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CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS WITH COMPLEX ORDER ASSOCIATED WITH GENERALIZED BESSEL FUNCTIONS

T. AL-HAWARY[1], A. AMOURAH[2], M. K. AOUF[3] and B. A. FRASIN[*,4]

Abstract

In this paper we obtain the necessary and sufficient conditions for generalized Bessel functions of the first kind $zu_p(z)$ to be in the classes $S(b,\lambda,\beta)$ and $R(b,\lambda,\beta)$ of analytic functions with complex order and also give the necessary and sufficient conditions for $z(2-u_p(z))$ to be in the classes $TS(b,\lambda,\beta)$ and $TR(b,\lambda,\beta)$. Furthermore, we give the necessary and sufficient conditions for $J(k,c)$ to be in the class $TR(b,\lambda,\beta)$ provided that the function f is in the class $R^{\tau}(A,B)$. Finally, we give conditions for the integral operator $G(k,c,z)=\int_0^z(2-u_p(t))dt$ to be in the class $TR(b,\lambda,\beta)$. Several corollaries and consequences of the main results are also obtained.

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1 Introduction

Bessel functions are needful in many branches of applied mathematics and mathematical physics, for example, those in acoustics, angular resolution, radio physics, hydrodynamics, and signal processing. Therefore, these special functions have been studied extensively, see [6, 9, 10, 11, 18, 20, 26, 27].

[1]Department of Applied Science, Ajloun College, Al-Balqa Applied University, Ajloun 26816. Jordan, e-mail: tariq-amh@bau.edu.jo
[2]Department of Mathematics, Faculty of Science and Technology, Irbid National University, Irbid, Jordan, e-mail: alaammour@yahoo.com
[3]Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt, e-mail: mkaouf127@yahoo.com
[4*] *Corresponding author*, Faculty of Science, Department of Mathematics, Al al-Bayt University, Mafraq, Jordan, e-mail: bafrasin@yahoo.com

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$. Further, let T be the subclass of $\mathcal{A}$ consisting of functions of the form:

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad z \in \mathcal{U}. \tag{2}$$

A function $f \in \mathcal{A}$ is said to be starlike of complex order $b(b \in \mathbb{C}^* = \mathbb{C} \diagdown \{0\})$, that is $f \in S(b)$ if it satisfies the inequality

$$\operatorname{Re}\left\{1 + \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right)\right\} > 0, \ \ (z \in \mathcal{U},\, b \in \mathbb{C}^*) \tag{3}$$

Also, a function $f \in \mathcal{A}$ is said to be convex of complex order $b(b \in \mathbb{C}^*)$, that is $f \in C(b)$ if it satisfies the inequality

$$\operatorname{Re}\left\{1 + \frac{1}{b}\frac{zf''(z)}{f'(z)}\right\} > 0 \quad (z \in \mathcal{U},\, b \in \mathbb{C}^*). \tag{4}$$

The class $S(b)$ was introduced and studied by Nasr and Aouf [**?**], (see also [30]) and the class $C(b)$ was introduced and studied by Wiatrowski [34], (see also [4] and [29]).

Furthermore a function $f \in \mathcal{A}$ is said to be close-to-convex of complex order $b(b \in \mathbb{C}^*)$ if it satisfies the inequality

$$\operatorname{Re}\left\{1 + \frac{1}{b}\left(f'(z) - 1\right)\right\} > 0 \quad (z \in \mathcal{U},\, b \in \mathbb{C}^*). \tag{5}$$

The class $R(b)$ was introduced and studied by Halim [22] and Owa [31] (see Aouf and Mostafa [5]).

Finally, let $S(b, \lambda, \beta)$ denote the class of functions $f \in \mathcal{A}$ which satisfies the inequality

$$\left|\frac{1}{b}\left(\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)f(z) + \lambda f'(z)} - 1\right)\right| < \beta, \quad (z \in \mathcal{U},\, b \in \mathbb{C}^*, 0 < \beta \leq 1, 0 \leq \lambda \leq 1). \tag{6}$$

Also let $R(b, \lambda, \beta)$ denote the class of functions $f \in \mathcal{A}$ which satisfies the inequality

$$\left|\frac{1}{b}\left(f'(z) + \lambda z f''(z) - 1\right)\right| < \beta, \quad (z \in \mathcal{U}, b \in \mathbb{C}^*, 0 < \beta \leq 1, 0 \leq \lambda \leq 1). \tag{7}$$

The classes $S(b, \lambda, \beta)$ and $R(b, \lambda, \beta)$ was introduced and studied by Altintas et al. [[2] with $n = 1$].

We note that:

(1) $S(b,0,1) \subset S(b)$, $S(b,1,1) \subset C(b)$ and $R(b,0,1) \subset R(b)$,
(2) $S(b,0,\beta) = S(b,\beta) = \left\{ f \in \mathcal{A} \colon \left| \frac{1}{b}\left(\frac{zf'(z)}{f(z)} - 1\right)\right| < \beta, \quad z \in \mathcal{U} \right\}$,
(3) $S(b,1,\beta) = C(b,\beta) = \left\{ f \in \mathcal{A} \colon \left| \frac{1}{b}\left(\frac{zf''(z)}{f'(z)} - 1\right)\right| < \beta, \quad z \in \mathcal{U} \right\}$,
(4) $R(b,0,\beta) = R(b,\beta) = \left\{ f \in \mathcal{A} \colon \left| \frac{1}{b}\left(f'(z) - 1\right)\right| < \beta, \quad z \in \mathcal{U} \right\}$.

Further, we define the classes $TS(b,\lambda,\beta)$ and $TR(b,\lambda,\beta)$ by
$TS(b,\lambda,\beta) = S(b,\lambda,\beta) \cap T$ and $TR(b,\lambda,\beta) = R(b,\lambda,\beta) \cap T$.
We note that $TS(1,\lambda,\beta) = TS(\lambda,\beta)$ (see [2]).

A function $f \in \mathcal{A}$ is said to be in the class $R^{\tau}(A,B), \tau \in \mathbb{C}^*$ and $-1 \leq B \leq A \leq 1$ if it satisfies the inequality

$$\left| \frac{f'(z) - 1}{(A-B)\tau - B[f'(z) - 1]} \right| < 1, \quad z \in \mathcal{U}. \tag{8}$$

The class $R^{\tau}(A,B)$ was introduced and studied by Dixit and Pal [14].

The generalized Bessel function w_p (see, [9]) is defined as a particular solution of the linear differential equation

$$z^2 w''(z) + bzw'(z) + [cz^2 - p^2 + (1-b)p]w(z) = 0, \tag{9}$$

where $b, p, c \in \mathbb{C}$. The analytic function w_p has the form:

$$w_p(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (c)^n}{n!\Gamma(p + n + \frac{b+1}{2})} \cdot \left(\frac{z}{2}\right)^{2n+p}, \qquad z \in \mathbb{C}. \tag{10}$$

Now, the generalized and normalized Bessel function u_p is defined with the transformation

$$\begin{aligned} u_p(z) &= 2^p \Gamma(p + \frac{b+1}{2}) z^{-p/2} w_p(z^{1/2}) \\ &= \sum_{n=0}^{\infty} \frac{(-c/4)^n}{(k)_n n!} z^n, \end{aligned} \tag{11}$$

where $k = p + (b+1)/2 \neq 0, -1, -2, \ldots$ and $(a)_n$ is the well-known Pochhammer symbol, defined in terms of the Euler Gamma function for $a \neq 0, -1, -2, \ldots$ by

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & \text{if } n = 0 \\ a(a+1)(a+2)\ldots(a+n-1), & \text{if } n \in \mathbb{N} = \{1, 2, \cdots\}. \end{cases}$$

The function u_p is analytic on $\mathbb{C}$ and satisfies the second linear differential equation

$$4z^2 u''(z) + 2(2p + b + 1)zu'(z) + czu(z) = 0. \tag{12}$$

Using the Hadamard product, we now consider the linear operator $J(k,c) : A \to A$ defined by

$$\begin{aligned} J(k,c)f(z) &= zu_p(z) * f(z) \\ &= \sum_{n=2}^{\infty} \frac{(-c/4)^n}{(k)_{n-1}(n-1)!} a_n z^n, \end{aligned} \tag{13}$$

where $*$ denote the convolution or Hadamard product of two series.

Motivated by results on connections between various subclasses of analytic univalent functions by using hypergeometric functions (see ,[7, 8, 19, 24, 33]), Struve functions (see [21, 23, 35]), Poisson distribution series (see [1, 12, 15, 16, 17, 25]) and Pascal distribution series (see [13, 16]), we determine necessary and sufficient conditions for the function $zu_p(z)$ to be in the classes $S(b,\lambda,\beta)$ and $R(b,\lambda,\beta)$ and also give necessary and sufficient condition for $z(2-u_p(z))$ to be in these classes. Also, we give necessary and sufficient condition for $J(k,c)$ to be in the class $TR(b,\lambda,\beta)$ provided that the function f is in the class $R^{\tau}(A,B)$. Finally, we give conditions for the integral operator $G(k,c,z)=\int\limits_0^z(2-u_p(t))dt$ to be in the class $TR(b,\lambda,\beta)$.

To establish our main results, we need the following lemmas.

Lemma 1. *([3], with $n=1$) A sufficient condition for a function of the form (1) to be in the class $S(b,\lambda,\beta)$ is that*

$$\sum_{n=2}^{\infty}\left[\lambda n^2+n(1+\lambda(\beta|b|-2))+(\beta|b|-1)(1-\lambda)\right]|a_n|\leq\beta|b|. \tag{14}$$

Lemma 2. ([3],*with* $n=1$) *Let the function $f(z)$ defined by (2). Then $f(z)\in TS(b,\lambda,\beta)$ if and only if (14) is satisfied.*

Lemma 3. *([3], with $n=1$)A sufficient condition for a function of the form (1) to be in the class $R(b,\lambda,\beta)$ is that*

$$\sum_{n=2}^{\infty}n\left[\lambda n+(1-\lambda)\right]|a_n|\leq\beta|b|. \tag{15}$$

Lemma 4. *([3], with* $n=1$) *Let the function $f(z)$ defined by (2). Then $f(z)\in TS(b,\lambda,\beta)$ if and only if (15) is satisfied.*

Lemma 5. *[14] If $f\in R^{\tau}(A,B)$ is of the form (1). Then*

$$|a_n|\leq\frac{(A-B)|\tau|}{n}\quad(n\geq 2). \tag{16}$$

The result is sharp for the function $f(z)$ given by

$$f(z)=\int\limits_0^z\left(1+\frac{(A-B)\tau t^{n-1}}{1+Bt^{n-1}}\right)dt\quad(\ z\in\mathfrak{U},\ n\geq 2). \tag{17}$$

Lemma 6. *[11] If $b,p,c\in\mathbb{C}$ and $k=0,-1,-2,\cdots$, then the function $u_p(z)$ satisfies the recursive relations*

$$\begin{aligned} u_p'(z) &= \frac{\left(\frac{-c}{4}\right)}{k}u_{p+1}(z)\quad(z\in\mathbb{C}),\\ u_p''(z) &= \frac{\left(\frac{-c}{4}\right)^2}{k(k+1)}u_{p+2}(z)\quad(z\in\mathbb{C}). \end{aligned} \tag{18}$$

2 The necessary and sufficient conditions

Unless otherwise mentioned, we shall assume in this paper that $b, \tau \in \mathbb{C}^*, 0 < \beta \leq 1, 0 \leq \lambda \leq 1, zu_p(z)$ is given by (11) and $J(k,c)$ is given by (13).

We obtain the sufficient condition for $zu_p(z)$ defined by (11) to be in the classes $S(b, \lambda, \beta)$.

Theorem 1. *If $c < 0$, $k > 0 (k \neq 0, -1, -2, \cdots)$, then $zu_p(z)$ is in the class $S(b, \lambda, \beta)$ if*

$$\beta\,|b|\,u_p''(1)\big((1+\lambda)(\beta\,|b|+1)\big)u_p'(1) + \beta\,|b|\,(u_p(1)-1) \leq \beta\,|b|\,. \tag{19}$$

Proof. Since

$$zu_p(z) = z + \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!} z^n, \tag{20}$$

according to (14), we must show that

$$\sum_{n=2}^{\infty} \left[\lambda n^2 + n(1+\lambda\,(\beta\,|b|-2)) + (\beta\,|b|-1)\,(1-\lambda)\right] \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!} \leq \beta\,|b|\,. \tag{21}$$

Writing $n = (n-1)+1$ and $n^2 = (n-1)(n-2) + 3(n-1) + 1$, we have

$$\begin{aligned}
&\lambda \sum_{n=2}^{\infty} (n-1)(n-2) \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!} + 3\lambda \sum_{n=2}^{\infty} (n-1) \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!} \\
&+\lambda \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!} + (1+\lambda\,(\beta\,|b|-2)) \sum_{n=2}^{\infty} (n-1) \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!} \\
&+(1+\lambda\,(\beta\,|b|-2)) \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!} + (\beta\,|b|-1)(1-\lambda) \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!} \\
= \; &\lambda \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!} + (1+\lambda\,(\beta\,|b|+1)) \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!} \\
&+\beta\,|b| \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!} \\
= \; &\lambda \frac{(-c/4)^2}{k(k+1)} \sum_{n=0}^{\infty} \frac{(-c/4)^n}{(k+2)_n n!} + (1+\lambda\,(\beta\,|b|+1)) \frac{(-c/4)}{k} \sum_{n=0}^{\infty} \frac{(-c/4)^{n-1}}{(k)_n (n-1)!} \\
&+\beta\,|b| \sum_{n=0}^{\infty} \frac{(-c/4)^{n+1}}{(k)_{n+1}(n+1)!} \\
= \; &\lambda u_p''(1) + (1+\lambda\,(\beta\,|b|+1)) u_p'(1) + \beta\,|b|\,(u_p(1)-1).
\end{aligned}$$

But this last expression is bounded above by $\beta\,|b|$ if (19) holds. This completes the proof. □

Corollary 1. *If $c < 0$, $k > 0 (k \neq 0, -1, -2, \cdots)$, then $z(2 - u_p(z))$ is in the class $TS(b, \lambda, \beta)$ if and only if the condition (19) is satisfied.*

Proof. Since

$$z(2 - u_p(z)) = z - \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!} z^n. \tag{22}$$

By using the same techniques given in proof of Theorem 1, we have Corollary 1. □

Putting $\lambda = 0, 1$ in Theorem 1 and Corollary 1, respectively, we obtain the following corollaries.

Corollary 2. *If $c < 0$, $k > 0 (k \neq 0, -1, -2, \cdots)$, then $zu_p(z)$ is in the class $S(b, \beta)$ if*

$$u_p'(1) + \beta |b| (u_p(1) - 1) \leq \beta |b|. \tag{23}$$

Corollary 3. *If $c < 0$, $k > 0 (k \neq 0, -1, -2, \cdots)$, then $zu_p(z)$ is in the class $C(b, \beta)$ if*

$$u_p''(1) + (\beta |b| + 2) u_p'(1) + \beta |b| (u_p(1) - 1) \leq \beta |b|. \tag{24}$$

Corollary 4. *If $c < 0$, $k > 0 (k \neq 0, -1, -2, \cdots)$, then $z(2 - u_p(z))$ is in the class $TS(b, \beta)$ if and only if the condition (23) is satisfied.*

Corollary 5. *If $c < 0$, $k > 0 (k \neq 0, -1, -2, \cdots)$, then $z(2 - u_p(z))$ is in the class $TC(b, \beta)$ if and only if the condition (24) is satisfied.*

Theorem 2. *If $c < 0$, $k > 0 (k \neq 0, -1, -2, \cdots)$. Then $zu_p(z)$ is in the class $R(b, \lambda, \beta)$ if*

$$\lambda u_p''(1) + (1 + 2\lambda) u_p'(1) + (u_p(1) - 1) \leq \beta |b|. \tag{25}$$

Proof. In view of (15), we must show that

$$\sum_{n=2}^{\infty} n [\lambda n + (1 - \lambda)] \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!} \leq \beta |b|. \tag{26}$$

As in the proof of Theorem 1, we have

$$\begin{aligned}
&\sum_{n=2}^{\infty} n\left[\lambda n+(1-\lambda)\right]\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!}\\
=\ &\lambda\sum_{n=2}^{\infty}(n-1)(n-2)\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!}+3\lambda\sum_{n=2}^{\infty}(n-1)\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!}\\
&+(1-\lambda)\sum_{n=2}^{\infty}(n-1)\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!}+(1-\lambda)\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!}\\
&+\lambda\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!}\\
=\ &\lambda\sum_{n=3}^{\infty}\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-3)!}+(1+2\lambda)+\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-2)!}+\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!}\\
=\ &\lambda\frac{(-c/4)^{2}}{k(k+1)}\sum_{n=0}^{\infty}\frac{(-c/4)^{n-1}}{(k+2)_{n}n!}+(1+2\lambda)\frac{(-c/4)}{k}\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(k+1)_{n}n!}\\
&+\sum_{n=0}^{\infty}\frac{(-c/4)^{n+1}}{(k)_{n+1}(n+1)!}\\
=\ &\lambda u_p''(1)+(1+2\lambda)u_p'(1)+(u_p(1)-1).
\end{aligned}$$

But this last expression is bounded above by $\beta\,|b|$ if (25) holds. This completes the proof. □

By using a similar method as in the proof of Corollary 1, we obtain the following corollary.

Corollary 6. *If $c<0,\ k>0(k\neq 0,-1,-2,\cdots)$, then $z(2-u_p(z))$ is in the class $TR(b,\lambda,\beta)$ if and only if the condition (25) is satisfied.*

Putting $\lambda=0$ in Theorem 2 and Corollary 6, respectively, we obtain the following corollaries.

Corollary 7. *If $c<0,\ k>0(k\neq 0,-1,-2,\cdots)$, then $zu_p(z)$ is in the class $R(b,\beta)$ if*

$$u_p'(1)+u_p(1)-1\leq\beta\,|b|\,. \tag{27}$$

Corollary 8. *If $c<0,\ k>0(k\neq 0,-1,-2,\cdots)$, then $z(2-u_p(z))$ is in the class $TR(b,\beta)$ if and only if (27) is satisfied.*

Theorem 3. *If $c<0,\ k>0(k\neq 0,-1,-2,\cdots)$, then $zu_p(z)$ is in the class $S(b,\lambda,\beta)$ if*

$$e^{(\frac{-c}{4k})}\Big[\frac{\lambda c^2}{16k^2}+\big(1+\lambda(\beta\,|b|+1)\big)\big(\frac{-c}{4k}\big)+\beta\,|b|\,(1-e^{(\frac{c}{4k})})\Big]\leq\beta\,|b|\,. \tag{28}$$

Proof. We note that

$$(k)_{n-1} = k(k+1)(k+2)\cdots(k+n-2) \geq k(k+1)^{n-2} \geq k^{n-1}, \qquad (n \in \mathbb{N}). \quad (29)$$

From (14), (20) and (29), we have

$$\begin{aligned}
&\sum_{n=2}^{\infty}\left[\lambda n^2 + n(1+\lambda(\beta\,|b|-2)) + (\beta\,|b|-1)(1-\lambda)\right]\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!}\\
=\;& \lambda\sum_{n=2}^{\infty}(n-1)(n-2)\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!} + 3\lambda\sum_{n=2}^{\infty}(n-1)\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!}\\
&+\lambda\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!} + (1+\lambda(\beta\,|b|-2))\sum_{n=2}^{\infty}(n-1)\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!}\\
&+(1+\lambda(\beta\,|b|-2))\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!} + (\beta\,|b|-1)(1-\lambda)\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!}\\
=\;& \lambda\sum_{n=3}^{\infty}\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-3)!} + (1+\lambda(\beta\,|b|+1))\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-2)!}\\
&+\beta\,|b|\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!}\\
\leq\;& \lambda\sum_{n=3}^{\infty}\frac{(-c/4k)^{n-1}}{(n-3)!} + (1+\lambda(\beta\,|b|+1))\sum_{n=2}^{\infty}\frac{(-c/4k)^{n-1}}{(n-2)!} + \beta\,|b|\sum_{n=2}^{\infty}\frac{(-c/4k)^{n-1}}{(n-1)!}\\
=\;& \lambda\left(\frac{\lambda c^2}{16k^2}e^{(\frac{-c}{4k})}\right) + (1+\lambda(\beta\,|b|+1))\left(\frac{-c}{4k}\right)e^{(\frac{-c}{4k})} + \beta\,|b|\left(e^{(\frac{-c}{4k})}-1\right)\\
=\;& e^{(\frac{-c}{4k})}\Big[\frac{\lambda c^2}{16k^2} + \big(1+\lambda(\beta\,|b|+1)\big)\Big(\frac{-c}{4k}\Big) + \beta\,|b|\,(1-e^{(\frac{c}{4k})})\Big].
\end{aligned}$$

Therefore, we see that the last expression is bounded above by $\beta\,|b|$ if (28) holds. This completes the proof. □

Putting $\lambda = 0$ and $\lambda = 0$, respectively in Theorem 3, we obtain the following corollaries.

Corollary 9. *If $c < 0$, $k > 0(k \neq 0, -1, -2, \cdots)$, then $zu_p(z)$ is in the class $S(b, \beta)$ if*

$$e^{(\frac{-c}{4k})}\Big[\frac{-c}{4k} + \beta\,|b|\,(1-e^{(\frac{c}{4k})})\Big] \leq \beta\,|b|\,. \quad (30)$$

Corollary 10. *If $c < 0$, $k > 0(k \neq 0, -1, -2, \cdots)$, then $z(2 - u_p(z))$ is in the class $C(b, \beta)$ if*

$$e^{(\frac{-c}{4k})}\Big[\frac{c^2}{16k^2} + (\beta\,|b|+2)\Big(\frac{-c}{4k}\Big) + \beta\,|b|\,(1-e^{(\frac{c}{4k})})\Big] \leq \beta\,|b|\,. \quad (31)$$

Theorem 4. *If $c < 0$, $k > 0(k \neq 0, -1, -2, \cdots)$, then $zu_p(z)$ is in the class $R(b, \lambda, \beta)$ if*

$$e^{(\frac{-c}{4k})}\Big[\frac{\lambda c^2}{16k^2} + 3\lambda\Big(\frac{-c}{4k}\Big) + (1-e^{(\frac{c}{4k})})\Big] \leq \beta\,|b|\,. \quad (32)$$

Proof. From (15), (20) and (29), we have

$$\begin{aligned}
&\sum_{n=2}^{\infty} n\big[\lambda n + (1-\lambda)\big]\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!} \\
= \;& \lambda\sum_{n=2}^{\infty}(n-1)(n-2)\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!} + 3\lambda\sum_{n=2}^{\infty}(n-1)\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!} \\
& +\lambda\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!} + (1-\lambda)\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!} \\
= \;& \lambda\sum_{n=3}^{\infty}\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-3)!} + 3\lambda\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-2)!} + \sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!} \\
\le \;& \lambda\sum_{n=0}^{\infty}\frac{(-c/4k)^{n-1}}{(n-3)!} + 3\lambda\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-2)!} + \sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!} \\
= \;& \lambda\left(\frac{c^2}{16k^2}e^{(\frac{-c}{4k})}\right) + 3\lambda\left(\frac{-c}{4k}\right)e^{(\frac{-c}{4k})} + \left(e^{(\frac{-c}{4k})} - 1\right) \\
= \;& e^{(\frac{-c}{4k})}\big[\frac{\lambda c^2}{16k^2} + 3\lambda(\frac{-c}{4k}) + (1 - e^{(\frac{c}{4k})})\big].
\end{aligned}$$

Therefore, we see that the last expression is bounded above by $\beta\,|b|$ if (32) satisfied. This completes the proof. □

Putting $\lambda = 0$ and $\lambda = 0$, respectively in Theorem 4, we obtain the following corollaries.

Corollary 11. *If $c < 0$, $k > 0(k \neq 0, -1, -2, \cdots)$, then $zu_p(z)$ is in the class $S(b,\beta)$ if*

$$e^{(\frac{-c}{4k})} - 1 \le \beta\,|b|\,.$$

Corollary 12. *If $c < 0$, $k > 0(k \neq 0, -1, -2, \cdots)$, then $zu_p(z)$ is in the class $C(b,\beta)$ if*

$$e^{(\frac{-c}{4k})}\big[\frac{c^2}{16k^2} + 3(\frac{-c}{4k}) + (1 - e^{(\frac{c}{4k})})\big] \le \beta\,|b|\,.$$

The proof of Theorems 5 and 6 (below) is much akin to that of Theorems 3 and 4, therefore the details may be omitted.

Theorem 5. *If $c < 0$, $k > 0(k \neq 0, -1, -2, \cdots)$, then $z(2 - u_p(z))$ is in the class $TS(b,\lambda,\beta)$ if*

$$e^{(\frac{-c}{4k})}\big[\frac{\lambda c^2}{16k^2} + (1 + \lambda(\beta\,|b| + 1))(\frac{-c}{4k}) + \beta\,|b|\,(1 - e^{(\frac{c}{4k})})\big] \le \beta\,|b|\,. \tag{33}$$

Theorem 6. *If $c < 0$, $k > 0(k \neq 0, -1, -2, \cdots)$, then $z(2 - u_p(z))$ is in the class $TR(b,\lambda,\beta)$ if*

$$e^{(\frac{-c}{4k})}\big[\frac{\lambda c^2}{16k^2} + 3\lambda(\frac{-c}{4k}) + (1 - e^{(\frac{c}{4k})})\big] \le \beta\,|b|\,. \tag{34}$$

3 Inclusion Properties

Making use of Lemma 5, we will study the action of the Bessel function on the class $TR(b,\lambda,\beta)$.

Theorem 7. *Let $c<0$, $k>0(k\neq 0,-1,-2,\cdots)$. If $f\in R^{\tau}(A,B)$, then $J(k,c)f$ is in the class $TR(b,\lambda,\beta)$ if and only if*

$$(A-B)\left|\tau\right|\left[\lambda u_p'(1)+u_p(1)-1\right]\leq \beta\left|b\right|. \tag{35}$$

Proof. In view of (15), it suffices to show that

$$\sum_{n=2}^{\infty} n\left[\lambda n+(1-\lambda)\right]\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!}\left|a_n\right|\leq \beta\left|b\right|. \tag{36}$$

Since $f\in \mathcal{R}^{\tau}(A,B)$, then by Lemma 5, we get

$$\left|a_n\right|\leq \frac{(A-B)\left|\tau\right|}{n}\quad (n\geq 1). \tag{37}$$

Thus we must show that

$$\begin{aligned}&\sum_{n=2}^{\infty} n[\lambda n+(1-\lambda))]\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!}\\ \leq\ &(A-B)\left|\tau\right|\left[\sum_{n=2}^{\infty} n\left[\lambda n+(1-\lambda)\right]\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!}\right].\end{aligned} \tag{38}$$

We have

$$\begin{aligned}&(A-B)\left|\tau\right|\times\\ &\left[\lambda\sum_{n=2}^{\infty}(n-1)\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!}+\lambda\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!}\right.\\ &\qquad\left.+(1-\lambda)\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!}\right]\\ =\ &(A-B)\left|\tau\right|\left[\lambda\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-2)!}+\sum_{n=2}^{\infty}\frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!}\right]\\ =\ &(A-B)\left|\tau\right|\left[\lambda\sum_{n=0}^{\infty}\frac{(-c/4)^{n+1}}{(k)_{n+1}n!}+\sum_{n=0}^{\infty}\frac{(-c/4)^{n+1}}{(k)_{n+1}(n+1)!}\right]\\ =\ &(A-B)\left|\tau\right|\left[\lambda u_p'(1)+u_p(1)-1\right].\end{aligned}$$

Therefore, we see that the last expression is bounded above by $\beta\left|b\right|$ if (35) satisfied. This completes the proof. □

Putting $\lambda=0$ in Theorem 7, we obtain the following corollary.

Corollary 13. *Let $c<0$, $k>0(k\neq 0,-1,-2,\cdots)$. If $f\in\mathcal{R}^{\tau}(A,B)$, then $J(k,c)f$ is in the class $TR(b,\beta)$ if and only if*

$$(A-B)\left|\tau\right|\left[u_p(1)-1\right]\leq\beta\left|b\right|.$$

4 An integral operator

In this section, we obtain the necessary and sufficient condition for the integral operator $G(k,c,z)$ defined by

$$G(k,c,z) = \int_0^z (2 - u_p(t))dt, \tag{39}$$

to be in the class $TR(b,\lambda,\beta)$.

Theorem 8. *If* $c < 0$, $k > 0 (k \neq 0, -1, -2, \cdots)$, *then the integral operator* $G(k,c,z)$ *is in the class* $TR(b,\lambda,\beta)$ *if and only if*

$$\lambda u_p'(1) + u_p(1) - 1 \leq \beta \left| b \right|. \tag{40}$$

Proof. Since

$$G(k,c,z) = z - \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(k)_{n-1}} \frac{z^n}{n!}, \tag{41}$$

then in view of (15), we need only to show that

$$\sum_{n=2}^{\infty} n \left[\lambda n + (1-\lambda)\right] \frac{(-c/4)^{n-1}}{(k)_{n-1} n!} \leq \beta \left| b \right| \tag{42}$$

or, equivalently

$$\sum_{n=2}^{\infty} \left[\lambda n + (1-\lambda)\right] \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!} \leq \beta \left| b \right|. \tag{43}$$

We have

$$\begin{aligned}
&\sum_{n=2}^{\infty} \left[\lambda n + (1-\lambda)\right] \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!} \\
=\ & \lambda \sum_{n=2}^{\infty} (n-1) \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!} + \lambda \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!} \\
& + (1-\lambda) \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!} \\
=\ & \lambda \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-2)!} + \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!} \\
=\ & \lambda \sum_{n=0}^{\infty} \frac{(-c/4)^{n+1}}{(k)_{n+1} n!} + \sum_{n=0}^{\infty} \frac{(-c/4)^{n+1}}{(k)_{n+1}(n+1)!} \\
=\ & \lambda u_p'(1) + u_p(1) - 1.
\end{aligned}$$

Therefore, we see that the last expression is bounded above by $\beta \left| b \right|$ if (39) satisfied. This completes the proof. □

Putting $\lambda = 0$ in Theorem 8, we obtain the following corollary.

Let $c < 0$, $k > 0 (k \neq 0, -1, -2, \cdots)$, then the integral operator $G(k, c, z)$ defined by (39) is in the class $TR(b, \beta)$ if and only if

$$u_p(1) - 1 \leq \beta |b| . \tag{44}$$

5 Conclusion

Necessary and sufficient conditions for generalized Bessel functions of the first kind $zu_p(z)$ to be in the classes $S(b, \lambda, \beta)$ and $R(b, \lambda, \beta)$ of analytic functions with complex order were obtained. Also we give the necessary and sufficient conditions for $z(2 - u_p(z))$ to be in the classes $TS(b, \lambda, \beta)$ and $TR(b, \lambda, \beta)$. Furthermore, we give the necessary and sufficient conditions for $J(k, c)$ to be in the class $TR(b, \lambda, \beta)$ provided that the function f is in the class $R^{\tau}(A, B)$. Finally, we give conditions for the integral operator $G(k, c, z) = \int_0^z (2 - u_p(t))dt$ to be in the class $TR(b, \lambda, \beta)$.

References

[1] Ahmad, M., Frasin, B., Murugusundaramoorthy, G. and Al-khazaleh, A., *An application of Mittag–Leffler-type Poisson distribution on certain subclasses of analytic functions associated with conic domains*, Heliyon, **7** (2021) e08109.

[2] Altintas, O., *On a subclass of certain starlike functions with negative coefficients*, Math. Japon. 36 (1991), 489–495.

[3] Altintas, O., Ozkan, O. and Srivastava, H.M., *Neighborhoods of a class of analytic functions with negative coefficients*, Appl. Math. Letters, **13** (2000), 63–67.

[4] Aouf, A.K., *p-Valent classes related to convex functions of complex order*, Rocky Mountain J. Math. **15** (1985), no. 4, 855–865.

[5] Aouf, A.K. and Mostafa, A.O., *Certain bounded functions of complex order*, Math. Vesnik **62** (2010), no. 2, 109–116.

[6] Aouf, A.K., Mostafa, A.O. and Zayed, H.M., *Convolution properties for some subclasses of meromorphic functions of complex order*, Abstract Appl. Math. Vol. 2015, Art ID 973613, 1-6.

[7] Aouf, A.K., Mostafa, A.O. and Zayed, H.M., *Necessity and sufficiency for hypergeometric functions to be in the subclass of analytic functions*, J. Egyptian Math. Soc. **23** (2015), 476-481.

[8] Aouf, A.K., Mostafa, A.O. and Zayed, H.M., *Some properties of uniformly starlike and convex hypergeometric functions*, Azerbaijan J. Math., **9** (2019), no. 2, 3-18.

[9] Baricz, A., *Geometric properties of generalized Bessel functions*, Publ. Math. Debrecen **73** (2008), no. 1-2, 155-178.

[10] Baricz, A., *Geometric properties of generalized Bessel functions of complex order*, Mathematica, **48(71)** (2006), 13-18.

[11] Baricz, A., *Generalized Bessel functions of the first kind*, Lecture Notes in Math. Vol. 1994, Springer-Verlag, 2010.

[12] El-Ashwah, R.M. and Kota, W.Y., *Some condition on a Poisson distribution series to be in subclasses of univalent functions*, Acta Universitatis Apulensis **51** (2017), 89-103.

[13] El-Deeb, S.M., Bulboacă, T. and Dziok, J., *Pascal distribution series connected with certain subclasses of univalent functions*, Kyungpook Math. J. **59** (2019), 301–314.

[14] Dixit, K.K. and Pal, S.K., *On a class of univalent functions related to complex order*, Indian J. Pure Appl. Math. **26** (1995), no. 9, 889-896.

[15] Frasin, B.A., *On certain subclasses of analytic functions associated with Poisson distribution series*, Acta Univ. Sapientiae, Mathematica **11** (2019), 78-86.

[16] Frasin, B.A., *Subclasses of analytic functions associated with Pascal distribution series*, Adv. Theory Nonlinear Anal. Appl. **4** (2020), no. 2, 92-99.

[17] Frasin, B.A. and Gharaibeh, M M., *Subclass of analytic functions associated with Poisson distribution series*, Afr. Mat. **31** (2020), 1167-1173.

[18] Frasin, B.A. and Aldawish, I., *On subclasses of uniformly spirallike functions associated with generalized Bessel functions*, J. Funct. Spaces, Volume 2019, Article ID 1329462, 1-6.

[19] Frasin, B.A., Al-Hawary, T. and Yousef, F., *Necessary and sufficient conditions for hypergeometric functions to be in a subclass of analytic functions*, Afr. Mat. **30** (2019), no. 1-2, 223-230.

[20] Frasin, B.A., Yousef, F., Al-Hawary, T. and Aldawish, I., *Application of generalized Bessel functions to classes of analytic functions*, Afr. Mat. **32** (2021) 431-439.

[21] Frasin, B.A., Al-Hawary, T., Yousef, F. and Aldawish, I., *On Subclasses of analytic functions associated with Struve functions*, Nonlinear Funct. Anal. Appl. **27** (2022), no. 1, 99-110.

[22] Halim, A., *On a class of functions of complex order*, Tamkang J. Math., **30** (1999), no. 2, 147-153.

[23] Janani, T. and Murugusundaramoorthy, G., *Inclusion results on subclasses of starlike and convex functions associated with Struve functions*, Italian J. Pure Applied Math. **32** (2014), 467-476.

[24] Merkes, E.P. and Scott, W.T., *Starlike hypergeometric functions*, Proc. Amer. Math. Soc. **12** (1961), no. 5, 223-230.

[25] Murugusundaramoorthy, G., *Subclasses of starlike and convex functions involving Poisson distribution series*, Afr. Mat. **28** (2017), 1357-1366.

[26] Murugusundaramoorthy, G. and Janani, T., *An application of generalized Bessel functions on certain subclasses of analytic functions*, Turk. J. Anal. Number Theory **3** (2015), no. 1, 1-6.

[27] Murugusundaramoorthy, G., Vijaya, K. and Kathuri, M., *A note on subclasses of analytic and convex functions associated with Bessel functions*, J. Nonlinear Funct. Anal. Vol. 2014, Art.11, 2014.

[28] Naeem, M., Hussain, S., Müge Sakar, F., Mahmood, T. and Rasheed, A., *Subclasses of uniformly convex and starlike functions associated with Bessel functions*, Turk. J. Math. **43** (2019), no. 5, 2433–2443.

[29] Nasr, M.A. and Aouf, A.K., *On convex functions of complex order*, Bull. Fac. Sci. Mansoura Univ. **9** (1982), 565-582.

[30] Nasr, M.A. and Aouf, A.K., *Bounded starlike functions of complex order*, Proc. Indian Acad. Sci. (Math. Sci.) **92** (1983), no.2, 97-102. Nasr, M.A. and Aouf, A.K., *Starlike function of complex order*, J. Natar Sci. Math. **25** (1985), no.1, 1-12.

[31] Owa, S., *Notes on starlike , convex and close-to-convex functions of complex order*, in: H.M-Srivastava and S. Owa (Eds.), Univalent Functions, Fractional Calculus and Their Applications, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto 1989, 199-218.

[32] Sakar, F.M. and Canbulat, A., *Quasi-subordinations for a subfamily of bi-univalent functions associated with q-analogue of Bessel function*, J. Math. Anal. **12** (2021), no. 1, 1-12.

[33] Silverman, H., *Starlike and convexity properties for hypergeometric functions*, J. Math. Anal. Appl. **172** (1993), no.2, 574–581.

[34] Wiatrowski, P., *The sufficient of a certain family holomorphic functions*, Zeszyty Nauk. Univ. Lodzk, Nauki, Math. Przyrd. Ser.II, Zeszyt Math. **39** (1971), 75-85.

[35] Yagmur, N. and Orhan, H., *Starlikeness and convexity of generalized Struve functions*, Abstr. Appl. Anal. (2013), Article ID 954513, 6 pages.