ON THE CIRCULAR NUMERICAL RANGE OF 5-BY-5 PARTIAL ISOMETRIES

Mohammed BENHARRAT*1 and Mehdi NAIMI2

Abstract

We prove, in some cases in term of kippenhahn curve, that if 5-by-5 partial isometry whose numerical range is a circular disc then its center is must be the origin. This gives a partial affirmative answer of the Conjecture 5.1. of [H. I. Gau et al., Linear and Multilinear Algebra, 64 (1) 2016, 14–35.], for the five dimensional case.

2000 Mathematics Subject Classification: 47A12; 47A56.

Key words: numerical range, partial isometric, $S_n$ operators.

1 Introduction

Let $A$ be an $n \times n$ complex matrix, its numerical range $W(A)$ is, by definition, the set of complex numbers

$$W(A) = \{ \langle Ax, x \rangle : \ x \in \mathbb{C}^n, \ ||x|| = 1 \}.$$

It is well known that $W(A)$ is a nonempty compact convex subset of $\mathbb{C}$, also contains all the eigenvalues of $A$ and therefore its convex hull, see for instance [6]. The matrix $A$ is said to be a partial isometry if it is isometric on the orthogonal complement of the kernel of $A$, $Ker(A)$. Assume that $A$ is a partial isometry whose numerical range $W(A)$ is a circular disc. The question is whether the center of $W(A)$ must be the origin. Gau et al. [5], gave an affirmative answer if the dimension is at most 4, as follows,

---

1** Corresponding author, Ecole Nationale Polytechnique d’Oran-Maurice Audin (Ex. ENSET d’Oran), BP 1523 Oran-El M’naouar, 31000 Oran, Algérie, e-mail: mohammed.benharrat@enp-oran.dz

2Ecole Nationale Polytechnique d’Oran-Maurice Audin (Ex. ENSET d’Oran), BP 1523 Oran-El M’naouar, 31000 Oran, Algérie, e-mail: mehdi.naimi@univ-mosta.dz

Authors gratefully acknowledge the financial support from the Laboratory of Fundamental and Applicable Mathematics of Oran (LMFAO) and the Algerian research project: PRFU, no. C00L03ES310120220003 (D.G.R.S.D.T).
Theorem 1. [5, Theorem 2.1] If $A$ is an $n$-by-$n$ $(n \leq 4)$ partial isometry with $W(A) = \{z \in \mathbb{C} : |z-a| \leq r\}$, $(r > 0)$, then $a = 0$.

Also have conjectured that theorem remains valid if $A$ is an $n$-by-$n$ partial isometry, see [5, Conjecture 5.1]. By the same procedure of [5], we give an affirmative answer for this conjecture for 5-by-5 partial isometry in some cases in terms of kippenhahn curve, to be more specific, our main theorem reads as follows,

Theorem 2. Let $A$ be a $5 \times 5$ partial isometry matrix $W(A) = \{z \in \mathbb{C} : |z-a| \leq r\}$, $(r > 0)$. If the kippenhahn curve $C_R(A)$ has one of the following shapes,

(i) $C_R(A)$ consists of three point and an ellipse.

(ii) $C_R(A)$ consists of two ellipses and a point.

(iii) $C_R(A)$ consists of a curve of degree 4 with double tangent and an ellipse.

Then $a = 0$.

Two successful approaches to establishing this result are a canonical decomposition of $n \times n$ partial isometry matrix and the Kippenhahn’s result for the numerical range of $n \times n$ matrix.

It well known that the numerical range of an $n \times n$ matrix $A$ is completely determined by its Kippenhahn polynomial $P_A(x, y, z) = \det(x\text{Re}(A) + y\text{Im}(A) + zI_n)$, where $\text{Re}(A) = (A + A^*)/2$ and $\text{Im}(A) = (A - A^*)/2i$ are the real and the imaginary part of $A$, respectively. $I_n$ denotes the $n \times n$ identity matrix and $i$ is the complex number $i^2 = -1$. Let $C(A)$ be the dual of the algebraic curve defined to be the zero set of $P_A(x, y, z) = 0$, on the complex projective plane $\mathbb{CP}^2$, which consists of all equivalence classes of points in $\mathbb{C}^3 \setminus \{(0, 0, 0)\}$ under the equivalence relation $\sim$, this relation is defined by $(x, y, z) \sim (x', y', z')$ if and only if there is a nonzero $\lambda \in \mathbb{C}$ such that $(x, y, z) = \lambda(x', y', z')$. Kippenhahn showed that $W(A)$ is the convex hull of the real points of $C(A)$, see [8] and its English translation [9] for a detailed discussion of the connections between the polynomial $P_A$, and the numerical range of $A$. This characterization is used by many authors to answer the question when the numerical range of a matrix is an elliptic disc. For $2 \times 2$ matrices a complete description of the numerical range is well known, that is $W(A)$ elliptic disk (with possibly degenerate interior), see [6]. In [8] Kippenhahn showed that there are four classes of shapes which the numerical range of matrices of order three. This was improved in [7] by expressing the conditions in terms of the eigenvalues and entries of $A$, which are easier to apply. By the same procedure, these results are generalized for $4 \times 4$ matrices. Let us mention here, that numerous results are known in this direction only for some special classes of matrices, for partial isometry, nilpotent, doubly stochastic matrices (etc...), see [11], [10]. But no unifying and general theory is not yet available.

In this paper, firstly, with a similar approach used in [2], we will give necessary and sufficient conditions for 5-by-5 matrix $A$ to have an ellipse in the associated
real Kippenhahn curve $C_R(A)$. We also express those conditions in terms of eigenvalues and entries of $A$, the main difficult is the heavy computations. All these conditions will be useful for construct a $5 \times 5$ matrix with an elliptic numerical range (Section 2.). Secondly, we establish the result of Theorem 2 in the special case of $S_5$-matrices ($A$ is $S_n$-matrix if $A$ is a contraction, the eigenvalues of $A$ are all in the open unit disc $D$ and $\text{rank}(I_n - A^*A) = 1$) (Section 3.). Finally, using the results of two preceding sections, we give the proof of Theorem 2 (Section 4.).

## 2 Necessary and sufficient conditions for $5 \times 5$ matrix to have an ellipse in its real Kippenhahn curve

Let $A$ be a $5 \times 5$ complex matrix. Here we give necessary and sufficient conditions for which the associated curve $C_R(A)$ contains an ellipse or a circle. It is well known that, by Schur’s theorem, every square matrix is unitarily equivalent to an upper triangular matrix. So, without loss of generality, we can assume that

$$A = \begin{bmatrix} \lambda_1 & a_{12} & a_{13} & a_{14} & a_{15} \\ 0 & \lambda_2 & a_{23} & a_{24} & a_{25} \\ 0 & 0 & \lambda_3 & a_{34} & a_{35} \\ 0 & 0 & 0 & \lambda_4 & a_{45} \\ 0 & 0 & 0 & 0 & \lambda_5 \end{bmatrix},$$

(1)

where $\lambda_j = \alpha_j + i\beta_j$, with $\alpha_j$ and $\beta_j$ are real for $j = 1, 2, 3, 4, 5$.

Then, we have

$$P_A(x, y, z) = \text{det}(x\text{Re}(A) + y\text{Im}(A) + zI_5) = \begin{vmatrix} \psi_1(x, y) & \frac{a_{12}}{2}(x - iy) & \frac{a_{13}}{2}(x - iy) & \frac{a_{14}}{2}(x - iy) & \frac{a_{15}}{2}(x - iy) \\ (x + iy)\frac{a_{12}}{2} & \psi_2(x, y) & \frac{a_{23}}{2}(x - iy) & \frac{a_{24}}{2}(x - iy) & \frac{a_{25}}{2}(x - iy) \\ (x + iy)\frac{a_{13}}{2} & (x + iy)\frac{a_{23}}{2} & \psi_3(x, y) & \frac{a_{34}}{2}(x - iy) & \frac{a_{35}}{2}(x - iy) \\ (x + iy)\frac{a_{14}}{2} & (x + iy)\frac{a_{24}}{2} & (x + iy)\frac{a_{34}}{2} & \psi_4(x, y) & \frac{a_{45}}{2}(x - iy) \\ (x + iy)\frac{a_{15}}{2} & (x + iy)\frac{a_{25}}{2} & (x + iy)\frac{a_{35}}{2} & (x + iy)\frac{a_{45}}{2} & \psi_5(x, y) \end{vmatrix},$$

(2)

where $\psi_j(x, y) = \alpha_j x + \beta_j y + z$, for $j = 1, 2, 3, 4, 5$. By straightforward calculus, we obtain

$$P_A(x, y, z) = \prod_{i=1}^{5} (\alpha_i x + \beta_i y + z) - \frac{x^2 + y^2}{4}Q(x, y, z),$$

(2)
Lemma 1. Let $A$ be a $5 \times 5$ matrix. Then the Kippenhahn curve $C_R(A)$ consists of two ellipses, one with foci $\lambda_1, \lambda_2$ and minor axis of length $r$, the other with foci $\lambda_3, \lambda_4$ and minor axis of length $s$, and $\lambda_5$ if and only if

$$P_A(x, y, z) = [(\alpha_1 x + \beta_1 y + z)(\alpha_2 x + \beta_2 y + z) - \frac{x^2}{4}(x^2 + y^2)][(\alpha_3 x + \beta_3 y + z)(\alpha_4 x + \beta_4 y + z) - \frac{s^2}{4}(x^2 + y^2)](\alpha_5 x + \beta_5 y + z),$$

where $\lambda_j = \alpha_j + i\beta_j, j = 1, 2, 3, 4, 5$ and the $\alpha_j's$ and $\beta_j's$ are real.

Proof. Let $B = \begin{bmatrix} \lambda_1 & r \\ 0 & \lambda_2 \end{bmatrix} \oplus \begin{bmatrix} \lambda_3 & s \\ 0 & \lambda_4 \end{bmatrix} \oplus \lambda_5$. As $C_R(A) = C_R(B)$, by duality the polynomials $P_A$ and $P_B$ are the same, therefore

$$P_A(x, y, z) = [(\alpha_1 x + \beta_1 y + z)(\alpha_2 x + \beta_2 y + z) - \frac{x^2}{4}(x^2 + y^2)][(\alpha_3 x + \beta_3 y + z)(\alpha_4 x + \beta_4 y + z) - \frac{s^2}{4}(x^2 + y^2)](\alpha_5 x + \beta_5 y + z).$$

The converse is clear. 

Using the above lemma, we can prove the following theorem.
Theorem 3. Let $A$ be in upper-triangular form (1). Then the Kippenhahn curve $C_R(A)$ consists of two ellipses, one with foci $\lambda_p$, $\lambda_q$ and minor axes of length $r$, the other with foci $\lambda_t$, $\lambda_v$ and minor axes of length $s$, and a point $\lambda_w$ if and only if

(a) $r^2 + s^2 = \sum_{S_{lm}} |a_{lm}|^2$.

(b) $r^2(\lambda_w + \lambda_t + \lambda_v) + s^2(\lambda_w + \lambda_q + \lambda_p) = \sum_{S_{ijklm}} |a_{klm}|^2(\lambda_i + \lambda_j + \lambda_k) - \sum_{S_{klm}} a_{klm}a_{lm}a_{km}$.

(c) $r^2(\lambda_w\lambda_t + \lambda_w\lambda_v + \lambda_t\lambda_v) + s^2(\lambda_w\lambda_p + \lambda_w\lambda_q + \lambda_p\lambda_q)$

$= \sum_{S_{ijklm}} |a_{klm}|^2(\lambda_i\lambda_j + \lambda_i\lambda_k + \lambda_j\lambda_k) - \sum_{S_{ijklm}} (\lambda_i + \lambda_j)a_{klm}a_{lm}a_{km} + \sum_{S_{ijklm}} a_{ijklm}a_{ijkl}$.

(d) $r^2\lambda_w\lambda_t\lambda_v + s^2\lambda_w\lambda_p\lambda_q = \sum_{S_{ijklm}} |a_{lm}|^2\lambda_i\lambda_j\lambda_k - \sum_{S_{ijklm}} \lambda_i\lambda_ja_{klm}a_{km} + \sum_{S_{ijklm}} a_{ijklm}a_{ijkl}a_{ik} - a_{12a_{23a_{34a_{45}}}a_{15}}$.

(e) $r^2\alpha_w\alpha_t\alpha_v + s^2\alpha_w\alpha_p\alpha_q - \frac{r^2s^2}{4}\alpha_w = \sum_{S_{ijklm}} |a_{lm}|^2\alpha_i\alpha_j\alpha_k - \sum_{S_{ijklm}} \frac{1}{4} Re(a_{ijklm}a_{km})\alpha_i + \frac{1}{4} \sum_{S_{ijklm}} \frac{1}{2} Re(a_{ijklm}a_{km}a_{jm})\alpha_i$.

(f) $r^2\beta_w\beta_t\beta_v + s^2\beta_w\beta_p\beta_q - \frac{r^2s^2}{4}\beta_w = \sum_{S_{ijklm}} |a_{lm}|^2\beta_i\beta_j\beta_k - \sum_{S_{ijklm}} \frac{1}{4} Im(a_{ijklm}a_{km})\beta_i\beta_j - \sum_{S_{ijklm}} \frac{1}{4} Im(a_{ijklm}a_{km}a_{ml})\beta_i + \sum_{S_{ijklm}} \frac{1}{4} Im(a_{ijklm}a_{km}a_{ik}) + \frac{1}{4} Im(a_{12a_{23a_{34a_{45}}}a_{15}})$.

(g) $r^2(\alpha_w\alpha_t + \alpha_w\alpha_v + \alpha_w\alpha_q) + s^2(\alpha_w\alpha_p + \alpha_w\alpha_q + \alpha_p\alpha_q) - \frac{r^2s^2}{4}$

$= \sum_{S_{ijklm}} |a_{lm}|^2(\alpha_i\alpha_j + \alpha_i\alpha_k + \alpha_j\alpha_k) - \sum_{S_{ijklm}} Re(a_{ijklm}a_{km})(\alpha_i + \alpha_j) - \frac{1}{4} \sum_{i=1}^5 P_i + \frac{1}{2} \sum_{S_{ijklm}} Re(a_{ijklm}a_{km})a_{jm}$.
where \( P_i = \sum_{s_{jkm}s_{lm}} |a_{jk}|^2|a_{lm}|^2 - \sum_{s_{jkl}s_{jml}} \text{Re}(a_{jk}a_{kl}\overline{a}_{jm}\overline{a}_{ml}) - \sum_{s_{jkl}s_{jkm}} \text{Re}(a_{jk}a_{lm}\overline{a}_{jl}\overline{a}_{lk}) \), for every \( i = 1, \ldots, 5 \).

**Proof.** By lemma 1, we have,

\[
P_A(x, y, z) = (\alpha_w x + \beta_w y + z)(\alpha_p x + \beta_p y + z)(\alpha_q x + \beta_q y + z) \\
(\alpha_x + \beta_t y + z)(\alpha_v x + \beta_v y + z) \\
- \frac{x^2 + y^2}{4} \text{Re}(\alpha_t x + \beta_t y + z)(\alpha_v x + \beta_v y + z) \\
+ s^2(\alpha_w x + \beta_w y + z)(\alpha_v x + \beta_v y + z)(\alpha_q x + \beta_q y + z) \\
- \frac{x^2 + y^2}{4} \text{Re}(\alpha_t x + \beta_t y + z)/.
\]

Comparing the previous formula of \( P_A(x, y, z) \) with (2), we obtain,

\[
Q(x, y, z) = r^2(\alpha_w x + \beta_w y + z)(\alpha_v x + \beta_v y + z) \\
+ s^2(\alpha_w x + \beta_w y + z)(\alpha_v x + \beta_v y + z)(\alpha_q x + \beta_q y + z) \\
- \frac{x^2 + y^2}{4} \text{Re}(\alpha_t x + \beta_t y + z).
\]

Computing the coefficients of \( x^3, y^3, z^3, x^2y, xy^2, x^2z, xz^2, y^2z, yz^2, xyz \) by identification, we find, respectively

1. \( r^2\alpha_w\alpha_t\alpha_v + s^2\alpha_w\alpha_p\alpha_q - \frac{r^2s^2}{4}\alpha_w = \sum_{s_{ijkl}} |a_{lm}|^2\alpha_i\alpha_j\alpha_k \\
- \sum_{s_{ijkl}} \text{Re}(a_{kl}a_{lm}\overline{a}_{km})\alpha_i\alpha_j - \frac{1}{4} \sum_{i=1}^5 P_i\alpha_i + \frac{1}{2} \sum_{s_{ijkl}} \text{Re}(a_{jk}a_{kl}\overline{a}_{jm}\overline{a}_{ml})\alpha_i \\
+ \frac{1}{4} \sum_{s_{ijkl}} \text{Re}(a_{ij}a_{kl}\overline{a}_{jm}\overline{a}_{ml}) - \frac{1}{4} \sum_{s_{ijkl}} \text{Im}(a_{ij}a_{kl}\overline{a}_{jm}\overline{a}_{ml}) \\
- \frac{1}{4} \sum_{s_{ijkl}} \text{Im}(a_{ij}a_{kl}\overline{a}_{jm}\overline{a}_{ml}) + \frac{1}{4} \text{Re}(a_{123434455\overline{15}}).
\]

2. \( r^2\beta_w\beta_t\beta_v + s^2\beta_w\beta_p\beta_q - \frac{r^2s^2}{4}\beta_w = \sum_{s_{ijkl}} |a_{lm}|^2\beta_i\beta_j\beta_k \\
- \sum_{s_{ijkl}} \text{Im}(a_{kl}a_{lm}\overline{a}_{km})\beta_i\beta_j - \frac{1}{4} \sum_{i=1}^5 P_i\beta_i - \frac{1}{2} \sum_{s_{ijkl}} \text{Re}(a_{jk}a_{kl}\overline{a}_{jm}\overline{a}_{ml})\beta_i \\
+ \frac{1}{4} \sum_{s_{ijkl}} \text{Im}(a_{kl}a_{lm}\overline{a}_{km})\beta_i - \frac{1}{4} \sum_{s_{ijkl}} \text{Im}(a_{ij}a_{kl}\overline{a}_{jm}\overline{a}_{ml}) \\
- \frac{1}{4} \sum_{s_{ijkl}} \text{Im}(a_{ij}a_{kl}\overline{a}_{jm}\overline{a}_{ml}) + \frac{1}{4} \text{Im}(a_{123434455\overline{15}}).
\]

3. \( r^2 + s^2 = \sum_{s_{lm}} |a_{lm}|^2. \)
4. \( r^2(\beta_w \alpha_t \alpha_v + \beta_t \alpha_w \alpha_v + \alpha_v \beta_w \beta_t) + s^2(\beta_w \alpha_p \alpha_q + \beta_p \alpha_w \alpha_q + \beta_q \alpha_w \alpha_p) - \frac{r^2 s^2}{4} \beta_w \)
\[
= \sum_{s \overline{s}} |a_{lm}|^2 (\alpha_i \beta_j + \alpha_j \beta_i) - \sum_{s \overline{s}} \operatorname{Im}(a_{kl} a_{lm} \overline{c_{km}}) \alpha_i \alpha_j \\
- \sum_{s \overline{s}} \operatorname{Re}(a_{kl} a_{lm} \overline{c_{km}})(\alpha_i \beta_j + \alpha_j \beta_i) + \frac{1}{2} \sum_{s \overline{s}} \operatorname{Re}(a_{jk} a_{kl} a_{lm} \overline{c_{jm}}) \beta_i \\
+ \sum_{s \overline{s}} \operatorname{Im}(a_{jk} a_{kl} a_{lm} \overline{c_{jm}}) \alpha_i + \frac{1}{4} \sum_{s \overline{s}} \operatorname{Im}(a_{kl} a_{lm} \overline{c_{km}})|a_{ij}|^2 - \frac{1}{4} \sum_{i=1}^5 P_i \beta_i \\
- \frac{1}{4} \sum_{s \overline{s}} \operatorname{Im}(a_{ij} a_{jk} a_{kl} a_{lm} \overline{c_{ml}}) - \frac{1}{4} \sum_{s \overline{s}} \operatorname{Im}(a_{ij} a_{jk} a_{kl} a_{lm} \overline{c_{ik}}) \\
- \frac{3}{4} \operatorname{Im}(a_{12} a_{23} a_{34} a_{45} \overline{a_{15}}).
\]

5. \( r^2(\alpha_w \beta_v + \alpha_t \beta_w \beta_v + \alpha_v \beta_w \beta_t) + s^2(\alpha_w \beta_p \beta_q + \alpha_p \beta_w \beta_q + \alpha_q \beta_w \beta_p) - \frac{r^2 s^2}{4} \alpha_w \)
\[
= \sum_{s \overline{s}} |a_{lm}|^2 (\alpha_i \beta_j + \alpha_j \beta_i) - \sum_{s \overline{s}} \operatorname{Re}(a_{kl} a_{lm} \overline{c_{km}})(\alpha_i \beta_j) \\
- \sum_{s \overline{s}} \operatorname{Im}(a_{kl} a_{lm} \overline{c_{km}})(\alpha_i \beta_j) + \frac{1}{4} \sum_{i=1}^5 P_i \alpha_i \\
- \frac{1}{2} \sum_{s \overline{s}} \operatorname{Re}(a_{jk} a_{kl} a_{lm} \overline{c_{jm}}) \alpha_i + \sum_{s \overline{s}} \operatorname{Im}(a_{jk} a_{kl} a_{lm} \overline{c_{jm}}) \beta_i \\
+ \frac{1}{4} \sum_{s \overline{s}} \operatorname{Re}(a_{kl} a_{lm} \overline{c_{km}})|a_{ij}|^2 - \frac{1}{4} \sum_{s \overline{s}} \operatorname{Re}(a_{ij} a_{jk} a_{kl} a_{lm} \overline{c_{ml}}) \\
- \frac{1}{4} \sum_{s \overline{s}} \operatorname{Re}(a_{ij} a_{jk} a_{lm} \overline{c_{im}} a_{ik}) + \frac{3}{4} \operatorname{Re}(a_{12} a_{23} a_{34} a_{45} \overline{a_{15}}).
\]

6. \( r^2(\alpha_w \alpha_v + \alpha_w \alpha_v + \alpha_v \alpha_t) + s^2(\alpha_w \alpha_p + \alpha_w \alpha_q + \alpha_p \alpha_q) - \frac{r^2 s^2}{4} \)
\[
= \sum_{s \overline{s}} |a_{lm}|^2 (\alpha_i \alpha_j + \alpha_j \alpha_k + \alpha_j \alpha_k) - \sum_{s \overline{s}} \operatorname{Re}(a_{kl} a_{lm} \overline{c_{km}})(\alpha_i + \alpha_j) \\
- \frac{1}{4} \sum_{i=1}^5 P_i + \frac{1}{2} \sum_{s \overline{s}} \operatorname{Re}(a_{jk} a_{kl} a_{lm} \overline{c_{jm}}).
\]

7. \( r^2(\alpha_w + \alpha_t + \alpha_v) + s^2(\alpha_w + \alpha_p + \alpha_q) \)
\[
= \sum_{s \overline{s}} |a_{lm}|^2 (\alpha_i + \alpha_j + \alpha_k) - \sum_{s \overline{s}} \operatorname{Re}(a_{kl} a_{lm} \overline{c_{km}}).
\]

8. \( r^2(\beta_w \beta_t + \beta_w \beta_v + \beta_t \beta_v) + s^2(\beta_w \beta_p + \beta_p \beta_v) - \frac{r^2 s^2}{4} \beta_w \)
\[
= \sum_{s \overline{s}} |a_{lm}|^2 (\beta_i \beta_j + \beta_j \beta_k + \beta_j \beta_k) - \sum_{s \overline{s}} \operatorname{Im}(a_{kl} a_{lm} \overline{c_{km}})(\beta_i + \beta_j) \\
- \frac{1}{4} \sum_{i=1}^5 P_i - \frac{1}{2} \sum_{s \overline{s}} \operatorname{Re}(a_{jk} a_{kl} a_{lm} \overline{c_{jm}}).
\]

9. \( r^2(\beta_w + \beta_t + \beta_v) + s^2(\beta_w + \beta_p + \beta_q) \)
\[
= \sum_{s \overline{s}} |a_{lm}|^2 (\beta_i + \beta_j + \beta_k) - \sum_{s \overline{s}} \operatorname{Im}(a_{kl} a_{lm} \overline{c_{km}}).
\]
It is seen that

\[ \mu \]

Let \( A \) be a \( n \times n \) irreducible matrix. Then the Kippenhahn curve \( C_R(A) \) consists of an ellipse and a curve of degree 4 with a double tangent and foci at \( \lambda_1, \lambda_2 \) and \( \lambda_3, \lambda_4, \lambda_5 \). The last theorem was obtained in [1, Theorem 2.2], but, there exists a gap in this Theorem, we shall point out that in its proof the errors is exactly on the left side of equation \( (4') \) and \( (5') \) and we should also add the condition \( (e) \), \( (f) \), and \( (g) \) to ensure the converse implication.

- If we take \( s = 0 \) in the previous theorem, then \( C_R(A) \) consists of ellipse and three points.

In the following, we can see that \( C_R(A) \) contains an ellipse and a curve of degree 4 with a double tangent. Using the same conditions derived in [13], to determine whether the numerical range boundary has a flatness for a \( 3 \times 3 \) irreducible matrix \( A \), these conditions are given in term of geometrical properties of flatness.

Let \( L \) be the supporting line of the convex set \( W(A) \) containing the flatness and perpendicular to the line which pass through the origin and forms angle \( \theta \) from the positive \( x \)-axis, and let \( \mu \) be the (signed) distance from the origin to \( L \). It is seen that \( \mu \) is the largest eigenvalue of \( \text{Re}(e^{-i\theta}A) \) (cf.[13], [8]).

**Lemma 2.** Let \( A \) be a \( 5 \times 5 \) matrix. Then the Kippenhahn curve \( C_R(A) \) consists of one ellipse with foci \( \lambda_1, \lambda_2 \) and minor axis of length \( r \), and a curve of degree 4 with a double tangent and foci at \( \lambda_3, \lambda_4, \lambda_5 \) if and only if the following conditions hold

(i) there exist \( \theta \in [0, 2\pi] \) and a real \( \mu \) such that

\[
P_A(x, y, z) = \left( (\alpha_1 x + \beta_1 y + z)(\alpha_2 x + \beta_2 y + z) - \frac{r^2}{4}(x^2 + y^2) \right)
\]

\[
\times \left[ (\alpha_3 x + \beta_3 y + z)(\alpha_4 x + \beta_4 y + z)(\alpha_5 x + \beta_5 y + z)
- (\alpha_3 x + \beta_3 y + z)(x^2 + y^2)(\text{Re}(e^{-i\theta}\lambda_1) + \mu)(\text{Re}(e^{-i\theta}\lambda_5) + \mu)
- (\alpha_3 x + \beta_3 y + z)(x^2 + y^2)(\text{Re}(e^{-i\theta}\lambda_3) + \mu)(\text{Re}(e^{-i\theta}\lambda_5) + \mu)
- (\alpha_5 x + \beta_5 y + z)(x^2 + y^2)(\text{Re}(e^{-i\theta}\lambda_3) + \mu)(\text{Re}(e^{-i\theta}\lambda_4) + \mu)
+ (x - iy)(x^2 + y^2)(\text{Re}(e^{-i\theta}\lambda_3) + \mu)(\text{Re}(e^{-i\theta}\lambda_4) + \mu)(\text{Re}(e^{-i\theta}\lambda_5) + \mu)e^{i\theta}
+ (x + iy)(x^2 + y^2)(\text{Re}(e^{-i\theta}\lambda_3) + \mu)(\text{Re}(e^{-i\theta}\lambda_4) + \mu)(\text{Re}(e^{-i\theta}\lambda_5) + \mu)e^{-i\theta}. \right]
\]
(ii) \( (\text{Re}(e^{-i\theta} \lambda_3) + \mu)(\text{Re}(e^{-i\theta} \lambda_4) + \mu)(\text{Re}(e^{-i\theta} \lambda_5) + \mu) \neq 0. \)

(iii) \( \lambda_j(\text{Re}(e^{-i\theta} \lambda_k) + \mu) + \lambda_k(\text{Re}(e^{-i\theta} \lambda_j) + \mu) - 2(\text{Re}(e^{-i\theta} \lambda_j) + \mu)(\text{Re}(e^{-i\theta} \lambda_k) + \mu)e^{i\theta} \neq \lambda_i((\text{Re}(e^{-i\theta} \lambda_k) + \mu) + (\text{Re}(e^{-i\theta} \lambda_j) + \mu)), \) for every \( 3 \leq i \neq j \neq k \leq 5. \)

Proof. Let \( B = \begin{bmatrix} \lambda_1 & r \\ 0 & \lambda_2 \end{bmatrix} \oplus C \) where \( C \) is 3-by-3 irreducible matrix whose Kippenhahn curve is of degree 4 with a double tangent, and eigenvalues at \( \lambda_3, \lambda_4 \) and \( \lambda_5. \)

Since \( C_R(A) = C_R(B), \) the polynomials \( P_A \) and \( P_B \) are the same.

\[
P_B(x, y, z) = \left( (\alpha_1 x + \beta_1 y + z)(\alpha_2 x + \beta_2 y + z) - \frac{r^2}{4}(x^2 + y^2) \right) P_C(x, y, z)
\]

Put \( C \) in an upper triangular form

\[
\begin{bmatrix}
\lambda_3 & a & b \\
0 & \lambda_4 & c \\
0 & 0 & \lambda_5
\end{bmatrix}
\]

By assumption \( \partial W(C) \) has a flat of portion (containing in the supporting line \( L \)), let \( \theta \in [0, 2\pi] \) be the angular between \( x \)-axis and the line which pass through the origin and perpendicular to \( L, \) so \( e^{-i\theta} C \) has a vertical flatness. According to Kippenhahn’s classification, \( \text{Re}(e^{-i\theta} C) \) must have a multiple eigenvalue, so there exist a real \( \mu \) such that

\[
\text{Re}(e^{-i\theta} C) + I\mu = \begin{bmatrix}
\text{Re}(e^{-i\theta} \lambda_3) + \mu & e^{-i\theta}a/2 & e^{-i\theta}b/2 \\
e^{i\theta}a/2 & \text{Re}(e^{-i\theta} \lambda_4) + \mu & e^{-i\theta}c/2 \\
e^{i\theta}b/2 & e^{i\theta}c/2 & \text{Re}(e^{-i\theta} \lambda_5) + \mu
\end{bmatrix}
\]

has rank one, because if otherwise \( \text{Re}(e^{-i\theta} C) + I\mu \) has zero rank, then \( \text{Re}(e^{-i\theta} C) \) and \( \text{Im}(e^{-i\theta} C) \) commutes, and \( C \) is therefore reducible, while due to the latter all \( 2 \times 2 \) minors of \( \text{Re}(e^{-i\theta} C) + I\mu \) are equal to zero. Consequently,

\[
\begin{align*}
|a|^2 &= 4(\text{Re}(e^{-i\theta} \lambda_3) + \mu)(\text{Re}(e^{-i\theta} \lambda_4) + \mu), \\
|b|^2 &= 4(\text{Re}(e^{-i\theta} \lambda_3) + \mu)(\text{Re}(e^{-i\theta} \lambda_5) + \mu), \\
|c|^2 &= 4(\text{Re}(e^{-i\theta} \lambda_4) + \mu)(\text{Re}(e^{-i\theta} \lambda_5) + \mu), \\
a\overline{b} &= 2(\text{Re}(e^{-i\theta} \lambda_3) + \mu)e^{i\theta}.
\end{align*}
\]

(3)

It is easy to see from equations above that if one of off-diagonal \( a, b \) or \( c \) is zero, then at least two of them are equal to zero, this contradict the irreducibility of \( C, \) so \( abc \neq 0. \)
On the other hand,

\[
P_C(x, y, z) = \det \begin{bmatrix}
(\alpha_3x + \beta_3y + z) & a/2(x - iy) & b/2(x - iy) \\
\bar{a}/2(x + iy) & (\alpha_4x + \beta_4y + z) & c/2(x - iy) \\
\bar{b}/2(x + iy) & \bar{c}/2(x + iy) & (\alpha_5x + \beta_5y + z)
\end{bmatrix}
\]

\[
= (\alpha_3x + \beta_3y + z)(\alpha_4x + \beta_4y + z)(\alpha_5x + \beta_5y + z)
\]

\[
- \frac{1}{4}(x^2 + y^2)[(\alpha_3x + \beta_3y + z)|c|^2 + (\alpha_4x + \beta_4y + z)|b|^2 + (\alpha_5x + \beta_5y + z)|a|^2]
\]

\[
+ \frac{1}{8}(x^2 + y^2)(x - iy)a\bar{c} + \frac{1}{8}(x^2 + y^2)(x + iy)\bar{a}\bar{c}.
\]

Combining this relation with (3) and \(a\bar{c} \neq 0\) we get successively (i) and (ii). Moreover the polynomial \(P_C\) (and therefore the matrix \(C\)) is irreducible. Thus, \(P_C\) cannot be factored into three linear factors, or into a quadratic factor and a linear one. We note that linear factors in the left-hand side of the equation \(P_C(x, y, z) = 0\) are corresponding always to eigenvalues of the matrix \(C\) (see [9], [8] and [13]).

Assume that \(P_C(x, y, z)\) has a linear factor \((\alpha_i x + \beta_i y + z)\) which corresponds to \(\lambda_i\), \(i = 3, 4, 5\). Also,

\[
P_C(x, y, z) = (\alpha_3x + \beta_3y + z)(\alpha_4x + \beta_4y + z)(\alpha_5x + \beta_5y + z)
\]

\[
- (\alpha_3x + \beta_3y + z)(x^2 + y^2)\mu_4\mu_5
\]

\[
- (\alpha_4x + \beta_4y + z)(x^2 + y^2)\mu_3\mu_5
\]

\[
- (\alpha_5x + \beta_5y + z)(x^2 + y^2)\mu_3\mu_4
\]

\[
+ (x - iy)(x^2 + y^2)\mu_3\mu_4 e^{i\theta}
\]

\[
+ (x + iy)(x^2 + y^2)\mu_3\mu_4 e^{-i\theta},
\]

where \(\mu_i = \text{Re}(e^{-i\theta}\lambda_i) + \mu\), we can see that for two by two equal index \(i, j, k \in \{1, 2, 3\}\), \((\alpha_i x + \beta_i y + z)\) gives,

\[
(\alpha_j x + \beta_j y + z)(x^2 + y^2)\mu_i\mu_k
\]

\[
+ (\alpha_k x + \beta_k y + z)(x^2 + y^2)\mu_i\mu_j
\]

\[
- (x - iy)(x^2 + y^2)\mu_i\mu_j e^{i\theta}
\]

\[
- (x + iy)(x^2 + y^2)\mu_i\mu_j e^{-i\theta}.
\]

This means that the coefficients of \(z, \alpha_i x, \beta_i y\) in the last polynomial are equals, which gives

\[
\beta_i[\alpha_j \mu_i \mu_k + \alpha_k \mu_i \mu_j - 2\mu_i \mu_j \mu_k \cos(\theta)] = \alpha_i[\beta_j \mu_i \mu_k + \beta_k \mu_i \mu_j - 2\mu_i \mu_j \mu_k \sin(\theta)] \tag{4}
\]

\[
\alpha_j \mu_i \mu_k + \alpha_k \mu_i \mu_j - 2\mu_i \mu_j \mu_k \cos(\theta) = \alpha_i[\mu_i \mu_k + \mu_i \mu_j] \tag{5}
\]

and

\[
\beta_j \mu_i \mu_k + \beta_k \mu_i \mu_j - 2\mu_i \mu_j \mu_k \sin(\theta) = \beta_i[\mu_i \mu_k + \mu_i \mu_j]. \tag{6}
\]

One can see that (4), (5) and (6) are equivalent to \(\lambda_j \mu_k + \lambda_k \mu_j - 2\mu_j \mu_k e^{i\theta} = \lambda_i(\mu_k + \mu_j)\) and hence (iii).

The converse is obvious. \(\square\)
In what follows, set \( \mu_j = \Re(e^{-i\theta_j} \lambda_j) + \mu \) for every \( j = 1, \ldots, 5 \).

**Theorem 4.** Let \( A \) be in upper-triangular form (1). Then \( C_{R}(A) \) consists of one ellipse with foci \( \lambda_q, \lambda_p \) and minor axis of length \( r \), and a curve of degree 4 with a double tangent and foci at \( \lambda_w, \lambda_t \) and \( \lambda_v \) if and only if there exist \( \theta \in [0, 2\pi] \) and a real \( \mu \) such that

(a) \( r^2 + 4(\mu_4 \mu_v + \mu_w \mu_v + \mu_w \mu_t) = \sum_{i=1}^{5} |a_{il}|^2 \).

(b) \( r^2(\lambda_w + \lambda_t + \lambda_v) + 4[(\lambda_w \mu_t \mu_v + \lambda_t \mu_w \mu_v + \lambda_v \mu_t \mu_w) - 2\mu_w \mu_t e^{i\theta}] \)
\[= \sum_{i=1}^{5} |a_{il}|^2 (\lambda_i + \lambda_j + \lambda_k) - \sum_{i=1}^{5} \lambda_i a_{klm} a_{lkm}. \]

(c) \( r^2(\lambda_w \lambda_t + \lambda_t \lambda_v + \lambda_w \lambda_v) + 4[\lambda_p \lambda_q(\mu_t \mu_v + \mu_w \mu_v + \mu_t \mu_w) \]
\[= \sum_{i=1}^{5} |a_{il}|^2 (\lambda_i + \lambda_j + \lambda_k) - \sum_{i=1}^{5} \lambda_i a_{klm} a_{lkm}. \]

(d) \( r^2(\lambda_w \lambda_t \lambda_v) + 4[\lambda_p \lambda_q(\mu_t \mu_v + \mu_w \mu_v + \mu_t \mu_w) - 2\mu_w \mu_t e^{i\theta}] \)
\[= \sum_{i=1}^{5} |a_{il}|^2 (\lambda_i + \lambda_j + \lambda_k) - \sum_{i=1}^{5} \lambda_i a_{klm} a_{lkm}. \]

(e) \( r^2[\alpha_w \alpha_t \alpha_v - (\alpha_w \mu_t \mu_v + \alpha_t \mu_w \mu_v + \alpha_v \mu_w \mu_t) + 2\mu_w \mu_t \cos(\theta)] \)
\[= \sum_{i=1}^{5} |a_{il}|^2 \alpha_i \alpha_j \alpha_k - \sum_{i=1}^{5} \Re(a_{ij} a_{il} a_{km} a_{lkm} a_{m}) + \frac{1}{4} \sum_{i=1}^{5} P_i \alpha_i \]
\[+ \frac{1}{2} \sum_{i=1}^{5} \Re(a_{ij} a_{kl} a_{jm} a_{ljm} a_{m} a_{k}) \]
\[+ \frac{1}{4} \sum_{i=1}^{5} \Re(a_{ij} a_{kl} a_{jm} a_{ljm} a_{m} a_{k}) - \frac{1}{4} \Re(a_{12} a_{23} a_{34} a_{45} a_{15}). \]

(f) \( r^2[\beta_w \beta_t \beta_v - (\beta_w \mu_t \mu_v + \beta_t \mu_w \mu_v + \beta_v \mu_w \mu_t) + 2\mu_w \mu_t \sin(\theta)] \)
\[= \sum_{i=1}^{5} |a_{il}|^2 \beta_i \beta_j \beta_k - \sum_{i=1}^{5} \Im(a_{ij} a_{kl} a_{km} a_{lkm} a_{m} a_{k}) + \frac{1}{4} \sum_{i=1}^{5} P_i \beta_i \]
\[+ \frac{1}{2} \sum_{i=1}^{5} \Re(a_{ij} a_{kl} a_{jm} a_{ljm} a_{m} a_{k}) \beta_i \]
\[+ \frac{1}{4} \sum_{i=1}^{5} \Im(a_{ij} a_{kl} a_{jm} a_{ljm} a_{m} a_{k}) - \frac{1}{4} \Re(a_{12} a_{23} a_{34} a_{45} a_{15}). \]

\( \}\)
(g) \[ r^2[(\alpha x + \alpha w x + \alpha w \alpha t) - (\mu t \mu v + \mu w \mu v + \mu w \mu t)] \\
+ 4\alpha w\alpha q(\mu t \mu v + \mu w \mu v + \mu w \mu t) \\
+ 4(\alpha p + \alpha q)[(\alpha u \mu t \mu v + \alpha t \mu w \mu v + \alpha v \mu w \mu t) - 2\mu w \mu t \mu v \cos(\theta)] \\
= \sum_{S_{ijklm}} |a_{im}|^2(\alpha t \alpha j + \alpha t \alpha k + \alpha j \alpha k) - \sum_{S_{ijklm}} \Re(a_{ik}a_{lm}a_{km})(\alpha i + \alpha j) \\
- \frac{1}{4} \sum_{i=1}^5 P_i + \frac{1}{2} \sum_{S_{ijklm}} \Re(a_{ik}a_{km}a_{jm}a_{lk}).
\]

(h) \( \mu w \mu t \mu v \neq 0 \).

(i) \( \lambda_j \mu_k + \lambda_k \mu_j - 2\mu_j \mu_k e^{i\theta} \neq \lambda_i(\mu_k + \mu_j) \) for every \( i, j, k \in \{w, t, v\} : i \neq j \neq k \),

where \( P_i = \sum_{S_{ijklm}} |a_{ik}|^2 |a_{lm}|^2 - \sum_{S_{ijklm}} \Re(a_{ik}a_{lm}a_{km}) - \sum_{S_{ijklm}} \Re(a_{ik}a_{km}a_{jm}). \)

for every \( i = 1, .., 5. \)

Proof. Taking conditions from lemma 2, we have,

\[
P_A(x, y, z) = \left[(\alpha_p x + \beta_p y + z)(\alpha_q x + \beta_q y + z) - \frac{y^2}{4}(x^2 + y^2)\right] \\
\times [(\alpha_u x + \beta_w y + z)(\alpha_e x + \beta v y + z) \\
- (\alpha w x + \beta w y + z)(\alpha t x + \beta t y + z)(\alpha v x + \beta v y + z) \\
- (\alpha t x + \beta t y + z)(\alpha w x + \beta w y + z) \\
- (\alpha v x + \beta v y + z)(\alpha t x + \beta t y + z)(\mu w \mu t) \\
+ (x - iy)(x^2 + y^2)(\mu w \mu t \mu e^{i\theta}) \]
\]

\[
\mu w \mu t \mu v \neq 0, \text{ and } \lambda_j \mu_k + \lambda_k \mu_j - 2\mu_j \mu_k e^{i\theta} \neq \lambda_i(\mu_k + \mu_j) \text{ for every } i, j, k \in \{w, t, v\} : i \neq j \neq k. \]

Comparing the previous formula of \( P_A(x, y, z) \) with polynomial (2). We obtain

\[
Q(x, y, z) = r^2[(\alpha w x + \beta w y + z)(\alpha t x + \beta t y + z)(\alpha v x + \beta v y + z) \\
- (x^2 + y^2)[(\alpha w x + \beta w y + z)(\alpha t x + \beta t y + z)(\alpha v x + \beta v y + z) \\
+ 2(x^2 + y^2)(x \mu w \mu t \mu v \cos(\theta) + y \mu w \mu t \mu v \sin(\theta)) \\
+ 4(\alpha_p x + \beta_p y + z)(\alpha_q x + \beta_q y + z) \\
+ ((\alpha w x + \beta w y + z)(\mu t \mu v + (\alpha t x + \beta t y + z)(\mu w t \mu v) \\
- 2(x \mu w \mu t \mu v \cos(\theta) + y \mu w \mu t \mu v \sin(\theta))].
\]

Computing the coefficients of \( x^3, y^3, z^3, x^2 y, xy^2, x^2 z, xz^2, y^2 z, yz^2, xy z \). By identification, we get, respectively,

1. \[ r^2[\alpha w \alpha t \alpha v - (\alpha w \mu t \mu v + \alpha t \mu w \mu v + \alpha v \mu w \mu t) + 2\mu w \mu t \mu v \cos(\theta)] \\
+ 4\alpha w \alpha q[(\alpha w \mu t \mu v + \alpha t \mu w \mu v + \alpha v \mu w \mu t) - 2\mu w \mu t \mu v \cos(\theta)] \\
= \sum_{S_{ijklm}} |a_{im}|^2 \alpha_i \alpha_j \alpha_k - \sum_{S_{ijklm}} \Re(a_{ik}a_{lm}a_{km}) \alpha_i \alpha_j - \frac{5}{4} \sum_{i=1}^5 P_i \alpha_i
\]
On the circular numerical range of 5-by-5 partial isometries

\[ + \frac{1}{2} \sum_{ijklm} Re(a_{jk}a_{kl}a_{lm}a_{jm})a_i \]
\[ + \frac{1}{4} \sum_{ijklm} Re(a_{kl}a_{lm}a_{km})a_{ij}^2 - \frac{1}{4} \sum_{ijklm} Re(a_{ij}a_{jk}a_{kl}a_{im}a_{ml}) \]
\[ - \frac{1}{4} \sum_{ijklm} Re(a_{ij}a_{jk}a_{lm}a_{im}a_{ik}) - \frac{1}{4} Re(a_{12}a_{23}a_{34}a_{45}a_{15}). \]

2. \[ r^2[\beta_w\beta_v - (\alpha_w\mu_v + \beta_v\mu_w + \alpha_v\mu_w + \alpha_v\mu_w) + 2\mu_w\mu_v \sin(\theta)] \]
\[ + 4\beta_u\beta_v[(\alpha_w\mu_v + \beta_v\mu_w + \alpha_v\mu_w + \beta_v\mu_w) - 2\mu_w\mu_v \sin(\theta)] \]
\[ = \sum_{ijklm} |a_{lm}|^2 \beta_i \beta_j - \sum_{ijklm} Im(a_{kl}a_{lm}a_{km}) \beta_i \beta_j - \frac{1}{4} \sum_{ijklm} P_i \beta_i \]
\[ - \frac{1}{2} \sum_{ijklm} Re(a_{jk}a_{kl}a_{lm}a_{jm}) \beta_i \]
\[ + \frac{1}{4} \sum_{ijklm} Im(a_{kl}a_{lm}a_{km})a_{ij}^2 - \frac{1}{4} \sum_{ijklm} Im(a_{ij}a_{jk}a_{kl}a_{im}a_{ml}) \]
\[ - \frac{1}{4} \sum_{ijklm} Im(a_{ij}a_{jk}a_{lm}a_{im}a_{ik}) + \frac{1}{4} Im(a_{12}a_{23}a_{34}a_{45}a_{15}). \]

3. \[ r^2 + 4(\mu_t\mu_v + \mu_w\mu_v + \mu_w\mu_t) = \sum_{ijklm} |a_{lm}|^2. \]

4. \[ r^2[(\beta_w\alpha_t\alpha_v + \beta_t\alpha_w\alpha_v + \beta_v\alpha_w\alpha_t) + 2\mu_w\mu_t \sin(\theta)] \]
\[ - (\beta_w\mu_t\mu_v + \beta_t\mu_w\mu_v + \beta_v\mu_w\mu_t)] + 4\alpha_p\alpha_q[(\alpha_w\mu_v + \beta_v\mu_w + \alpha_v\mu_w - 2\mu_w\mu_v \cos(\theta)] \]
\[ + 4(\alpha_p\beta_q + \alpha_q\beta_p)[(\alpha_w\mu_v + \alpha_t\mu_w + \alpha_v\mu_t - 2\mu_w\mu_v \cos(\theta)] \]
\[ = \sum_{ijklm} |a_{lm}|^2 (\alpha_i \alpha_j \beta_k + \alpha_i \beta_k \beta_j + \alpha_j \alpha_k \beta_i) - \sum_{ijklm} Im(a_{kl}a_{lm}a_{km}) \alpha_i \alpha_j \]
\[ - \sum_{ijklm} Re(a_{kl}a_{lm}a_{km})(\alpha_i \beta_j + \alpha_j \beta_i) + \frac{1}{2} \sum_{ijklm} Re(a_{jk}a_{kl}a_{lm}a_{jm}) \beta_i \]
\[ + \frac{1}{4} \sum_{ijklm} P_i \beta_i - \frac{1}{4} \sum_{ijklm} Im(a_{ij}a_{jk}a_{kl}a_{im}a_{jm}) \alpha_i \alpha_j \]
\[ - \frac{1}{4} \sum_{ijklm} Im(a_{ij}a_{jk}a_{lm}a_{im}a_{ik}) - \frac{1}{4} \sum_{ijklm} Im(a_{ij}a_{jk}a_{lm}a_{im}a_{ik}) \]
\[ - \frac{3}{4} Im(a_{12}a_{23}a_{34}a_{45}a_{15}). \]

5. \[ r^2[(\alpha_w\beta_v + \alpha_t\beta_v + \alpha_v\beta_w + \beta_t\beta_v + 2\mu_w\mu_v \cos(\theta) - (\alpha_w\mu_v + \alpha_t\mu_v + \alpha_v\mu_t + \beta_t\mu_v \cos(\theta)] \]
\[ + 4\beta_v \beta_q[(\alpha_w\mu_v + \alpha_t\mu_v + \alpha_v\mu_v + \beta_t\mu_v + \alpha_v\mu_t - 2\mu_w\mu_v \sin(\theta)] \]
\[ = \sum_{ijklm} |a_{lm}|^2 (\alpha_i \beta_j \beta_k + \alpha_j \beta_k \beta_i + \alpha_k \beta_i \beta_j) - \sum_{ijklm} Re(a_{kl}a_{lm}a_{km}) \beta_i \beta_j \]
\[ - \sum_{ijklm} Im(a_{kl}a_{lm}a_{km})(\alpha_i \beta_j + \alpha_j \beta_i) - \frac{1}{4} \sum_{ijklm} P_i \alpha_i \]
\[ - \frac{1}{2} \sum_{ijklm} Re(a_{jk}a_{kl}a_{lm}a_{jm}) \alpha_i \]
Note that the combination of (1), (2), (4) and (5) is equivalent to (d), (e) and (f), because (1) − (5) − i(2) + i(4) yields (d).

(6), (8) and (10) is equivalent to (c) and (g), since (6) − (8) + i(10) yields (c). (7) and (9) is equivalent to (b), it follows from (7) + i(9).

This completes the proof. \[\square\]
3 On the circular numerical range of $S_5$ matrices

Recall that an n-by-n matrix $A$ is said to be of class $S_n$ if $A$ is a contraction, the eigenvalues of $A$ are all in the open unit disc $D$ and $\text{rank}(I_n - A^*A) = 1$. Two unitary equivalent $S_n$-matrices have the following useful characterization.

Lemma 3. [4, Theorem 4.1] Let $A_1$ and $A_2$ be two $n$-dimensional operators with $A_2$ in $S_n$. Then $A_1$ is unitary equivalent to $A_2$ if and only if $A_1$ is a contraction and $W(A_1) = W(A_2)$.

Now, we establish the result of Theorem 2 in the special case of $S_5$-matrices.

Theorem 5. Let $A$ be a non-invertible $S_5$ matrix with $W(A) = \{z \in \mathbb{C} : |z - a| \leq r\}$, $(r > 0)$. If kippenhahn curve $C_R(A)$ has one of the following shapes,

(i) $C_R(A)$ consists of three points and an ellipse.

(ii) $C_R(A)$ consists of two ellipses and a point.

Then $a = 0$.

Proof. Without loss of generality, we assume that $A$ is an upper triangular matrix. The assumption on the numerical range of $A$ implies that the origin $a$ is an eigenvalue with algebraic multiplicity at least 2. So, by [3, Corollary 1.3] $A$ takes the following form

$$A = \begin{bmatrix} a & 1 - a^2 & -a\sqrt{1-a^2} & 0 & 0 \\ 0 & a & \sqrt{1-a^2} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{1-|b|^2} & -b\sqrt{1-|c|^2} \\ 0 & 0 & 0 & b & \sqrt{1-|b|^2}\sqrt{1-|c|^2} \\ 0 & 0 & 0 & 0 & c \end{bmatrix}.$$

Moreover, we can take $a$ positive by a suitable rotation, thus $W(A)$ is symmetric with respect to the real axis, which means that $W(A) = W(A^*)$, ($A^*$ is the adjoint matrix of $A$), as we mentioned below $A$ is of class $S_5$ and therefore by Lemma 3 $A$ and $A^*$ are unitary equivalent, moreover one can see that the eigenvalues $b$ and $c$ of $A$ must be real or complex conjugates. Let

$$B = A - aI_5 = \begin{bmatrix} 0 & 1 - a^2 & -a\sqrt{1-a^2} & 0 & 0 \\ 0 & 0 & \sqrt{1-a^2} & 0 & 0 \\ 0 & 0 & -a & \sqrt{1-|b|^2} & -b\sqrt{1-|c|^2} \\ 0 & 0 & 0 & b - a & \sqrt{1-|b|^2}\sqrt{1-|c|^2} \\ 0 & 0 & 0 & 0 & c - a \end{bmatrix}.$$
Consider the homogeneous Kippenhahn polynomial $P_B(x, y, z) = \det(x\Re B + y\Im B + zI_5)$ of degree 5 on the complex projective plane $\mathbb{CP}^2$. Since by hypotheses $W(B)$ is a circular disc with center 0 and radius $r$, then $C_R(B)$ has one of two possible shapes,

(i) A circle with center 0 and radius $r$ together with three points $-a, b-a, c-a$
inside it.

(ii) A circle with center 0 and radius $r$ together with a point $(-a, b-a$ or $c-a)$ and an ellipse with (minor axis length $s \leq 2r$) and the two remaining points as the foci, all inside the circle.

Applying condition (d) of Theorem 3 to the upper-triangular matrix $B$ yields to

\[
4r^2(-a(b-a)(c-a)) = (1-a^2)^2(-a)(b-a)(c-a) - (1-a^2)(-a)\sqrt{1-a^2}\sqrt{1-a^2}(b-a)(c-a) = 0.
\]

Then either $a = 0$, $a = b$ or $a = c$. If it is the first case so it will done, otherwise if it is one of the two latter cases, the condition (c) of Theorem 3 gives,

\[
4r^2(-a(b-a) + (b-a)(c-a) - a(c-a)) = (1-a^2)^2(-a(b-a) + (b-a)(c-a) - a(c-a)) + a^2(1-a^2)(b-a)(c-a) + (1-a^2)(b-a)(c-a) - (1-a^2)^2(-a(b-a) - a(c-a)) = 0.
\]

Thus $a = b$ if $a = c$ and vise versa. By the condition (b) of Theorem 3

\[
4r^2(-a + (b-a) + (c-a)) + s^2(0 + 0 + \lambda) = (1-a^2)^2(-a + b-a + c-a) + a^2(1-a^2)(b-a + c-a) + (1-a^2)(b-a + c-a)
\]

\[
+ (1-b^2)(c-a) + b^2(1-c^2)(b-a)
\]

\[
- a(1-b^2)(1-c^2) + a(1-a^2)^2 + b(1-b^2)(1-c^2).
\]

where $\lambda$ takes one of the eigenvalue $-a, b-a$ or $c-a$. Assume that $a = b = c$, we get $(4r^2 + s^2)(-a) = 0$ or $ar^2 = 0$. Hence $a = 0$. This complete the proof. □

**Remark 2.** It is well known that for every $S_n$ matrix $A$, $\Re(A)$ have only simple eigenvalues see [4, Corollary 2.7], then $C_R(A)$ not contains a curve of degree 4 with double tangent.

### 4 Proof of Theorem 2

It is well known that a $n$-by-$n$ partial isometry $A$ can be represented on $\text{Ker}(A) \oplus \text{Ker}(A)^\perp$, by

\[
A = \begin{bmatrix}
0 & B \\
0 & C
\end{bmatrix}
\]
On the circular numerical range of 5-by-5 partial isometries

with $B$ and $C$ satisfying $B^*B + C^*C = I_{\ker(A)^\perp}$, where $I_{\ker(A)^\perp}$ is the identity matrix on $\ker(A)^\perp$, see [5, Proposition 2.1]. Also, the irreducibility of a partial isometry can be characterized by,

**Lemma 4.** [5, lemma 2.8] Let

$$A = \begin{bmatrix} 0_m & B \\ 0 & C \end{bmatrix} \quad \text{on} \quad \mathbb{C}^n = \mathbb{C}^m \oplus \mathbb{C}^{n-m}, \quad (1 \leq m \leq n).$$

(a) If $k = \text{rank}B < m$, then $A$ is unitarily similar to $0_{m-k} \oplus A_1$ for some matrix $A_1 = \begin{bmatrix} 0_k & B_1 \\ 0 & C_1 \end{bmatrix}$ on $\mathbb{C}^{n-m+k} = \mathbb{C}^{k} \oplus \mathbb{C}^{n-m}$, with $\text{rank}B_1 = k$.

(b) If $m > \lceil n/2 \rceil$, the largest integer less than or equal to $n/2$, then $A$ is unitarily similar to $0_{2m-n} \oplus A_2$ for some matrix $A_2 = \begin{bmatrix} 0_{n-m} & B_2 \\ 0 & C_2 \end{bmatrix}$ on $\mathbb{C}^{2(n-m)} = \mathbb{C}^{m} \oplus \mathbb{C}^{n-m}$.

The next proposition relates partial isometries with $S_n$-matrices.

**Proposition 1.** [5, Proposition 2.3] Let $A$ be an $n$-by-$n$ matrix. Then $A$ is an irreducible partial isometry with $\dim \ker A = 1$ if and only if $A$ is of class $S_n$ with 0 in $\sigma(A)$.

Now, we are ready to establish our main theorem.

**Proof of Theorem 2.** Let $A$ be an $5 \times 5$ partial isometry with $W(A) = \{ z \in \mathbb{C} : |z - a| \leq r \}$, \quad (r > 0).

First let us remark that if $A$ is reducible, then $A$ is unitarily similar to $A_1 \oplus A_2$, where $A_1$ and $A_2$ are two partial isometries with order at most 4. Since one of $W(A_1)$ or $W(A_2)$ must be equal to that of $A$, so by Theorem 1 it follows that $a = 0$.

Now, we assume that $A$ is irreducible. According to the dimension of the kernel of $A$, we distinguish three cases.

**Case 1.** $\dim \ker A = 1$. By Proposition 1, $A$ is non-invertible $S_5$-matrix, so according to Theorem 5 and Remark 2, $a = 0$.

**Case 2.** $\dim \ker A = 2$. Since $W(A)$ is a circular disc centered at $a$, we may assume that

$$A = \begin{bmatrix} 0 & B \\ 0 & C \end{bmatrix} = \begin{bmatrix} 0 & 0 & k & l & t \\ 0 & 0 & g & h & j \\ 0 & 0 & b & e & f \\ 0 & 0 & 0 & a & d \\ 0 & 0 & 0 & 0 & a \end{bmatrix} \quad \text{on} \quad \mathbb{C}^2 \oplus \mathbb{C}^3,$$
with
\[
I_3 = B^*B + C^*C
\]
\[
= \begin{bmatrix}
|k|^2 + |g|^2 + |b|^2 & \overline{kl} + \overline{gh} + \overline{be} & \overline{kt} + \overline{gj} + \overline{bf} \\
\overline{lk} + \overline{gh} + \overline{eb} & |l|^2 + |e|^2 + |a|^2 & \overline{lt} + \overline{hh} + \overline{ae} + \overline{ad} \\
\overline{lk} + \overline{gh} + \overline{eb} & \overline{lt} + \overline{hh} + \overline{ae} + \overline{ad} & |t|^2 + |j|^2 + |f|^2 + |d|^2 + |a|^2
\end{bmatrix}.
\]

As in the proof of Theorem 5, \(a\) is positive and \(C_R(A)\) has one of the three possible shapes.

(i) \(C_R(A)\) contains a circle (with center \(a\), radius \(r\)) and three points \(0,0,b\).

(ii) \(C_R(A)\) is a circle (with center \(a\), radius \(r\)), together with an ellipse and a point.

(iii) \(C_R(A)\) contains a circle (with center \(a\), radius \(r\)) and a curve of degree 4 with a double tangent.

Applying condition (d) of Theorem 3 and Theorem 4 to \(A - aI_3\) we get
\[
4r^2(a.a.(b-a)) = |d|^2a^2(b-a) - a^2ed\overline{f} + a(b-a)hd\overline{j} + a(b-a)ld\overline{t} - aged\overline{j} - aked\overline{l} = |d|^2a^2(b-a) - a^2ed\overline{f} + ad(b-a)(h\overline{j} + l\overline{t}) - aed(g\overline{j} + k\overline{t}) = |d|^2a^2(b-a) - a^2ed\overline{f} - ad(b-a)(e\overline{f} + a\overline{d}) + aedb\overline{f} = 0.
\]

Thus \(a = 0\) or \(b = a\). If \(b = a\), by condition (c) of Theorem 3 and Theorem 4
\[
4r^2(a^2) = a^2(|e|^2 + |f|^2 + |d|^2) + 2aed\overline{f} + aeg\overline{h} + aek\overline{l} + afg\overline{j} + afk\overline{t} + adh\overline{j} + adl\overline{t} + edg\overline{j} + edk\overline{t}.
\]
\[
= a^2(|e|^2 + |f|^2 + |d|^2) + 2aed\overline{f} + ae(g\overline{h} + k\overline{l}) + af(g\overline{j} + k\overline{t}) + ad(h\overline{j} + l\overline{t}) + ed(g\overline{j} + k\overline{t}) = 0
\]

and therefore \(a = 0\).

Case 3. \(\dim \ker A > 2\), then it follows from Lemma 4 that \(A\) is reducible, then \(a = 0\).

This completes the proof of the theorem.

Remark 3. In order to give a complete answer to the conjecture of Gau et al, in dimension 5, it remains to study the case when \(C_R(A)\) is an ellipse and a curve of order 6, consisting of an oval and a curve of three cups. Based on the factoribility of \(P_A\), Kippenhahn in [8] gave a fully classification of the numerical range of \(3 \times 3\) matrices, also a pertinent tests were offered in [13]. However, there is no much results about the connection between concrete description of the curve \(C_R(A)\) and \(P_A\) when \(W(A)\) is an oval. Thus, this case is still an open question.
References


