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DIAMOND-ALPHA INEQUALITIES WITH TWO PARAMETERS ON TIME SCALES

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Abstract

In this article, we give a new version of n-tuple diamond-alpha Hölder inequality on time scales, this result generalizes some results known in the literature. Second, we present n-tuple diamond-alpha inequality with two parameters on time scales. It is a tool to generalize integral inequalities on time scales, its goal is to create an inequality restarted from the left side with a parameter to a right side with two parameters. Moreover, new Minkowski integral inequality with two parameters is given, and some interesting integral inequalities are found.

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1 Introduction and preliminaries

The Hölder and Minkowski inequalities are fundamental inequalities in computation in analysis and an indispensable tool for the study of the space L^q and its dual as spaces of sequences l_q. In recent years, the study of dynamic inequalities on Hölder and Minkowski time scales has received much attention, for more details, we refer to recent papers [2]-[10]. The goal of studying dynamic inequalities on time scales with diamond-alpha allows proving the delta, nabla and the both the integral form of differential inequalities and the discrete form of inequalities. The Minkowski inequality [7] states that, for $q \geq 1$, if

$$0 < \int_r^d \mid h(\tau) \mid^q \ \diamondsuit_\alpha \tau < \infty \quad and \quad 0 < \int_r^d \mid \phi(\tau) \mid^q \ \diamondsuit_\alpha \tau < \infty$$

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then

$$\left(\int_r^d \mid h(\tau)+\phi(\tau)\mid^q \diamondsuit_\alpha \tau\right)^{\frac{1}{q}} \leq \left(\int_r^d \mid h(\tau)\mid^q \diamondsuit_\alpha \tau\right)^{\frac{1}{q}} + \left(\int_r^d \mid \phi(\tau)\mid^q \diamondsuit_\alpha \tau\right)^{\frac{1}{q}}. \tag{1}$$

In 2010, the authors [6] gave the following Theorem.

Theorem 1. *(Theorem 3.2) Let* $\mathbf{T}$ *be a time scale,* $r, d \in \mathbf{T}$ *with* $r < d$, $\Phi, \Psi, \mu : [r, d] \times [r, d] \to \mathbf{R}$ *be* $\diamondsuit_\alpha$*-integrable functions, and* $\frac{1}{q} + \frac{1}{q'} = 1$ *with* $q > 1$*. Then*

$$\begin{aligned} &\int_r^d \int_r^d \mid \mu(t_1, t_2)\Phi(t_1, t_2)\Psi(t_1, t_2)\mid \diamondsuit_\alpha t_1 \diamondsuit_\alpha t_2 \leq \\ &\left(\int_r^d \int_r^d \mid \mu(t_1, t_2 \mid\mid \Phi(t_1, t_2)\mid^q \diamondsuit_\alpha t_1 \diamondsuit_\alpha t_2\right)^{\frac{1}{q}} \times \\ &\left(\int_r^d \int_r^d \mid \mu(t_1, t_2)\mid\mid \Psi(t_1, t_2)\mid^{q'} \diamondsuit_\alpha t_1 \diamondsuit_\alpha t_2\right)^{\frac{1}{q'}}. \end{aligned} \tag{2}$$

In [14], the authors gave some new three-tuple diamond-alpha integral Hölder's inequalities on time scales. The comprehensive development of the calculus of the diamond-alpha derivative and diamond-alpha integration is given in [11], [12]. Let $\mathbf{T}$ be a time scale and Φ be differentiable on $\mathbf{T}$ in the Δ and ∇ sense. For $\tau \in \mathbf{T}$, we define the diamond-alpha derivative $\Phi^{\diamondsuit_\alpha}(\tau)$ by

$$\Phi^{\diamondsuit_\alpha}(\tau) = \alpha\Phi^{\Delta}(\tau) + (1-\alpha)\Phi^{\nabla}(\tau).$$

If Φ is Δ and ∇ differentiable, then Φ is diamond-alpha differentiable.

Theorem 2. *[12, Theorem 3.2] Let* $0 \leq \alpha \leq 1$*. If* Φ *is both* Δ *and* ∇ *differentiable at* $\tau \in \mathbf{T}$*, then* Φ *is* $\diamondsuit_\alpha$ *differentiable at* τ *and*

$$\Phi^{\diamondsuit_\alpha}(\tau) = \alpha\Phi^{\Delta}(\tau) + (1-\alpha)\Phi^{\nabla}(\tau).$$

Definition 1. *[11, Definition 1] Let* $r, d \in \mathbf{T}$*, and* $\Phi : \mathbf{T} \to \mathbf{R}$*. Then, the diamond-*$\alpha$ *integral from* r *to* d *of* Φ *is defined by*

$$\int_r^d \Phi(t)\diamondsuit_\alpha t = \alpha \int_r^d \Phi(t)\Delta t + (1-\alpha)\int_r^d \Phi(t)\nabla t, \quad 0 \leq \alpha \leq 1.$$

provided that there exists delta and nabla integrals of Φ *on* $\mathbf{T}$*.*

The diamond-α integral of Φ exists when Φ is a continuous function. Let $r,\ d,\ e \in \mathbf{T}$, $\lambda, \beta \in \mathbf{R}$ and Φ, Ψ be continuous functions on $[r,\ d] \cap \mathbf{T} = [r,\ d]_{\mathbf{T}}$. Then, the following properties hold.

1. $\displaystyle\int_r^d (\lambda\Phi(z) + \beta\Psi(z))\diamondsuit_\alpha z = \lambda\int_r^d \Phi(z)\diamondsuit_\alpha z + \beta\int_r^d \Psi(z)\diamondsuit_\alpha z.$

2. $\displaystyle\int_r^d \Phi(z)\diamondsuit_\alpha z = -\int_d^r \Phi(z)\diamondsuit_\alpha z, \quad \int_r^r \Phi(z)\diamondsuit_\alpha z = 0.$

3. $\displaystyle\int_r^d \Phi(z)\diamondsuit_\alpha z = \int_r^e \Phi(z)\diamondsuit_\alpha z + \int_e^d \Phi(z)\diamondsuit_\alpha z.$

4. If $\Phi(z) \geq 0$ for all $z \in [r, d]_{\mathbf{T}}$, then $\displaystyle\int_r^d \Phi(z)\diamondsuit_\alpha z \geq 0.$

5. If $\Phi(z) \leq \Psi(z)$ for all $z \in [r, d]_{\mathbf{T}}$, then $\displaystyle\int_r^d \Phi(z)\diamondsuit_\alpha z \leq \int_r^d \Psi(z)\diamondsuit_\alpha z.$

6. If $\Phi(z) \geq 0$ for all $z \in [r, d]_{\mathbf{T}}$, then $\Phi(z) = 0$ if only if $\displaystyle\int_r^d \Phi(z)\diamondsuit_\alpha z = 0.$

Lemma 1. *[5, Theorem 1.1.21]. Let* $\mathbf{T}$ *be a time scale,* $r, d \in \mathbf{T}$ *with* $r < d$*, and* Φ*,* Ψ *be two positive functions. If* $\frac{1}{q} + \frac{1}{q'} = 1$ *with* $q > 1$*, then*

$$\int_r^d \Phi(\tau)\Psi(z)\diamondsuit_\alpha z \leq \left(\int_r^d \Phi^q(z)\diamondsuit_\alpha z\right)^{\frac{1}{q}} \left(\int_r^d \Psi^{q'}(z)\diamondsuit_\alpha z\right)^{\frac{1}{q'}}. \tag{3}$$

The inequality (3) is reversed for $0 < q < 1$.

2 m-tuple diamond-alpha Hölder inequality

To define the diamond-α integral for a function of m variables, we can extended the definition of the diamond-α integral for a function of two variables given in [6, page .4] and [13, page .2]. Throughout the paper, we assume that the m-Diamond-alpha integrals exist and are finite.
We present the next Lemma that is useful for proving our results.

Lemma 2. *Let* $\mathbf{T}$ *be a time scale,* $r, d \in \mathbf{T}$ *with* $r < d$*,* $q > 1$*,* $m \geq 1$ *and* $\Phi, \Psi, \mu : [r, d] \times [r, d]^{m-1} \to \mathbf{R}$ *be* $\diamondsuit_\alpha$*-integrable functions,* $\dfrac{1}{q} + \dfrac{1}{q'} = 1$*. Then*

$$\begin{aligned}
&\int_r^d \cdots \int_r^d \mid \mu(t_1, ..., t_m)\Phi(t_1, ..., t_m)\Psi(t_1, ..., t_m) \mid \diamondsuit_\alpha t_1 \ldots \diamondsuit_\alpha t_m \leq \\
&\left(\int_r^d \cdots \int_r^d \mid \mu(t_1, ..., t_m) \mid\mid \Phi(t_1, ..., t_m) \mid^q \diamondsuit_\alpha t_1 \ldots \diamondsuit_\alpha t_m\right)^{\frac{1}{q}} \times \\
&\left(\int_r^d \cdots \int_r^d \mid \mu(t_1, ..., t_m) \mid\mid \Psi(t_1, ..., t_m) \mid^{q'} \diamondsuit_\alpha t_1 \ldots \diamondsuit_\alpha t_m\right)^{\frac{1}{q'}}.
\end{aligned} \tag{4}$$

The inequality (4) is reversed for $0 < q < 1$.

Proof. Setting

$$B(t_{m+1}) = \left(\int_r^d \cdots \int_r^d \mid \mu(t_1, ..., t_{m+1}) \mid\mid \Phi(t_1, ..., t_{m+1}) \mid^q \diamondsuit_\alpha t_1 \ldots \diamondsuit_\alpha t_m\right)^{\frac{1}{q}},$$

$$H(t_{m+1}) = \left(\int_r^d \cdots \int_r^d \mid \mu(t_1, ..., t_{m+1}) \mid\mid \Psi(t_1, ..., t_{m+1}) \mid^{q'} \diamondsuit_\alpha t_1 \ldots \diamondsuit_\alpha t_m\right)^{\frac{1}{q'}},$$

one may then write the inequality (4),

$$\begin{aligned}&\int_r^d ... \int_r^d \mid \mu(t_1,...,t_{m+1})\Phi(t_1,...,t_{m+1})\Psi(t_1,...,t_{m+1}) \mid \diamondsuit_\alpha t_1 ...\diamondsuit_\alpha t_m \\ \le\ & B(t_{m+1})H(t_{m+1}).\end{aligned}$$

Now we assume that the inequality (4) is verified for m and using the Hölder's inequality (3), we get

$$\begin{aligned}&\int_r^d ... \int_r^d \mid \mu(t_1,...,t_{m+1})\Phi(t_1,...,t_{m+1})\Psi(t_1,...,t_{m+1}) \mid \diamondsuit_\alpha t_1 ...\diamondsuit_\alpha t_{m+1} \\ &\le \int_r^d B(t_{m+1})H(t_{m+1})\diamondsuit_\alpha t_{m+1} \\ &\le \left(\int_r^d B^q(t_{m+1})\diamondsuit_\alpha t_{m+1}\right)^{\frac{1}{q}} \left(\int_r^d H^{q'}(t_{m+1})\diamondsuit_\alpha t_{m+1}\right)^{\frac{1}{q'}} \\ &= \left(\int_r^d \int_r^d ... \int_r^d \mid \mu(t_1,...,t_{m+1}) \mid\mid \Phi(t_1,...,t_{m+1}) \mid^q \diamondsuit_\alpha t_1 ...\diamondsuit_\alpha t_{m+1}\right)^{\frac{1}{q}} \\ &\times \left(\int_r^d \int_r^d ... \int_r^d \mid \mu(t_1,...,t_{m+1}) \mid\mid \Psi(t_1,...,t_{m+1}) \mid^{q'} \diamondsuit_\alpha t_1 ...\diamondsuit_\alpha t_{m+1}\right)^{\frac{1}{q'}}.\end{aligned}$$

This complete the proof.

□

Remark 1. *Taking $m = 2$, we result the inequality (2) [6].*

Put $\alpha = 1$, we get the m-tuple *Delta*-Hölder inequality as below.

Corollary 1. *Let $\mathbf{T}$ be a time scale, $r, d \in \mathbf{T}$ with $r < d$, $q > 1$, $m \ge 1$ and $\Phi, \Psi, \mu : [r, d]^m \to \mathbf{R}$ be Δ-integrable functions. Then*

$$\begin{aligned}&\int_r^d \int_r^d ... \int_r^d \mid \mu(t_1,...,t_m)\Phi(t_1,...,t_m)\Psi(t_1,...,t_m) \mid \Delta t_1 ...\Delta t_m \le \\ &\left(\int_r^d \int_r^d ... \int_r^d \mid \mu(t_1,...,t_m) \mid\mid \Phi(t_1,...,t_m) \mid^q \Delta t_1 ...\Delta t_m\right)^{\frac{1}{q}} \times \\ &\left(\int_r^d \int_r^d ... \int_r^d \mid \mu(t_1,...,t_m) \mid\mid \Psi(t_1,...,t_m) \mid^{q'} \Delta t_1 ...\Delta t_m\right)^{\frac{1}{q'}}.\end{aligned} \tag{5}$$

Remark 2. *Take $\alpha = 0$, we obtain the Hölder's m-tuple ∇ inequality.*

Let $\mathbf{T} = \mathbf{R}$, we get the Hölder's m-tuple inequality as follows.

Corollary 2. *Let $r, d \in \mathbf{R}$ with $r < d$, $q > 1$, $m \geq 1$ and $\Phi, \Psi, \mu : [r,d]^m \to \mathbf{R}$ be integrable functions, thus*

$$\begin{aligned} &\int_r^d \int_r^d ... \int_r^d \mid \mu(t_1,...,t_m)\Phi(t_1,...,t_m)\Psi(t_1,...,t_m) \mid dt_1 ... dt_m \leq \\ &\left(\int_r^d \int_r^d ... \int_r^d \mid \mu(t_1,...,t_m) \mid\mid \Phi(t_1,...,t_m) \mid^q dt_1 ... dt_m\right)^{\frac{1}{q}} \times \\ &\left(\int_r^d \int_r^d ... \int_r^d \mid \mu(t_1,...,t_m)\mid\mid\Psi(t_1,...,t_m) \mid^{q'} dt_1 ... dt_m\right)^{\frac{1}{q'}}. \end{aligned} \tag{6}$$

Set $\mu = 1$, we deduce the following Corollary.

Corollary 3. *Let $\mathbf{T}$ be a time scale, $r, d \in \mathbf{T}$ with $r < d$, $q > 1$, $m \geq 1$ and $\Phi, \Psi : [r,d]^m \to \mathbf{R}$ be $\diamondsuit_\alpha$-integrable functions, then*

$$\begin{aligned} &\int_r^d \int_r^d ... \int_r^d \mid \Phi(t_1,...,t_m)\Psi(t_1,...,t_m) \mid \diamondsuit_\alpha t_1 ... \diamondsuit_\alpha t_m \leq \\ &\left(\int_r^d \int_r^d ... \int_r^d \mid \Phi(t_1,...,t_m) \mid^q \diamondsuit_\alpha t_1 ... \diamondsuit_\alpha t_m\right)^{\frac{1}{q}} \times \\ &\left(\int_r^d \int_r^d ... \int_r^d \Psi(t_1,...,t_m) \mid^{q'} \diamondsuit_\alpha t_1 ... \diamondsuit_\alpha t_m\right)^{\frac{1}{q'}}. \end{aligned} \tag{7}$$

2.1 m-Tuple diamond-alpha Minkowski inequality

Lemma 3. *Let $\mathbf{T}$ be a time scale, $r, d \in \mathbf{T}$ with $r < d$, $q \geq 1$, $m \geq 1$ and $\Phi, \Psi, \mu : [r,d]^m \to \mathbf{R}$ be $\diamondsuit_\alpha$-integrable functions, then*

$$\begin{aligned} &\left(\int_r^d ... \int_r^d \mid \mu(t_1,...,t_m) \mid \Phi(t_1,...,t_m) + \Psi(t_1,...,t_m) \mid^q \diamondsuit_\alpha t_1 ... \diamondsuit_\alpha t_m\right)^{\frac{1}{q}} \leq \\ &\left(\int_r^d ... \int_r^d \mid \mu(t_1,...,t_m)\mid\mid\Phi(t_1,...,t_m) \mid^q \diamondsuit_\alpha t_1 ... \diamondsuit_\alpha t_m\right)^{\frac{1}{q}} + \\ &\left(\int_r^d ... \int_r^d \mid \mu(t_1,...,t_m) \mid\mid \Psi(t_1,...,t_m) \mid^q \diamondsuit_\alpha t_1 ... \diamondsuit_\alpha t_m\right)^{\frac{1}{q}}. \end{aligned} \tag{8}$$

The inequality (8) is reversed for $0 < q < 1$.

Proof. If $q = 1$, we use the triangle inequality. For $q > 1$, we denote $X = (t_1, ..., t_m)$, we have

$$\mid \mu(X) \mid\mid \Phi(X) + \Psi(X) \mid^q = \mid \mu(X) \mid \mid \Phi(X) + \Psi(X) \mid \mid \Phi(X) + \Psi(X) \mid^{q-1}$$

$$\leq \mid \mu(X) \mid \mid \Phi(X) \mid \mid \Phi(X) + \Psi(X) \mid^{q-1} + \mid \mu(X) \mid \mid \Psi(X) \mid \mid \Phi(X) + \Psi(X) \mid^{q-1},$$

apply Hölder's inequality (4), we get

$$\int_r^d \dots \int_r^d \mid \mu(X) \mid^{\frac{1}{q}} \mid \Phi(X) \mid\mid \mu(X) \mid^{\frac{1}{q'}} \mid \Phi(X) + \Psi(X) \mid^{q-1} \diamondsuit_\alpha t_1 \dots \diamondsuit_\alpha t_m$$
$$\leq \left(\int_r^d \dots \int_r^d \mid \mu(X) \mid\mid \Phi(X) + \Psi(X) \mid^{\frac{p-1}{q'}} \diamondsuit_\alpha t_1 \dots \diamondsuit_\alpha t_m \right)^{\frac{1}{q'}} \times$$
$$\left(\int_r^d \dots \int_r^d \mid \mu(X) \mid\mid \Phi(X) \mid^{q} \diamondsuit_\alpha t_1 \dots \diamondsuit_\alpha t_m \right)^{\frac{1}{q}}.$$

This gives us

$$\int_r^d \dots \int_r^d \mid \mu(X) \mid\mid \Phi(X) + \Psi(X) \mid^{q} \diamondsuit_\alpha t_1 \dots \diamondsuit_\alpha t_m \leq$$
$$\left(\int_r^d \dots \int_r^d \mid \mu(X) \mid\mid \Phi(X) + \Psi(X) \mid^{q} \diamondsuit_\alpha t_1 \dots \diamondsuit_\alpha t_m \right)^{\frac{1}{q'}}$$
$$\times \left[\left(\int_r^d \dots \int_r^d \mid \mu(X) \mid\mid \Phi(X) \mid^{q} \diamondsuit_\alpha t_1 \dots \diamondsuit_\alpha t_m \right)^{\frac{1}{q}} + \left(\int_r^d \dots \int_r^d \mid \mu(X) \mid\mid \Psi(X) \mid^{q} \diamondsuit_\alpha t_1 \dots \diamondsuit_\alpha t_m \right)^{\frac{1}{q}} \right],$$

therefore

$$\left(\int_r^d \dots \int_r^d \mid \mu(X) \mid\mid \Phi(X) + \Psi(X) \mid^{q} \diamondsuit_\alpha t_1 \dots \diamondsuit_\alpha t_m \right)$$
$$\times \left(\int_r^d \dots \int_r^d \mid \mu(X) \mid\mid \Phi(X) + \Psi(X) \mid^{q} \diamondsuit_\alpha t_1 \dots \diamondsuit_\alpha t_m \right)^{-\frac{1}{q'}} \leq$$
$$\left(\int_r^d \dots \int_r^d \mid \mu(X) \mid\mid \Phi(X) \mid^{q} \diamondsuit_\alpha t_1 \dots \diamondsuit_\alpha t_m \right)^{\frac{1}{q}} +$$
$$\left(\int_r^d \dots \int_r^d \mid \mu(X) \mid\mid \Psi(X) \mid^{q} \diamondsuit_\alpha t_1 \dots \diamondsuit_\alpha t_m \right)^{\frac{1}{q}},$$

since $1 - \frac{1}{q'} = \frac{1}{q}$, we result the inequality (8). □

Remark 3. • *Using $m = 1$, we get the weight inequality form for (1) [7].*

- *By setting $m = 2$, we get the following weight inequality. For $q \geq 1$, we have*

$$\left(\int_r^d \int_r^d \mid \mu(t_1, t_2) \mid \mid \Phi(t_1, t_2) + \Psi(t_1, t_2) \mid^q \diamondsuit_\alpha t_1 \diamondsuit_\alpha t_2\right)^{\frac{1}{q}} \leq$$

$$\left(\int_r^d \int_r^d \mid \mu(t_1, t_2) \mid \mid \Phi(t_1, t_2) \mid^q \diamondsuit_\alpha t_1 \diamondsuit_\alpha t_2\right)^{\frac{1}{q}} + \left(\int_r^d \int_r^d \mid \mu(t_1, t_2) \mid \mid \Psi(t_1, t_2) \mid^q \diamondsuit_\alpha t_1 \diamondsuit_\alpha t_2\right)^{\frac{1}{q}}. \quad (9)$$

Inequality (9) is reversed for $0 < q < 1$.

3 Diamond-alpha inequality with two parameters of summation

Now let us present the first result with two parameters of summation and also give some results relative to this theorem.

Theorem 3. *Let $\mathbf{T}$ be a time scale, $r, d \in \mathbf{T}$ with $r < d$, $m \geq 1$ and $\Phi, \Psi : [r, d]^m \to \mathbf{R}$ be $\diamondsuit_\alpha$-integrable functions. If $0 < q_1 \leq q_2 < \infty$, the inequality*

$$\left(\int_r^d ... \int_r^d \mid \Phi(t_1, ..., t_m) \mid^{q_1} \mid \Psi(t_1, ..., t_m) \mid \diamondsuit_\alpha t_1 ... \diamondsuit_\alpha t_m\right)^{\frac{1}{q_1}} \leq \left(\int_r^d ... \int_r^d \mid \Psi(t_1, ..., t_m) \mid \diamondsuit_\alpha t_1 ... \diamondsuit_\alpha t_m\right)^{\frac{q_1 - q_2}{q_1 q_2}} \times \left(\int_r^d ... \int_r^d \mid \Phi(t_1, ..., t_m) \mid^{q_2} \mid \Psi(t_1, ..., t_m) \mid \diamondsuit_\alpha t_1 ... \diamondsuit_\alpha t_m\right)^{\frac{1}{q_2}}, \quad (10)$$

holds if the right side is finite.
The inequality (10) holds for $-\infty < q_2 \leq q_1 < 0$ and is reversed for $0 < q_2 \leq q_1 < \infty$.

Proof. Let $0 < q_1 \leq q_2 < \infty$, for $q_1 = q_2$ It has equality. Suppose that $q_1 \neq q_2$, then apply Hölder's m-tuple diamond-alpha inequality (7) for $\dfrac{q_2}{q_1} > 1$, we get

$$\int_r^d ... \int_r^d \mid \Phi(t_1, ..., t_m) \mid^{q_1} \mid \Psi(t_1, ..., t_m) \mid \diamondsuit_\alpha t_1 ... \diamondsuit_\alpha t_m$$

$$= \int_r^d ... \int_r^d \mid \Psi(X) \mid^{\frac{q_2 - q_1}{q_2}} \mid \Phi^{q_1}(X) \mid\mid \Psi(X) \mid^{\frac{q_1}{q_2}} \diamondsuit_\alpha t_1 ... \diamondsuit_\alpha t_m$$

$$\leq \left(\int_r^d ... \int_r^d \mid \Psi(t_1, ..., t_m) \mid \diamondsuit_\alpha t_1 ... \diamondsuit_\alpha t_m\right)^{\frac{q_2 - q_1}{q_2}} \times \left(\int_r^d ... \int_r^d \mid \Phi(t_1, ..., t_m) \mid^{q_2} \mid \Psi(t_1, ..., t_m) \mid \diamondsuit_\alpha t_1 ... \diamondsuit_\alpha t_m\right)^{\frac{q_1}{q_2}}.$$

□

Setting $\alpha = 1$, then we obtain the m-tuple Δ-integral inequality with two parameters q_1 and q_2 as follows.

Corollary 4. *Let $\mathbf{T}$ be a time scale, $r, d \in \mathbf{T}$ with $r < d$, $m \geq 1$ and $\Phi, \Psi : [r, d]^n \to \mathbf{R}$ be Δ-integrable functions. If $0 < q_1 \leq q_2 < \infty$, then*

$$\left(\int_r^d ... \int_r^d \mid \Phi(t_1, ..., t_m) \mid^{q_1} \mid \Psi(t_1, ..., t_m) \mid \Delta t_1 ... \Delta t_m\right)^{\frac{1}{q_1}} \leq \left(\int_a^b ... \int_a^b \mid \Psi(x_1, ..., x_n) \mid \Delta t_1 ... \Delta t_m\right)^{\frac{q_2 - q_1}{q_1} q_2} \left(\int_r^d ... \int_r^d \mid \Phi(t_1, ..., t_m) \mid^{q_2} \mid \Psi(t_1, ..., t_m) \mid \Delta t_1 ... \Delta t_m\right)^{\frac{1}{q_2}}. \tag{11}$$

The inequality (11) holds for $-\infty < q_2 \leq q_1 < 0$ and is reversed for $0 < q_2 \leq q_1 < \infty$.

Remark 4. *Putting $\alpha = 0$, we obtain the version of m-tuple ∇-integral inequality for two parameters q_1 and q_2.*

Let $\mathbf{T} = \mathbf{R}$, we get the m-tuple integral inequality as follows.

Corollary 5. *Let $a, b \in \mathbf{R}$ with $a < b$, $n \geq 1$ and $\Phi, \Psi : [a, b]^n \to \mathbf{R}$ be integrable functions. If $0 < p \leq q < \infty$, the inequality*

$$\left(\int_r^d ... \int_r^d \mid \Phi(t_1, ..., t_m) \mid^{q_1} \mid \Psi(t_1, ..., t_m) \mid dt_1 ... dt_m\right)^{\frac{1}{q_1}} \leq \left(\int_r^d ... \int_r^d \mid \Psi(t_1, ..., t_m) \mid dt_1 ... dt_m\right)^{\frac{q_2 - q_1}{q_1 q_2}} \times \left(\int_r^d ... \int_r^d \mid \Phi(t_1, ..., t_m) \mid^{q_2} \mid \Psi(t_1, ..., t_m) \mid dt_1 ... dt_m\right)^{\frac{1}{q_2}}. \tag{12}$$

The inequality (12) holds for $-\infty < q_2 \leq q_1 < 0$ and reversed for $0 < q_2 \leq q_1 < \infty$.

3.1 Diamond-alpha Minkowski inequality with two parameters of summation

Lemma 4. *Let* $\mathbf{T}$ *be a time scale,* $r, d \in \mathbf{T}$ *with* $r < d$*,* $m \geq 1$ *and* $\Phi, \Psi, \mu : [r, d]^m \to \mathbf{R}$ *be* $\diamondsuit_\alpha$*-integrable functions. If* $1 \leq q_1 \leq q_2 < \infty$*, the inequality*

$$\left(\int_r^d ... \int_r^d \mid \mu(t_1, ..., t_m) \mid \mid \Phi(t_1, ..., t_m) + \Psi(t_1, ..., t_m) \mid^{q_1} \diamondsuit_\alpha t_1 ... \diamondsuit_\alpha t_m\right)^{\frac{1}{q_1}} \leq \left(\int_a^b ... \int_a^b \mid \mu(x_1, ..., x_n) \mid \diamondsuit_\alpha t_1 ... \diamondsuit_\alpha t_m\right)^{\frac{q_2 - q_1}{q_1 q_2}} \times \left[\left(\int_r^d ... \int_r^d \mid \mu(t_1, ..., t_m) \mid \mid \Phi(t_1, ..., t_m) \mid^{q_2} \diamondsuit_\alpha t_1 ... \diamondsuit_\alpha t_m\right)^{\frac{1}{q_2}} + \left(\int_r^d ... \int_r^d \mid \mu(t_1, ..., t_m) \mid \mid \Psi(t_1, ..., t_m) \mid^{q_2} \diamondsuit_\alpha t_1 ... \diamondsuit_\alpha t_m\right)^{\frac{1}{q_2}}\right], \tag{13}$$

holds if the right side is finite.
The inequality (13) is reversed for $0 < q_2 < q_1 \leq 1$*.*

Proof. By using the inequalities (10) and (8), we result the inequality (13). □

3.2 New Minkowski's inequality in time scales

Putting $q_1 = 1$ and $q_2 = q$ in the inequality (13), we obtain a new weighted Minkowski inequality in time scales.

Corollary 6. *Let* $\mathbf{T}$ *be a time scale,* $r, d \in \mathbf{T}$ *with* $r < d$*,* $m \geq 1$ *and* $\Phi, \Psi, \mu : [r, d]^m \to \mathbf{R}$ *be* $\diamondsuit_\alpha$*-integrable functions.*
If $1 \leq q < \infty$*, the inequality*

$$\int_r^d ... \int_r^d \mid \mu(t_1, ..., t_m) \mid \mid \Phi(t_1, ..., t_m) + \Psi(t_1, ..., t_m) \mid \diamondsuit_\alpha t_1 ... \diamondsuit_\alpha t_m \leq \left(\int_r^d ... \int_r^d \mid \mu(t_1, ..., t_m) \mid \diamondsuit_\alpha t_1 ... \diamondsuit_\alpha t_m\right)^{\frac{q-1}{q}} \times \left[\left(\int_r^d ... \int_r^d \mid \mu(t_1, ..., t_m) \mid \mid \Phi(t_1, ..., t_m) \mid^{q} \diamondsuit_\alpha t_1 ... \diamondsuit_\alpha t_m\right)^{\frac{1}{q}} + \left(\int_r^d ... \int_r^d \mid \mu(t_1, ..., t_m) \mid \mid \Psi(t_1, ..., t_m) \mid^{q} \diamondsuit_\alpha t_1 ... \diamondsuit_\alpha t_m\right)^{\frac{1}{q}}\right]. \tag{14}$$

The inequality (14) is reversed for $0 < q < 1$*.*

Set $\mu = 1$ in the inequality (14), we get a new Minkowski's inequality in time scales.

Corollary 7. *Let $\mathbf{T}$ be a time scale, $r, d \in \mathbf{T}$ with $-\infty < r < d < +\infty$, $m \geq 1$ and $\Phi, \Psi : [r, d]^m \to \mathbf{R}$ be $\diamondsuit_\alpha$-integrable functions.*
If $1 \leq q < \infty$, the inequality

$$\int_r^d \dots \int_r^d \mid \Phi(t_1, ..., t_m) + \Psi(t_1, ..., t_m) \mid \diamondsuit_\alpha t_1 \dots \diamondsuit_\alpha t_m \leq (d-r)^{m(\frac{q-1}{q})} \times \Bigg[\left(\int_r^d \dots \int_r^d \mid \Phi(t_1, ..., t_m) \mid^q \diamondsuit_\alpha t_1 \dots \diamondsuit_\alpha t_m \right)^{\frac{1}{q}} + \left(\int_r^d \dots \int_r^d \mid \Psi(t_1, ..., t_m) \mid^q \diamondsuit_\alpha t_1 \dots \diamondsuit_\alpha t_m \right)^{\frac{1}{q}} \Bigg]. \quad (15)$$

The inequality (15) is reversed for $0 < q < 1$.

4 Integral inequality with one variable

Now we present some interesting cases of the Theorem 3.

Lemma 5. *Let $\mathbf{T}$ be a time scale, $r, d \in \mathbf{T}$ with $r < d$ and $\Phi, \Psi : [r, d] \to \mathbf{R}$ be $\diamondsuit_\alpha$-integrable functions. If $0 < q_1 \leq q_2 < \infty$, then*

$$\left(\int_r^d \mid \Psi(t) \mid\mid \Phi(t) \mid^{q_1} \diamondsuit_\alpha t \right)^{\frac{1}{q_1}} \leq \left(\int_r^d \mid \Psi(t) \mid \diamondsuit_\alpha t \right)^{\frac{q_2 - q_1}{q_1 q_2}} \left(\int_r^d \mid \Psi(t) \mid\mid \Phi(t) \mid^{q_2} \diamondsuit_\alpha t \right)^{\frac{1}{q_2}} (16)$$

The inequality (16) holds for $-\infty < q_2 \leq q_1 < 0$ and is reversed for $0 < q_2 \leq q_1 < \infty$.

Remark 5. *The above integral inequality (16) coincide with the integral inequality (3.1) in [2].*

Taking $\alpha = 1$, we get the version of Δ-integral inequality as follows.

Corollary 8. *Let $\mathbf{T}$ be a time scale, $r, d \in \mathbf{T}$ with $r < d$ and $\Phi, \Psi : [r, d] \to \mathbf{R}$ be Δ-integrable functions. If $0 < q_1 \leq q_2 < \infty$, then*

$$\left(\int_r^d \mid \Psi(t) \mid\mid \Phi(t) \mid^{q_1} \Delta t \right)^{\frac{1}{q_1}} \leq \left(\int_r^d \mid \Psi(t) \mid \Delta t \right)^{\frac{q_2 - q_1}{q_1 q_2}} \left(\int_r^d \mid \Psi(t) \mid \Phi(t) \mid^{q_2} \Delta t \right)^{\frac{1}{q_2}}. \quad (17)$$

The inequality (17) holds for $-\infty < q_2 \leq q_1 < 0$ and reversed for $0 < q_2 \leq q_1 < \infty$.

Remark 6. *The above integral inequality (17) coincide with the integral inequality (3.3) in [4].*

Let $\mathbf{T} = \mathbf{R}$, we get the following integral inequality.

Corollary 9. *Let $r, d \in \mathbf{R}$ with $r < d$ and $\Phi, \Psi : [r, d] \to \mathbf{R}$ be integrable functions. If $0 < q_1 \leq q_2 < \infty$, then*

$$\left(\int_r^d \mid \Psi(t) \mid\mid \Phi(t) \mid^{q_1} dt\right)^{\frac{1}{q_1}} \leq \left(\int_r^d \mid \Psi(t) \mid dt\right)^{\frac{q_2-q_1}{q_1 q_2}} \left(\int_r^d \mid \Psi(t) \mid\mid \Phi(t) \mid^{q_2} dt\right)^{\frac{1}{q_2}}. \tag{18}$$

The inequality (18) holds for $-\infty < q_2 \leq q_1 < 0$ and reversed for $0 < q_2 \leq q_1 < \infty$.

Remark 7. *The inequality (18) coincide with the inequalities in [1], [3].*

4.1 Diamond-alpha Minkowski inequality with two parameters of summation

Setting $m = 1$ in the Lemma 4, we get the following Corollary.

Corollary 10. *Let $\mathbf{T}$ be a time scale, $r, d \in \mathbf{T}$ with $r < d$ and $\Phi, \Psi, \mu : [r, d] \to \mathbf{R}$ be $\diamondsuit_\alpha$-integrable functions. If $1 \leq q_1 \leq q_2 < \infty$, thus*

$$\begin{gathered}\left(\int_r^d \mid \mu(t) \mid \mid \Phi(t) + \Psi(t) \mid^{q_1} \diamondsuit_\alpha t\right)^{\frac{1}{q_1}} \leq \left(\int_r^d \mid \mu(t) \mid \diamondsuit_\alpha t\right)^{\frac{q_2-q_1}{q_2 q_2}} \times \\ \left[\left(\int_r^d \mid \mu(t) \mid \mid \Phi(t) \mid^{q_2} \diamondsuit_\alpha x\right)^{\frac{1}{q_2}} + \left(\int_r^d \mid \mu(t) \mid \mid \Psi(t) \mid^{q_2} \diamondsuit_\alpha x\right)^{\frac{1}{q_2}}\right].\end{gathered} \tag{19}$$

The inequality (19) is reversed for $0 < q_2 < q_1 \leq 1$.

We present some remarks about the inequality (19).

1. Taking $q_1 = 1$ and $q_2 = q$, we deduce a new weight Minkowski's inequality in time scales for $1 \leq q < \infty$

$$\begin{gathered}\int_r^d \mid \mu(t) \mid \mid \Phi(t) + \Psi(t) \mid \diamondsuit_\alpha t \leq \left(\int_r^d \mid \mu(t) \mid \diamondsuit_\alpha t\right)^{\frac{q-1}{q}} \times \\ \left[\left(\int_r^d \mid \mu(t) \mid \mid \Phi(t) \mid^{q} \diamondsuit_\alpha t\right)^{\frac{1}{q}} + \left(\int_r^d \mid \mu(t) \mid \mid \Psi(t) \mid^{q} \diamondsuit_\alpha t\right)^{\frac{1}{q}}\right].\end{gathered} \tag{20}$$

   The inequality (20) is reversed for $0 < q < 1$.

2. Taking $\mu = 1$ and $-\infty < r < d < +\infty$, thus gives us for $1 \leq q_1 \leq q_2 < \infty$

$$\begin{gathered}\left(\int_r^d \mid \Phi(t) + \Psi(t) \mid^{q_1} \diamondsuit_\alpha t\right)^{\frac{1}{q_1}} \leq (d-r)^{\frac{q_2-q_1}{q_1 q_2}} \times \\ \left[\left(\int_r^d \mid \Phi(t) \mid^{q_2} \diamondsuit_\alpha t\right)^{\frac{1}{q_2}} + \left(\int_r^d \mid \Psi(t) \mid^{q_2} \diamondsuit_\alpha t\right)^{\frac{1}{q_2}}\right].\end{gathered} \tag{21}$$

   The inequality (21) is reversed for $0 < q_2 < q_1 \leq 1$.

3. Putting $q_1 = 1$ and $q_2 = q$ in the inequality (21), we get a new refinement to Minkowski's inequality in time scales. For $1 \leq q$

$$\int_r^d |\ \Phi(t) + \Psi(t)\ |\ \diamondsuit_\alpha t$$
$$\leq \ (d-r)^{\frac{q-1}{q}} \left[\left(\int_r^d |\ \Phi(t)\ |^q\ \diamondsuit_\alpha t \right)^{\frac{1}{q}} + \left(\int_r^d |\ \Psi(t)\ |^q\ \diamondsuit_\alpha t \right)^{\frac{1}{q}} \right]. \quad (22)$$

The inequality (22) is reversed for $0 < q < 1$.

4. For $\mathbf{T} = \mathbf{R}$, we get for $1 \leq q$

$$\int_r^d |\ \Phi(t) + \Psi(t)\ |\ dt$$
$$\leq \ (d-r)^{\frac{q-1}{q}} \left[\left(\int_r^d |\ \Phi(t)\ |^q\ dt \right)^{\frac{1}{q}} + \left(\int_r^d |\ \Psi(t)\ |^q\ dt \right)^{\frac{1}{q}} \right]. \quad (23)$$

The inequality (23) is reversed for $0 < q < 1$.

5 Integral inequality with two variables

From the Theorem 3, we get the following results.

Lemma 6. *Let $\mathbf{T}$ be a time scale, $r, d \in \mathbf{T}$ with $r < d$ and $\Phi, \Psi : [r,d] \times [r,d] \to \mathbf{R}$ be $\diamondsuit_\alpha$-integrable functions. If $0 < q_1 \leq q_2 < \infty$, then*

$$\left(\int_r^d \int_r^d |\ \Phi(t_1,t_2)\ |^{q_1} |\ \Psi(t_1,t_2)\ |\ \diamondsuit_\alpha t_1 \diamondsuit_\alpha t_2 \right)^{\frac{1}{q_1}}$$
$$\leq \ \left(\int_r^d \int_r^d |\ \Psi(t_1,t_2)\ |\ \diamondsuit_\alpha t_1 \diamondsuit_\alpha t_2 \right)^{\frac{q_2-q_1}{q_1 q_2}} \times$$
$$\times \left(\int_r^d \int_r^d |\ \Phi(t_1,t_2)\ |^{q_2} |\ \Psi(t_1,t_2)\ |\ \diamondsuit_\alpha t_1 \diamondsuit_\alpha t_2 \right)^{\frac{1}{q_2}}. \quad (24)$$

The inequality (24) holds for $-\infty < q_2 \leq q_1 < 0$ and reversed for $0 < q_2 \leq q_1 < \infty$.

Put $\alpha = 1$, thus we obtain the version of Δ-integral inequality as follows.

Corollary 11. *Let $\mathbf{T}$ be a time scale, $r, d \in \mathbf{T}$ with $r < d$ and $\Phi, \Psi : [r,d] \times [r,d] \to \mathbf{R}$ be Δ-integrable functions. If $0 < q_1 \leq q_2 < \infty$, then*

$$\left(\int_r^d \int_r^d |\ \Phi(t_1,t_2)\ |^{q_1} |\ \Psi(t_1,t_2)\ |\ \Delta t_1 \Delta t_2 \right)^{\frac{1}{q_1}} \leq$$
$$\left(\int_r^d \int_r^d |\ \Psi(t_1,t_2)\ |\ \Delta t_1 \Delta t_2 \right)^{\frac{q_2-q_1}{q_1 q_2}} \left(\int_r^d \int_r^d |\ \Phi(t_1,t_2)\ |^{q_2} |\ \Psi(t_1,t_2)\ |\ \Delta t_1 \Delta t_2 \right)^{\frac{1}{q_2}}. \quad (25)$$

The inequality (25) holds for $-\infty < q_2 \leq q_1 < 0$ *and reversed for* $0 < q_2 \leq q_1 < \infty$.

Let $\mathbf{T} = \mathbf{R}$, we deduce the following integral inequality.

Corollary 12. *Let* $r, d \in \mathbf{R}$ *with* $r < d$ *and* $\Phi, \Psi : [r, d] \times [r, d] \to \mathbf{R}$ *be integrable functions. If* $0 < q_1 \leq q_2 < \infty$*, then*

$$\left(\int_r^d \int_r^d \mid \Phi(t_1, t_2) \mid^{q_1} \mid \Psi(t_1, t_2) \mid dt_1 dt_2 \right)^{\frac{1}{q_1}} \leq$$
$$\left(\int_r^d \int_r^d \mid \Psi(t_1, t_2) \mid dt_1 dt_2 \right)^{\frac{q_2 - q_1}{q_1 q_2}} \left(\int_r^d \int_r^d \mid \Phi(t_1, t_2) \mid^{q_2} \mid \Psi(t_1, t_2) \mid dt_1 dt_2 \right)^{\frac{1}{q_2}}. \tag{26}$$

The inequality (26) holds for $-\infty < q_2 \leq q_1 < 0$ *and reversed for* $0 < q_2 \leq q_1 < \infty$.

Proposition 1. *Taking* $\Psi = 1$ *in inequalities (16) and (24), we obtain simple and recently diamond-alpha inequalities with two parameters of summation. For* $0 < q_1 \leq q_2 < \infty$*, then*

$$\left(\int_r^d \mid \Phi(t) \mid^{q_1} \diamondsuit_\alpha t \right)^{\frac{1}{q_1}} \leq (d - r)^{\frac{q_2 - q_1}{q_1 q_2}} \left(\int_r^d \mid \Phi(t) \mid^{q_2} \diamondsuit_\alpha t \right)^{\frac{1}{q_2}}. \tag{27}$$

$$\left(\int_r^d \int_r^d \mid \Phi(t_1, t_2) \mid^{q_1} \diamondsuit_\alpha t_1 \diamondsuit_\alpha t_2 \right)^{\frac{1}{q_1}}$$
$$\leq \quad (d - r)^{2(\frac{q_2 - q_1}{q_1 q_2})} \left(\int_r^d \int_r^d \mid \Phi(t_1, t_2) \mid^{q_2} \diamondsuit_\alpha t_1 \diamondsuit_\alpha t_2 \right)^{\frac{1}{q_2}}. \tag{28}$$

Inequalities (27) and (28) hold for $-\infty < q_2 \leq q_1 < 0$ *and reversed for* $0 < q_2 \leq q_1 < \infty$.

Proposition 2. *Setting* $\mathbf{T} = \mathbf{R}$*, we get the following classical integral inequalities. For* $0 < q_1 \leq q_2 < \infty$ *or* $-\infty < q_2 \leq q_1 < 0$

$$\left(\int_r^d \mid \Phi(t) \mid^{q_1} dt \right)^{\frac{1}{q_1}} \leq (d - r)^{\frac{q_2 - q_1}{q_1 q_2}} \left(\int_r^d \mid \Phi(t) \mid^{q_2} dt \right)^{\frac{1}{q_2}}, \tag{29}$$

$$\left(\int_r^d \int_r^d \mid \Phi(t_1, t_2) \mid^{q_1} dt_1 dt_2 \right)^{\frac{1}{q_1}} \leq (d - r)^{2(\frac{q_2 - q_1}{q_1 q_2})} \left(\int_r^b \int_r^d \mid \Phi(t_1, t_2) \mid^{q_2} dt_1 dt_2 \right)^{\frac{1}{q_2}}. \tag{30}$$

The above inequalities are reversed for $0 < q_2 \leq q_2 < \infty$.

5.1 Double integral Minkowski's Diamond-alpha inequality with two parameters

Setting $m = 2$ in Lemma 4, we get the following Corollary.

Corollary 13. *Let* $\mathbf{T}$ *be a time scale,* $r, d \in \mathbf{T}$ *with* $r < d$ *and* $\Phi, \Psi, \mu : [r, d] \times [r, d] \to \mathbf{R}$ *be* $\Diamond_\alpha$*-integrable functions. If* $1 \le q_1 \le q_2 < \infty$*, then*

$$\begin{aligned} &\left(\int_r^d \int_r^d \mid \mu(t,y) \mid \mid \Phi(t,y) + \Psi(t,y) \mid^{q_1} \Diamond_\alpha t \Diamond_\alpha y\right)^{\frac{1}{q_1}} \\ \le\ & \left(\int_r^d \int_r^d \mid \mu(t,y) \mid \Diamond_\alpha t \Diamond_\alpha y\right)^{\frac{q_2-q_1}{q_1 q_2}} \\ &\times \left[\left(\int_r^d \int_r^d \mid \mu(t,y) \mid \mid \Phi(t,y) \mid^{q_2} \Diamond_\alpha t \Diamond_\alpha y\right)^{\frac{1}{q_2}} \right. \\ &\left. + \left(\int_r^d \int_r^d \mid \mu(t,y) \mid \mid \Psi(t,y) \mid^{q_2} \Diamond_\alpha t \Diamond_\alpha y\right)^{\frac{1}{q_2}}\right]. \end{aligned} \tag{31}$$

The inequality (31) is reversed for $0 < q_2 < q_1 \le 1$.

We present some interesting results concerning inequality (31).

1. Set $q_1 = 1$ and $q_2 = q$, we get weight Minkowski's inequality in time scales for $1 \le q < \infty$

$$\begin{aligned} &\int_r^d \int_r^d \mid \mu(t,y) \mid \mid \Phi(t,y) + \Psi(t,y) \mid \Diamond_\alpha t \Diamond_\alpha y \\ \le\ & \left(\int_r^d \int_r^d \mid \mu(t,y) \mid \Diamond_\alpha t \Diamond_\alpha y\right)^{\frac{q-1}{q}} \\ &\times \left[\left(\int_r^d \int_r^d \mid \mu(t,y) \mid \mid \Phi(t,y) \mid^{q} \Diamond_\alpha t \Diamond_\alpha y\right)^{\frac{1}{q}} \right. \\ &\left. + \left(\int_r^d \int_r^d \mid \mu(t,y) \mid \mid \Psi(t,y) \mid^{q} \Diamond_\alpha t \Diamond_\alpha y\right)^{\frac{1}{q}}\right]. \end{aligned} \tag{32}$$

The inequality (32) is reversed for $0 < q < 1$.

2. Setting $\mu = 1$ and $-\infty < r < d < +\infty$, we deduce that for $1 \le q_1 \le q_2 < \infty$

$$\begin{aligned} &\left(\int_r^d \int_r^d \mid \Phi(t,y) + \Psi(t,y) \mid^{q_1} \Diamond_\alpha t \Diamond_\alpha y\right)^{\frac{1}{q_1}} \le (d-r)^{2\left(\frac{q_2-q_1}{q_1 q_2}\right)} \\ \times &\left[\left(\int_r^d \int_r^d \mid \Phi(t,y) \mid^{q_2} \Diamond_\alpha t \Diamond_\alpha y\right)^{\frac{1}{q_2}} + \left(\int_r^d \int_r^d \mid \Psi(t,y) \mid^{q_2} \Diamond_\alpha t \Diamond_\alpha y\right)^{\frac{1}{q_2}}\right]. \end{aligned} \tag{33}$$

The inequality (33) is reversed for $0 < q_2 < q_1 \le 1$.

3. Taking $p = 1$ and $q_2 = q$ in the inequality (33), we obtain a refinement to Minkowski's inequality in time scales. For $1 \leq q < \infty$

$$\int_r^d \int_r^d \mid \Phi(t,y) + \Psi(t,y) \mid \diamondsuit_\alpha t \diamondsuit_\alpha y \leq (d-r)^{2\left(\frac{q-1}{q}\right)} \times \left[\left(\int_r^d \int_r^d \mid \Phi(t,y) \mid^q \diamondsuit_\alpha t \diamondsuit_\alpha y \right)^{\frac{1}{q}} + \left(\int_r^d \int_r^d \mid \Psi(t,y) \mid^q \diamondsuit_\alpha t \diamondsuit_\alpha y \right)^{\frac{1}{q}} \right]. \tag{34}$$

Inequality (34) is reversed for $0 < q < 1$.

4. Putting $\mathbf{T} = \mathbf{R}$, we get for $1 \leq q < \infty$

$$\int_r^d \int_r^d \mid \Phi(t,y) + \Psi(t,y) \mid dt\, dy \leq (d-r)^{2\left(\frac{q-1}{q}\right)} \times \left[\left(\int_r^d \int_r^d \mid \Phi(t,y) \mid^q dt\, dy \right)^{\frac{1}{q}} + \left(\int_r^d \int_r^d \mid \Psi(t,y) \mid^q dt\, dy \right)^{\frac{1}{q}} \right]. \tag{35}$$

Inequality (35) is reversed for $0 < q < 1$.

6 Conclusion

In this paper, using the n-tuple diamond-alpha Hölder inequality and the Minkowski inequality on time scales, we have established Diamond-alpha Hölder and Diamond-alpha Minkowski inequalities with two summation parameters on time scale, in additionally a new refinement of the Minkowski inequality on time scales with one and two variables. We present the delta, nabla and continuous inequalities as special cases of our main results.

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