CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS WITH COMPLEX ORDER ASSOCIATED WITH GENERALIZED BESSEL FUNCTIONS

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Abstract

In this paper we obtain the necessary and sufficient conditions for generalized Bessel functions of the first kind \( z u_p(z) \) to be in the classes \( S(b, \lambda, \beta) \) and \( R(b, \lambda, \beta) \) of analytic functions with complex order and also give the necessary and sufficient conditions for \( z(2 - u_p(z)) \) to be in the classes \( TS(b, \lambda, \beta) \) and \( TR(b, \lambda, \beta) \). Furthermore, we give the necessary and sufficient conditions for \( J(k, c) \) to be in the class \( TR(b, \lambda, \beta) \) provided that the function \( f \) is in the class \( R'(A, B) \). Finally, we give conditions for the integral operator \( G(k, c, z) = \int_0^z (2 - u_p(t))dt \) to be in the class \( TR(b, \lambda, \beta) \). Several corollaries and consequences of the main results are also obtained.

2000 Mathematics Subject Classification: 30C45, 33C10.

Key words: analytic functions, Hadamard product, generalized Bessel functions.

1 Introduction

Bessel functions are needful in many branches of applied mathematics and mathematical physics, for example, those in acoustics, angular resolution, radio physics, hydrodynamics, and signal processing. Therefore, these special functions have been studied extensively, see [6, 9, 10, 11, 18, 20, 26, 27].

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Let $A$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1)$$

which are analytic in the open unit disc $\mathcal{U} = \{ z : |z| < 1 \}$. Further, let $T$ be the subclass of $A$ consisting of functions of the form:

$$f(z) = z - \sum_{n=2}^{\infty} |a_n| z^n, \quad z \in \mathcal{U} \quad (2)$$

A function $f \in A$ is said to be starlike of complex order $b$ ($b \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$), that is $f \in S(b)$ if it satisfies the inequality

$$\text{Re} \left\{ 1 + \frac{1}{b} \left( \frac{zf'(z)}{f(z)} - 1 \right) \right\} > 0, \quad (z \in \mathcal{U}, \ b \in \mathbb{C}^*) \quad (3)$$

Also, a function $f \in A$ is said to be convex of complex order $b$ ($b \in \mathbb{C}^*$), that is $f \in C(b)$ if it satisfies the inequality

$$\text{Re} \left\{ 1 + \frac{1}{b} \frac{zf''(z)}{f'(z)} \right\} > 0 \quad (z \in \mathcal{U}, \ b \in \mathbb{C}^*). \quad (4)$$

The class $S(b)$ was introduced and studied by Nasr and Aouf [?], (see also [30]) and the class $C(b)$ was introduced and studied by Wiatrowski [34], (see also [4] and [29]).

Furthermore a function $f \in A$ is said to be close-to-convex of complex order $b$ ($b \in \mathbb{C}^*$) if it satisfies the inequality

$$\text{Re} \left\{ 1 + \frac{1}{b} (f'(z) - 1) \right\} > 0 \quad (z \in \mathcal{U}, \ b \in \mathbb{C}^*). \quad (5)$$

The class $R(b)$ was introduced and studied by Halim [22] and Owa [31] (see Aouf and Mostafa [5]).

Finally, let $S(b, \lambda, \beta)$ denote the class of functions $f \in A$ which satisfies the inequality

$$\left| \frac{1}{b} \left( \frac{zf'(z) + \lambda z^2 f''(z)}{(1 - \lambda)f(z) + \lambda f'(z)} - 1 \right) \right| < \beta, \quad (z \in \mathcal{U}, \ b \in \mathbb{C}^*, 0 < \beta \leq 1, 0 \leq \lambda \leq 1). \quad (6)$$

Also let $R(b, \lambda, \beta)$ denote the class of functions $f \in A$ which satisfies the inequality

$$\left| \frac{1}{b} (f'(z) + \lambda z f''(z) - 1) \right| < \beta, \quad (z \in \mathcal{U}, b \in \mathbb{C}^*, 0 < \beta \leq 1, 0 \leq \lambda \leq 1). \quad (7)$$

The classes $S(b, \lambda, \beta)$ and $R(b, \lambda, \beta)$ was introduced and studied by Altintas et al. [2] with $n = 1$.

We note that:
Certain subclasses of analytic functions with complex order

(1) $S(b, 0, 1) \subset S(b), S(b, 1, 1) \subset C(b)$ and $R(b, 0, 1) \subset R(b)$,
(2) $S(b, 0, \beta) = S(b, \beta) = \left\{ f \in \mathcal{A} : \frac{1}{b} \left( \frac{z f'(z)}{f(z)} - 1 \right) < \beta, ~ z \in \mathcal{U} \right\}$,
(3) $S(b, 1, \beta) = C(b, \beta) = \left\{ f \in \mathcal{A} : \frac{1}{b} \left( \frac{z f''(z)}{f'(z)} - 1 \right) < \beta, ~ z \in \mathcal{U} \right\}$,
(4) $R(b, 0, \beta) = R(b, \beta) = \left\{ f \in \mathcal{A} : \frac{1}{b} \left( f'(z) - 1 \right) < \beta, ~ z \in \mathcal{U} \right\}$. 

Further, we define the classes $T S(b, \lambda, \beta)$ and $T R(b, \lambda, \beta)$ by

$T S(b, \lambda, \beta) = S(b, \lambda, \beta) \cap T$ and $T R(b, \lambda, \beta) = R(b, \lambda, \beta) \cap T$.

We note that $T S(1, \lambda, \beta) = T S(\lambda, \beta)$ (see [2]).

A function $f \in \mathcal{A}$ is said to be in the class $R^*(A, B), \tau \in \mathbb{C}^*$ and $-1 \leq B \leq A \leq 1$ if it satisfies the inequality

$$\left| \frac{f'(z) - 1}{(A - B)\tau - B[f'(z) - 1]} \right| < 1, ~ z \in \mathcal{U}. \quad (8)$$

The class $R^*(A, B)$ was introduced and studied by Dixit and Pal [14].

The generalized Bessel function $w_p$ (see, [9]) is defined as a particular solution of the linear differential equation

$$z^2 w''(z) + bw'(z) + [cz^2 - p^2 + (1 - b)p]w(z) = 0, \quad (9)$$

where $b, p, c \in \mathbb{C}$. The analytic function $w_p$ has the form:

$$w_p(z) = \sum_{n=0}^{\infty} \frac{(-1)^n (c)^n}{n! \Gamma(p + n + \frac{b + 1}{2})} \frac{(\frac{z}{2})^{2n+p}}{4^n}, ~ z \in \mathbb{C}. \quad (10)$$

Now, the generalized and normalized Bessel function $u_p$ is defined with the transformation

$$u_p(z) = 2^p \frac{\Gamma(p + \frac{b + 1}{2})}{2^{p/2}} z^{-p/2} w_p(z^{1/2})$$

$$= \sum_{n=0}^{\infty} \frac{(-c/4)^n}{(k)_n n!} z^n, \quad (11)$$

where $k = p + (b + 1)/2 \neq 0, -1, -2, \ldots$ and $(a)_n$ is the well-known Pochhammer symbol, defined in terms of the Euler Gamma function for $a \neq 0, -1, -2, \ldots$ by

$$(a)_n = \frac{\Gamma(a + n)}{\Gamma(a)} = \left\{ \begin{array}{ll} 1, & \text{if } n = 0 \\ a(a + 1)(a + 2) \ldots (a + n - 1), & \text{if } n \in \mathbb{N} = \{1, 2, \ldots \}. \end{array} \right.$$-

The function $u_p$ is analytic on $\mathbb{C}$ and satisfies the second linear differential equation

$$4z^2 u''(z) + 2(2p + b + 1)zu'(z) + czu(z) = 0. \quad (12)$$

Using the Hadamard product, we now consider the linear operator $J(k, c) : A \to A$ defined by

$$J(k, c)f(z) = zu_p(z) \ast f(z)$$

$$= \sum_{n=2}^{\infty} \frac{(-c/4)^n}{(k)_{n-1} (n-1)!} a_n z^n, \quad (13)$$
where * denote the convolution or Hadamard product of two series.

Motivated by results on connections between various subclasses of analytic univalent functions by using hypergeometric functions (see [7, 8, 19, 24, 33]), Struve functions (see [21, 23, 35]), Poisson distribution series (see [1, 12, 15, 16, 17, 25]) and Pascal distribution series (see [13, 16]), we determine necessary and sufficient conditions for the function $z u_p(z)$ to be in the classes $S(b, \lambda, \beta)$ and $R(b, \lambda, \beta)$ and also give necessary and sufficient condition for $z(2-u_p(z))$ to be in these classes. Also, we give necessary and sufficient condition for $J(k, c)$ to be in the class $TR(b, \lambda, \beta)$ provided that the function $f$ is in the class $R_\tau(A, B)$. Finally, we give conditions for the integral operator $G(k, c, z) = \int_0^z (2-u_p(t)) dt$ to be in the class $TR(b, \lambda, \beta)$.

To establish our main results, we need the following lemmas.

Lemma 1. ([3], with $n = 1$) A sufficient condition for a function of the form (1) to be in the class $S(b, \lambda, \beta)$ is that

$$\sum_{n=2}^{\infty} \left[ (\lambda n^2 + n(1 + \lambda (|b| - 2))) + (\beta |b| - 1) (1 - \lambda) \right] |a_n| \leq \beta |b|. \quad (14)$$

Lemma 2. ([3], with $n = 1$) Let the function $f(z)$ defined by (2). Then $f(z) \in TS(b, \lambda, \beta)$ if and only if (14) is satisfied.

Lemma 3. ([3], with $n = 1$) A sufficient condition for a function of the form (1) to be in the class $R(b, \lambda, \beta)$ is that

$$\sum_{n=2}^{\infty} n \left[ \lambda n + (1 - \lambda) \right] |a_n| \leq \beta |b|. \quad (15)$$

Lemma 4. ([3], with $n = 1$) Let the function $f(z)$ defined by (2). Then $f(z) \in TS(b, \lambda, \beta)$ if and only if (15) is satisfied.

Lemma 5. [14] If $f \in R_\tau(A, B)$ is of the form (1). Then

$$|a_n| \leq \frac{(A - B) |\tau|}{n} \quad (n \geq 2). \quad (16)$$

The result is sharp for the function $f(z)$ given by

$$f(z) = \int_0^z \left( 1 + \frac{(A - B) \tau t^{n-1}}{1 + B t^{n-1}} \right) dt \quad (z \in \mathbb{U}, \ n \geq 2). \quad (17)$$

Lemma 6. [11] If $b, p, c \in \mathbb{C}$ and $k = 0, -1, -2, \ldots$, then the function $u_p(z)$ satisfies the recursive relations

$$u'_p(z) = \frac{\tau}{k} u_{p+1}(z) \quad (z \in \mathbb{C}),$$

$$u''_p(z) = \frac{\tau^2}{k(k+1)} u_{p+2}(z) \quad (z \in \mathbb{C}). \quad (18)$$
2 The necessary and sufficient conditions

Unless otherwise mentioned, we shall assume in this paper that \( b, \tau \in \mathbb{C}^*, 0 < \beta \leq 1, 0 \leq \lambda \leq 1, z_{u_p}(z) \) is given by (11) and \( J(k, c) \) is given by (13).

We obtain the sufficient condition for \( z_{u_p}(z) \) defined by (11) to be in the classes \( S(b, \lambda, \beta) \).

**Theorem 1.** If \( c < 0, k > 0(k \neq 0, -1, -2, \cdots) \), then \( z_{u_p}(z) \) is in the class \( S(b, \lambda, \beta) \) if

\[
\beta |b| u_p''(1)((1 + \lambda)(|b| + 1))u_p'(1) + \beta |b| (u_p(1) - 1) \leq \beta |b|. 
\] (19)

**Proof.** Since

\[
z_{u_p}(z) = z + \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!} z^n,
\] (20)

according to (14), we must show that

\[
\sum_{n=2}^{\infty} \left[ \lambda n^2 + n(1 + \lambda (|b| - 2)) + (|b| - 1)(1 - \lambda) \right] \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!} \leq \beta |b|. 
\] (21)

Writing \( n = (n - 1) + 1 \) and \( n^2 = (n - 1)(n - 2) + 3(n - 1) + 1 \), we have

\[
= \lambda \sum_{n=2}^{\infty} \frac{(n-1)(n-2)}{(k)_{n-1}(n-1)!} \left( \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!} \right) + 3 \lambda \sum_{n=2}^{\infty} \frac{(n-1)}{(k)_{n-1}(n-1)!} \left( \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!} \right) \\
+ \lambda \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!} + (1 + \lambda (|b| - 2)) \sum_{n=2}^{\infty} (n-1) \left( \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!} \right) \\
+ (1 + \lambda (|b| - 2)) \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!} + (|b| - 1)(1 - \lambda) \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!} \\
= \lambda \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!} + (1 + \lambda (|b| + 1)) \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!} \\
+ \beta |b| \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!} \\
= \lambda \sum_{n=2}^{\infty} \frac{(-c/4)^{n}}{(k)(k+1)n!} + (1 + \lambda (|b| + 1)) \sum_{n=0}^{\infty} \frac{(-c/4)^{n}}{k(k)_{n}(n-1)!} \\
+ \beta |b| \sum_{n=0}^{\infty} \frac{(-c/4)^{n+1}}{(k)_{n+1}(n+1)!} \\
= \lambda u_p''(1) + (1 + \lambda (|b| + 1))u_p'(1) + \beta |b| (u_p(1) - 1).
\]

But this last expression is bounded above by \( \beta |b| \) if (19) holds. This completes the proof. \( \square \)
Corollary 1. If \( c < 0, k > 0 (k \neq 0, -1, -2, \cdots) \), then \( z(2 - u_p(z)) \) is in the class \( TS(b, \lambda, \beta) \) if and only if the condition (19) is satisfied.

Proof. Since

\[
z(2 - u_p(z)) = z - \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!} z^n.
\]  

(22)

By using the same techniques given in proof of Theorem 1, we have Corollary 1.

Putting \( \lambda = 0, 1 \) in Theorem 1 and Corollary 1, respectively, we obtain the following corollaries.

Corollary 2. If \( c < 0, k > 0 (k \neq 0, -1, -2, \cdots) \), then \( z u_p(z) \) is in the class \( S(b, \beta) \) if

\[
u_p'(1) + \beta |b| (u_p(1) - 1) \leq \beta |b|.
\]

(23)

Corollary 3. If \( c < 0, k > 0 (k \neq 0, -1, -2, \cdots) \), then \( z u_p(z) \) is in the class \( C(b, \beta) \) if

\[
u_p''(1) + (\beta |b| + 2) u_p'(1) + \beta |b| (u_p(1) - 1) \leq \beta |b|.
\]

(24)

Corollary 4. If \( c < 0, k > 0 (k \neq 0, -1, -2, \cdots) \), then \( z(2 - u_p(z)) \) is in the class \( TS(b, \beta) \) if and only if the condition (23) is satisfied.

Corollary 5. If \( c < 0, k > 0 (k \neq 0, -1, -2, \cdots) \), then \( z(2 - u_p(z)) \) is in the class \( TC(b, \beta) \) if and only if the condition (24) is satisfied.

Theorem 2. If \( c < 0, k > 0 (k \neq 0, -1, -2, \cdots) \). Then \( z u_p(z) \) is in the class \( R(b, \lambda, \beta) \) if

\[
\lambda u_p''(1) + (1 + 2\lambda) u_p'(1) + (u_p(1) - 1) \leq \beta |b|.
\]

(25)

Proof. In view of (15), we must show that

\[
\sum_{n=2}^{\infty} n [\lambda n + (1 - \lambda)] \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!} \leq \beta |b|.
\]

(26)
As in the proof of Theorem 1, we have

\[
\sum_{n=2}^{\infty} n [\lambda n + (1 - \lambda)] \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!} = \lambda \sum_{n=2}^{\infty} (n-1)(n-2) \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!} + 3\lambda \sum_{n=2}^{\infty} (n-1) \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!} + (1 - \lambda) \sum_{n=2}^{\infty} (n-1) \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!}
\]

\[
+ \lambda \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!} + 3\lambda \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!} + (1 - \lambda) \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!}
\]

\[
= \lambda \sum_{n=3}^{\infty} \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-3)!} + (1 + 2\lambda) \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-2)!} + \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!}
\]

\[
+ \sum_{n=0}^{\infty} \frac{(-c/4)^{n+1}}{(k)_{n+1}(n+1)!}
\]

\[
= \lambda u'_p(1) + (1 + 2\lambda)u'_p(1) + (u_p(1) - 1).
\]

But this last expression is bounded above by \(\beta |b|\) if (25) holds. This completes the proof. \(\square\)

By using a similar method as in the proof of Corollary 1, we obtain the following corollary.

**Corollary 6.** If \(c < 0, k > 0(k \neq 0, -1, -2, \cdots)\), then \(z(2 - u_p(z))\) is in the class \(TR(b, \lambda, \beta)\) if and only if the condition (25) is satisfied.

Putting \(\lambda = 0\) in Theorem 2 and Corollary 6, respectively, we obtain the following corollaries.

**Corollary 7.** If \(c < 0, k > 0(k \neq 0, -1, -2, \cdots)\), then \(zu_p(z)\) is in the class \(R(b, \beta)\) if

\[
u'_p(1) + u_p(1) - 1 \leq \beta |b|.
\] (27)

**Corollary 8.** If \(c < 0, k > 0(k \neq 0, -1, -2, \cdots)\), then \(z(2 - u_p(z))\) is in the class \(TR(b, \beta)\) if and only if (27) is satisfied.

**Theorem 3.** If \(c < 0, k > 0(k \neq 0, -1, -2, \cdots)\), then \(zu_p(z)\) is in the class \(S(b, \lambda, \beta)\) if

\[
e^{(\lambda u'_p)} \left[ \frac{\lambda c^2}{16k^2} + (1 + \lambda(\beta |b| + 1))(\frac{-c}{4k}) + \beta |b| (1 - e^{(\lambda u'_p)}) \right] \leq \beta |b|.
\] (28)
Proof. We note that
\[(k)_{n-1} = k(k+1)(k+2) \cdots (k+n-2) \geq k(k+1)^{n-2} \geq k^{n-1}, \quad (n \in \mathbb{N}). \tag{29}\]

From (14), (20) and (29), we have
\[
\sum_{n=2}^{\infty} \left[ \lambda n^2 + n(1 + \lambda(\beta |b| - 2)) + (\beta |b| - 1)(1 - \lambda) \right] \frac{(-c/4)^{n-1}}{(n-1)!} (k)_{n-1}(n-1)! = \lambda \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(n-1)!} + 3\lambda \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(n-1)!} \\
+ \lambda \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(n-1)!} (1 + \lambda(\beta |b| - 2)) \sum_{n=2}^{\infty} (n-1) \frac{(-c/4)^{n-1}}{(n-1)!} \\
+ (1 + \lambda(\beta |b| - 2)) \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(n-1)!} (k)_{n-1}(n-1)! + (\beta |b| - 1)(1 - \lambda) \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(n-1)!} \\
= \lambda \sum_{n=3}^{\infty} \frac{(-c/4)^{n-1}}{(n-3)!} + (1 + \lambda(\beta |b| + 1)) \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(n-2)!} \\
+ \beta |b| \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(n-1)!} \\
\leq \lambda \sum_{n=3}^{\infty} \frac{(-c/4k)^{n-1}}{(n-3)!} + (1 + \lambda(\beta |b| + 1)) \sum_{n=2}^{\infty} \frac{(-c/4k)^{n-1}}{(n-2)!} + \beta |b| \sum_{n=2}^{\infty} \frac{(-c/4k)^{n-1}}{(n-1)!} \\
= \lambda \left( \frac{\lambda c^2}{16k^2} e^{\left( \frac{-c}{4k} \right)} \right) + (1 + \lambda(\beta |b| + 1)) \left( \frac{-c}{4k} \right) e^{\left( \frac{-c}{4k} \right)} + \beta |b| \left( e^{\left( \frac{-c}{4k} \right)} - 1 \right) \\
= e^{\left( \frac{-c}{4k} \right)} \left[ \frac{\lambda c^2}{16k^2} + (1 + \lambda(\beta |b| + 1)) \left( \frac{-c}{4k} \right) + \beta |b| (1 - e^{\left( \frac{-c}{4k} \right)}) \right].
\]

Therefore, we see that the last expression is bounded above by \(\beta |b|\) if (28) holds. This completes the proof. \(\square\)

Putting \(\lambda = 0\) and \(\lambda = 0\), respectively in Theorem 3, we obtain the following corollaries.

Corollary 9. If \(c < 0, k > 0(k \neq 0, -1, -2, \cdots)\), then \(z_{u_p}(z)\) is in the class \(S(b, \beta)\) if
\[
e^{\left( \frac{-c}{4k} \right)} \left[ \frac{-c}{4k} + \beta |b| (1 - e^{\left( \frac{-c}{4k} \right)} \right] \leq \beta |b|. \tag{30}\]

Corollary 10. If \(c < 0, k > 0(k \neq 0, -1, -2, \cdots)\), then \(z(2 - u_p(z))\) is in the class \(C(b, \beta)\) if
\[
e^{\left( \frac{-c}{4k} \right)} \left[ \frac{c^2}{16k^2} + (\beta |b| + 2) \left( \frac{-c}{4k} \right) + \beta |b| (1 - e^{\left( \frac{-c}{4k} \right)} \right] \leq \beta |b|. \tag{31}\]

Theorem 4. If \(c < 0, k > 0(k \neq 0, -1, -2, \cdots)\), then \(z_{u_p}(z)\) is in the class \(R(b, \lambda, \beta)\) if
\[
e^{\left( \frac{-c}{4k} \right)} \left[ \frac{\lambda c^2}{16k^2} + 3\lambda \left( \frac{-c}{4k} \right) + (1 - e^{\left( \frac{-c}{4k} \right)} \right] \leq \beta |b|. \tag{32}\]
Corollary 12. Certain subclasses of analytic functions with complex order.

Proof. From (15), (20) and (29), we have

\[
\sum_{n=2}^{\infty} n [\lambda n + (1 - \lambda)] \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!} = \lambda \sum_{n=2}^{\infty} (n-1)(n-2) \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!} + 3\lambda \sum_{n=2}^{\infty} (n-1) \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!}
\]

\[
+ \lambda \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!} + (1 - \lambda) \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!}
\]

\[
\leq \lambda \sum_{n=3}^{\infty} \frac{(-c/4k)^{n-1}}{(n-3)!} + 3\lambda \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-2)!} + \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!}
\]

\[
= \lambda \left( \frac{c^2}{16k^2} e^{(-c/k)} \right) + 3\lambda \left( \frac{-c}{4k} \right) e^{(-c/k)} + (e^{(-c/k)} - 1)
\]

\[
e^{\left( \frac{c^2}{16k^2} \right)} + 3\lambda (\frac{-c}{4k}) + (1 - e^{(-c/k)})
\]

Therefore, we see that the last expression is bounded above by \(\beta |b|\) if (32) satisfied. This completes the proof. \(\square\)

Putting \(\lambda = 0\) and \(\lambda = 0\), respectively in Theorem 4, we obtain the following corollaries.

Corollary 11. If \(c < 0, k > 0(k \neq 0, -1, -2, \cdots)\), then \(zu_p(z)\) is in the class \(S(b, \beta)\) if

\[e^{(\frac{c}{4k})} - 1 \leq \beta |b|.\]

Corollary 12. If \(c < 0, k > 0(k \neq 0, -1, -2, \cdots)\), then \(zu_p(z)\) is in the class \(C(b, \beta)\) if

\[e^{(\frac{c}{4k})} \left[ \frac{c^2}{16k^2} + 3\left( \frac{-c}{4k} \right) + (1 - e^{(-c/k)}) \right] \leq \beta |b|.\]

The proof of Theorems 5 and 6 (below) is much akin to that of Theorems 3 and 4, therefore the details may be omitted.

Theorem 5. If \(c < 0, k > 0(k \neq 0, -1, -2, \cdots)\), then \(z(2 - u_p(z))\) is in the class \(TS(b, \lambda, \beta)\) if

\[e^{(\frac{c}{4k})} \left[ \frac{\lambda c^2}{16k^2} + (1 + \lambda (|b| + 1))(\frac{-c}{4k}) + \beta |b| (1 - e^{(-c/k)}) \right] \leq \beta |b|. \ (33)\]

Theorem 6. If \(c < 0, k > 0(k \neq 0, -1, -2, \cdots)\), then \(z(2 - u_p(z))\) is in the class \(TR(b, \lambda, \beta)\) if

\[e^{(\frac{c}{4k})} \left[ \frac{\lambda c^2}{16k^2} + 3\lambda (\frac{-c}{4k}) + (1 - e^{(-c/k)}) \right] \leq \beta |b|. \ (34)\]
3 Inclusion Properties

Making use of Lemma 5, we will study the action of the Bessel function on the class $TR(b, \lambda, \beta)$.

**Theorem 7.** Let $c < 0$, $k > 0 (k \neq 0, -1, -2, \cdots )$. If $f \in R^\tau (A, B)$, then $J(k, c)f$ is in the class $TR(b, \lambda, \beta)$ if and only if

$$(A - B) |\tau| [\lambda u'_p(1) + u_p(1) - 1] \leq \beta |b|.$$  \hfill (35)

**Proof.** In view of (15), it suffices to show that

$$\sum_{n=2}^{\infty} n[\lambda n + (1 - \lambda)] \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!} |a_n| \leq \beta |b|.$$  \hfill (36)

Since $f \in R^\tau (A, B)$, then by Lemma 5, we get

$$|a_n| \leq \frac{(A - B) |\tau|}{n} (n \geq 1).$$  \hfill (37)

Thus we must show that

$$\sum_{n=2}^{\infty} n[\lambda n + (1 - \lambda)] \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!} \leq \frac{(A - B) |\tau|}{n} \sum_{n=2}^{\infty} n[\lambda n + (1 - \lambda)] \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!}.$$  \hfill (38)

We have

$$\left( A - B \right) |\tau| \times \left[ \lambda \sum_{n=2}^{\infty} (n-1) \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!} + \lambda \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!} \right] + (1 - \lambda) \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!}$$

$$= \left( A - B \right) |\tau| \left[ \lambda \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-2)!} + \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(k)_{n-1}(n-1)!} \right]$$

$$= \left( A - B \right) |\tau| \left[ \lambda \sum_{n=0}^{\infty} \frac{(-c/4)^{n+1}}{(k)_{n+1}n!} + \sum_{n=0}^{\infty} \frac{(-c/4)^{n+1}}{(k)_{n+1}(n+1)!} \right]$$

$$= \left( A - B \right) |\tau| [\lambda u'_p(1) + u_p(1) - 1].$$

Therefore, we see that the last expression is bounded above by $\beta |b|$ if (35) satisfied. This completes the proof. \hfill \Box

Putting $\lambda = 0$ in Theorem 7, we obtain the following corollary.

**Corollary 13.** Let $c < 0$, $k > 0 (k \neq 0, -1, -2, \cdots )$. If $f \in R^\tau (A, B)$, then $J(k, c)f$ is in the class $TR(b, \beta)$ if and only if

$$(A - B) |\tau| [u_p(1) - 1] \leq \beta |b|.$$
4 An integral operator

In this section, we obtain the necessary and sufficient condition for the integral operator \( G(k, c, z) \) defined by

\[
G(k, c, z) = \int_0^z (2 - u_p(t)) dt,
\]

(39)
to be in the class \( TR(b, \lambda, \beta) \).

**Theorem 8.** If \( c < 0, k > 0 (k \neq 0, -1, -2, \cdots) \), then the integral operator \( G(k, c, z) \) is in the class \( TR(b, \lambda, \beta) \) if and only if

\[
\lambda u_p'(1) + u_p(1) - 1 \leq \beta |b|.
\]

(40)

**Proof.** Since

\[
G(k, c, z) = z - \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1} z^{n}}{(k)_{n-1} n!},
\]

(41)
then in view of (15), we need only to show that

\[
\sum_{n=2}^{\infty} n \left[ \lambda n + (1 - \lambda) \right] \frac{(-c/4)^{n-1}}{(k)_{n-1} n!} \leq \beta |b|
\]

(42)
or, equivalently

\[
\sum_{n=2}^{\infty} \left[ \lambda n + (1 - \lambda) \right] \frac{(-c/4)^{n-1}}{(k)_{n-1} (n-1)!} \leq \beta |b|.
\]

(43)

We have

\[
\sum_{n=2}^{\infty} \left[ \lambda n + (1 - \lambda) \right] \frac{(-c/4)^{n-1}}{(k)_{n-1} (n-1)!} = \lambda \sum_{n=2}^{\infty} (n-1) \frac{(-c/4)^{n-1}}{(k)_{n-1} (n-1)!} + \lambda \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(k)_{n-1} (n-1)!} + (1 - \lambda) \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(k)_{n-1} (n-1)!} = \lambda \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(k)_{n-1} (n-2)!} + \sum_{n=2}^{\infty} \frac{(-c/4)^{n-1}}{(k)_{n-1} (n-1)!} = \lambda \sum_{n=0}^{\infty} \frac{(-c/4)^{n+1}}{(k)_{n+1} n!} + \sum_{n=0}^{\infty} \frac{(-c/4)^{n+1}}{(k)_{n+1} (n+1)!} = \lambda u_p'(1) + u_p(1) - 1.
\]

Therefore, we see that the last expression is bounded above by \( \beta |b| \) if (39) satisfied. This completes the proof. \( \square \)
Putting $\lambda = 0$ in Theorem 8, we obtain the following corollary.

Let $c < 0$, $k > 0 (k \neq 0, -1, -2, \cdots)$, then the integral operator $G(k, c, z)$ defined by (39) is in the class $TR(b, \beta)$ if and only if

$$u_p(1) - 1 \leq \beta |b|.$$  \hspace{1cm} (44)

5 Conclusion

Necessary and sufficient conditions for generalized Bessel functions of the first kind $zu_p(z)$ to be in the classes $S(b, \lambda, \beta)$ and $R(b, \lambda, \beta)$ of analytic functions with complex order were obtained. Also we give the necessary and sufficient conditions for $z(2 - u_p(z))$ to be in the classes $TS(b, \lambda, \beta)$ and $TR(b, \lambda, \beta)$. Furthermore, we give the necessary and sufficient conditions for $J(k, c)$ to be in the class $TR(b, \lambda, \beta)$ provided that the function $f$ is in the class $R^\tau(A, B)$. Finally, we give conditions for the integral operator $G(k, c, z) = \int_0^z (2 - u_p(t)) \, dt$ to be in the class $TR(b, \lambda, \beta)$.

References


Certain subclasses of analytic functions with complex order


