0$, $a=b$ or $a=c$. If it is the first case so it will done, otherwise if it is one of the two latter cases, the condition (c) of Theorem 3 gives,

$$\begin{aligned}
&4r^2(-a(b-a)+(b-a)(c-a)-a(c-a))\\
&\quad = (1-a^2)^2(-a(b-a)+(b-a)(c-a)-a(c-a))\\
&\quad + a^2(1-a^2)(b-a)(c-a)+(1-a^2)(b-a)(c-a)\\
&\quad - (1-a^2)^2(-a(b-a)-a(c-a))\\
&\quad = 0.
\end{aligned}$$

Thus $a=b$ if $a=c$ and vise versa. By the condition (b) of Theorem 3

$$\begin{aligned}
&4r^2(-a+(b-a)+(c-a)) + s^2(0+0+\lambda) = (1-a^2)^2(-a+b-a+c-a)\\
&\quad + a^2(1-a^2)(b-a+c-a)+(1-a^2)(b-a+c-a)\\
&\quad + (1-b^2)(c-a)+b^2(1-c^2)(b-a)\\
&\quad - a(1-b^2)(1-c^2)+a(1-a^2)^2+b(1-b^2)(1-c^2).
\end{aligned}$$

where λ takes one of the eigenvalue $-a, b-a$ or $c-a$. Assume that $a=b=c$, we get $(4r^2+s^2)(-a)=0$ or $ar^2=0$. Hence $a=0$. This complete the proof. □

Remark 2. *It is well known that for every S_n matrix A,* Re(A) *have only simple eigenvalues see [4, Corollary 2.7], then $C_R(A)$ not contains a curve of degree* 4 *with double tangent.*

4 Proof of Theorem 2

It is well known that a n-by-n partial isometry A can be represented on $Ker(A)\oplus Ker(A)^{\perp}$, by

$$A = \begin{bmatrix} 0 & B \\ 0 & C \end{bmatrix}$$

with B and C satisfying $B^*B + C^*C = I_{Ker(A)^\perp}$, where $I_{Ker(A)^\perp}$ is the identity matrix on $Ker(A)^\perp$, see [5, Proposition 2.1]. Also, the irreducibility of a partial isometry can be characterized by,

Lemma 4. *[5, lemma 2.8] Let*

$$A = \begin{bmatrix} 0_m & B \\ 0 & C \end{bmatrix} \quad on \quad \mathbb{C}^n = \mathbb{C}^m \oplus \mathbb{C}^{n-m}, \quad (1 \le m \le n).$$

(a) If $k = rankB < m$, then A is unitarily similar to $0_{m-k} \oplus A_1$ for some matrix

$$A_1 = \begin{bmatrix} 0_k & B_1 \\ 0 & C_1 \end{bmatrix} \quad on \quad \mathbb{C}^{n-m+k} = \mathbb{C}^k \oplus \mathbb{C}^{n-m}, \quad with \quad rankB_1 = k.$$

(b) If $m > [n/2]$, the largest integer less than or equal to $n/2$, then A is unitarily similar to $0_{2m-n} \oplus A_2$ for some matrix

$$A_2 = \begin{bmatrix} 0_{n-m} & B_2 \\ 0 & C_2 \end{bmatrix} \quad on \quad \mathbb{C}^{2(n-m)} = \mathbb{C}^{n-m} \oplus \mathbb{C}^{n-m}.$$

The next proposition relates partial isometries with S_n-matrices.

Proposition 1. *[5, Proposition 2.3] Let A be an n-by-n matrix. Then A is an irreducible partial isometry with $dimkerA = 1$ if and only if A is of class S_n with 0 in $\sigma(A)$.*

Now, we are ready to establish our main theorem.

Proof of Theorem 2. Let A be an 5×5 partial isometry with

$$W(A) = \{z \in \mathbb{C} : |z - a| \le r\}, \quad (r > 0).$$

First let us remark that if A is reducible, then A is unitarily similar to $A_1 \oplus A_2$, where A_1 and A_2 are two partial isometries with order at most 4. Since one of $W(A_1)$ or $W(A_2)$ must be equal to that of A , so by Theorem 1 it follows that $a = 0$.

Now, we assume that A is irreducible. According to the dimension of the kernel of A, we distinguish three cases.

Case 1. $\dim kerA = 1$. By Proposition 1, A is non-invrtible S_5-matrix, so according to Theorem 5 and Remark 2, $a = 0$.

Case 2. $\dim kerA = 2$. Since $W(A)$ is a circular disc centered at a, we may assume that

$$A = \begin{bmatrix} 0 & B \\ 0 & C \end{bmatrix} = \begin{bmatrix} 0 & 0 & k & l & t \\ 0 & 0 & g & h & j \\ 0 & 0 & b & e & f \\ 0 & 0 & 0 & a & d \\ 0 & 0 & 0 & 0 & a \end{bmatrix} \quad \text{on } \mathbb{C}^2 \oplus \mathbb{C}^3,$$

with

$$I_3 = B^*B + C^*C$$
$$= \begin{bmatrix} |k|^2+|g|^2+|b|^2 & \overline{k}l+\overline{g}h+\overline{b}e & \overline{k}t+\overline{g}j+\overline{b}f \\ \overline{l}k+\overline{h}g+\overline{e}b & |l|^2+|h|^2+|e|^2+|a|^2 & \overline{l}t+\overline{h}j+\overline{e}f+\overline{a}d \\ \overline{t}k+\overline{j}g+\overline{f}b & \overline{t}l+\overline{j}h+\overline{f}e+\overline{d}a & |t|^2+|j|^2+|f|^2+|d|^2+|a|^2 \end{bmatrix}.$$

As in the proof of Theorem5, a is positive and $C_R(A)$ has one of the three possible shapes.

(i) $C_R(A)$ contains a circle (with center a, radius r) and three points $0, 0, b$.

(ii) $C_R(A)$ is a circle (with center a, radius r) , together with an ellipse and a point.

(iii) $C_R(A)$ contains a circle (with center a, radius r) and a curve of degree 4 with a double tangent.

Applying condition (d) of Theorem 3 and Theorem 4 to $A - aI_5$ we get

$$\begin{aligned} 4r^2(a.a.(b-a)) &= |d|^2a^2(b-a) - a^2ed\overline{f} + a(b-a)hd\overline{j} + a(b-a)ld\overline{t} \\ &\quad - aged\overline{j} - aked\overline{t} \\ &= |d|^2a^2(b-a) - a^2ed\overline{f} + ad(b-a)(h\overline{j}+l\overline{t}) - aed(g\overline{j}+k\overline{t}) \\ &= |d|^2a^2(b-a) - a^2ed\overline{f} - ad(b-a)(e\overline{f}+a\overline{d}) + aedb\overline{f} \\ &= 0. \end{aligned}$$

Thus $a = 0$ or $b = a$. If $b = a$, by condition (c) of Theorem 3 and Theorem 4

$$\begin{aligned} 4r^2(a^2) &= a^2(|e|^2+|f|^2+|d|^2) + 2aed\overline{f} + aeg\overline{h} + aek\overline{l} + afg\overline{j} + afk\overline{t} \\ &\quad + adh\overline{j} + adl\overline{t} + edg\overline{j} + edk\overline{t}. \\ &= a^2(|e|^2+|f|^2+|d|^2) + 2aed\overline{f} + ae(g\overline{h}+k\overline{l}) + af(g\overline{j}+k\overline{t}) \\ &\quad + ad(h\overline{j}+l\overline{t}) + ed(g\overline{j}+k\overline{t}) \\ &= 0 \end{aligned}$$

and therefore $a = 0$.

Case 3. $\dim kerA > 2$, then it follows from Lemma 4 that A is reducible, then $a = 0$.

This completes the proof of the theorem. □

Remark 3. *In order to give a complete answer to the conjecture of Gau et al, in dimension 5, it remains to study the case when $C_R(A)$ is an ellipse and a curve of order 6, consisting of an oval and a curve of three cups. Based on the factoribility of P_A, Kippenhahn in [8] gave a fully classification of the numerical range of 3×3 matrices, also a pertinent tests were offered in [13]. However, there is no much results about the connection between concrete description of the curve $C_R(A)$ and P_A when $W(A)$ is an oval. Thus, this case is still an open question.*

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