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# INNOVATIONS OF SOME DYNAMIC ESTIMATES COMBINED ON TIME SCALES

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**Abstract**

We establish fractional versions of generalizations of the Schweitzer, Kantorovich, Pólya–Szegö, Cassels, Greub–Rheinboldt, and reverse Minkowski inequalities on time scales. We present that fractional Pólya–Szegö's dynamic inequality generalizes Cassels' inequality. Time scales calculus unifies and extends discrete, continuous, quantum versions of results.

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## 1 Introduction

The calculus of time scales was accomplished by Stefan Hilger [10]. A time scale is an arbitrary nonempty closed subset of the real numbers. Let $\mathbb{T}$ be a time scale, $a, b \in \mathbb{T}$ with $a < b$ and an interval $[a, b]_{\mathbb{T}}$ means the intersection of the real interval with the given time scale. The major aim of the calculus of time scales is to establish results in general, comprehensive, unified, and extended forms. The basic ideas about time scales calculus are given in the monographs [5, 6]. Dynamic inequalities may be extended by applying the diamond-$\alpha$ integral, which is defined as a linear operator of delta and nabla integrals on time scales. Several results concerning dynamic inequalities on time scales and fractional dynamic inequalities on time scales have been developed in the last two decades (see [1, 2, 3, 4, 14, 15, 16, 17]).

We generalize the following classical inequalities [12] within fractional calculus on time scales.

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First, we consider the inequality given by Schweitzer [18] such that

$$\left(\frac{1}{p}\sum_{k=1}^{p} x_k\right)\left(\frac{1}{p}\sum_{k=1}^{p}\frac{1}{x_k}\right) \leq \frac{(M+m)^2}{4Mm}, \tag{1}$$

where $0 < m \leq x_k \leq M$ for $k = 1, \ldots, p$.

In the same paper, Schweitzer has also shown that if functions $y \mapsto f(y)$ and $y \mapsto \frac{1}{f(y)}$ are integrable on $[a, b]$ and $0 < m \leq f(y) \leq M$ on $[a, b]$, then

$$\int_a^b f(y)dy \int_a^b \frac{1}{f(y)}dy \leq \frac{(M+m)^2}{4Mm}(b-a)^2. \tag{2}$$

Pólya and Szegö [13] proved that

$$\frac{\left(\sum\limits_{k=1}^{p} x_k^2\right)\left(\sum\limits_{k=1}^{p} y_k^2\right)}{\left(\sum\limits_{k=1}^{p} x_k y_k\right)^2} \leq \left(\frac{\sqrt{\frac{MN}{mn}} + \sqrt{\frac{mn}{MN}}}{2}\right)^2, \tag{3}$$

where $0 < m \leq x_k \leq M$ and $0 < n \leq y_k \leq N$ for $k = 1, \ldots, p$.

Kantorovich [11] proved that

$$\left(\sum_{k=1}^{p} x_k y_k^2\right)\left(\sum_{k=1}^{p}\frac{1}{x_k} y_k^2\right) \leq \frac{1}{4}\left(\sqrt{\frac{M}{m}} + \sqrt{\frac{m}{M}}\right)^2\left(\sum_{k=1}^{p} y_k^2\right)^2, \tag{4}$$

where $0 < m \leq x_k \leq M$ and $y_k \in \mathbb{R}$ for $k = 1, \ldots, p$, and he pointed out that inequality (4) is a particular case of (3).

Greub and Rheinboldt [9] proved that

$$\left(\sum_{k=1}^{p} x_k^2 z_k^2\right)\left(\sum_{k=1}^{p} y_k^2 z_k^2\right) \leq \frac{(MN+mn)^2}{4MNmn}\left(\sum_{k=1}^{p} x_k y_k z_k^2\right)^2, \tag{5}$$

where $0 < m \leq x_k \leq M < \infty$, $0 < n \leq y_k \leq N < \infty$ and $z_k \in \mathbb{R}$ for $k = 1, \ldots, p$ with $\sum\limits_{k=1}^{p} z_k^2 < \infty$.

Next, we consider the following two additive versions of Cassels' inequality as given in [8], and we generalize these inequalities within fractional calculus on time scales.

Let $x_k > 0$, $y_k > 0$ and $w_k \geq 0$ for $k = 1, \ldots, p$. Suppose that $\phi = \min\limits_{1\leq k\leq p}\left\{\frac{x_k}{y_k}\right\}$ and $\varphi = \max\limits_{1\leq k\leq p}\left\{\frac{x_k}{y_k}\right\}$. Then

$$0 \leq \left(\sum_{k=1}^{p} w_k x_k^2 \sum_{k=1}^{p} w_k y_k^2\right)^{\frac{1}{2}} - \sum_{k=1}^{p} w_k x_k y_k \leq \frac{(\sqrt{\varphi}-\sqrt{\phi})^2}{2\sqrt{\varphi\phi}}\sum_{k=1}^{p} w_k x_k y_k, \tag{6}$$

and

$$0 \leq \sum_{k=1}^{p} w_k x_k^2 \sum_{k=1}^{p} w_k y_k^2 - \left(\sum_{k=1}^{p} w_k x_k y_k\right)^2 \leq \frac{(\varphi-\phi)^2}{4\varphi\phi}\left(\sum_{k=1}^{p} w_k x_k y_k\right)^2. \tag{7}$$

## 2 Preliminaries

We need the following results concerning time scales calculus.

The following definition concerning the time scale $\Delta$-Riemann–Liouville type fractional integral is given in [2, 4].

For $\alpha \geq 1$, the time scale $\Delta$-Riemann–Liouville type fractional integral for a function $f \in C_{rd}$ is defined by

$$\mathcal{I}_a^\alpha f(t) = \int_a^t h_{\alpha-1}(t, \sigma(\tau)) f(\tau) \Delta\tau, \tag{8}$$

which is an integral on $[a, t)_{\mathbb{T}}$, see [7] and $h_\alpha : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$, $\alpha \geq 0$ are the coordinate wise rd-continuous functions, such that $h_0(t, s) = 1$,

$$h_{\alpha+1}(t, s) = \int_s^t h_\alpha(\tau, s) \Delta\tau, \ \forall s, t \in \mathbb{T}. \tag{9}$$

Notice that

$$\mathcal{I}_a^1 f(t) = \int_a^t f(\tau) \Delta\tau,$$

which is absolutely continuous in $t \in [a, b]_{\mathbb{T}}$, see [7].

The following definition concerning the time scale $\nabla$-Riemann–Liouville type fractional integral is given in [3, 4].

For $\alpha \geq 1$, the time scale $\nabla$-Riemann–Liouville type fractional integral for a function $f \in C_{ld}$ is defined by

$$\mathcal{J}_a^\alpha f(t) = \int_a^t \hat{h}_{\alpha-1}(t, \rho(\tau)) f(\tau) \nabla\tau, \tag{10}$$

which is an integral on $(a, t]_{\mathbb{T}}$, see [7] and $\hat{h}_\alpha : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$, $\alpha \geq 0$ are the coordinate wise ld-continuous functions, such that $\hat{h}_0(t, s) = 1$,

$$\hat{h}_{\alpha+1}(t, s) = \int_s^t \hat{h}_\alpha(\tau, s) \nabla\tau, \ \forall s, t \in \mathbb{T}. \tag{11}$$

Notice that

$$\mathcal{J}_a^1 f(t) = \int_a^t f(\tau) \nabla\tau,$$

which is absolutely continuous in $t \in [a, b]_{\mathbb{T}}$, see [7].

In this paper, it is assumed that all considerable integrals exist and are finite.

## 3 Main results

In order to present our main results, first we give a simple proof for an extension of Schweitzer's and Kantorovich's inequalities on time scales by using the $\Delta$-Riemann–Liouville type fractional integral.

**Theorem 1.** *Let $w, f, g \in C_{rd}([a,b]_{\mathbb{T}}, \mathbb{R}-\{0\})$ be $\Delta$-integrable functions. Assume that there exist four positive $\Delta$-integrable functions $f_1$, $f_2$, $g_1$ and $g_2$ such that:*

$$0 < f_1(y) \leq |f(y)| \leq f_2(y) < \infty \text{ and } 0 < g_1(y) \leq |g(y)| \leq g_2(y) < \infty,$$

*$y \in [a,x]_{\mathbb{T}}$, $\forall x \in [a,b]_{\mathbb{T}}$. Let $\alpha, \beta \geq 1$ and $h_{\alpha-1}(.,.), h_{\beta-1}(.,.) > 0$. If $\zeta > 1$ with $\frac{1}{\zeta}+\frac{1}{\eta}=1$, then we have the following inequality*

$$\frac{\left(\mathfrak{I}_a^{\alpha}\left(|w(x)||f(x)|^2\right)\mathfrak{I}_a^{\beta}\left((g_1g_2)(x)|w(x)|\right)\right)^{\frac{1}{\zeta}}\left(\mathfrak{I}_a^{\alpha}\left((f_1f_2)(x)|w(x)|\right)\mathfrak{I}_a^{\beta}\left(|w(x)||g(x)|^2\right)\right)^{\frac{1}{\eta}}}{\mathfrak{I}_a^{\alpha}\left(f_1(x)|(wf)(x)|\right)\mathfrak{I}_a^{\beta}\left(g_1(x)|(wg)(x)|\right)+\mathfrak{I}_a^{\alpha}\left(f_2(x)|(wf)(x)|\right)\mathfrak{I}_a^{\beta}\left(g_2(x)|(wg)(x)|\right)} \leq \frac{1}{\zeta^{\frac{1}{\zeta}}\eta^{\frac{1}{\eta}}}. \quad (12)$$

*Proof.* Using the given conditions, we have

$$\left(\frac{f_2(y)}{g_1(z)} - \frac{|f(y)|}{|g(z)|}\right) \geq 0,$$

and

$$\left(\frac{|f(y)|}{|g(z)|} - \frac{f_1(y)}{g_2(z)}\right) \geq 0, \quad y, z \in [a,x]_{\mathbb{T}},$$

which imply that

$$\left(\frac{f_1(y)}{g_2(z)} + \frac{f_2(y)}{g_1(z)}\right)\frac{|f(y)|}{|g(z)|} \geq \frac{|f(y)|^2}{|g(z)|^2} + \frac{f_1(y)f_2(y)}{g_1(z)g_2(z)}.$$

Multiplying both sides by $g_1(z)g_2(z)|g(z)|^2$, we have

$$f_1(y)g_1(z)|f(y)g(z)| + f_2(y)g_2(z)|f(y)g(z)| \geq g_1(z)g_2(z)|f(y)|^2 + f_1(y)f_2(y)|g(z)|^2. \quad (13)$$

Multiplying both sides of (13) by $h_{\alpha-1}(x,\sigma(y))|w(y)|h_{\beta-1}(x,\sigma(z))|w(z)|$ and double integrating over $y$ and $z$ from $a$ to $x$, respectively, we have

$$\begin{aligned}
&\mathfrak{I}_a^{\alpha}\left(f_1(x)|w(x)f(x)|\right)\mathfrak{I}_a^{\beta}\left(g_1(x)|w(x)g(x)|\right) \\
&\quad + \mathfrak{I}_a^{\alpha}\left(f_2(x)|w(x)f(x)|\right)\mathfrak{I}_a^{\beta}\left(g_2(x)|w(x)g(x)|\right) \\
&\quad \geq \mathfrak{I}_a^{\alpha}\left(|w(x)||f(x)|^2\right)\mathfrak{I}_a^{\beta}\left(g_1(x)g_2(x)|w(x)|\right) \\
&\quad\quad + \mathfrak{I}_a^{\alpha}\left(f_1(x)f_2(x)|w(x)|\right)\mathfrak{I}_a^{\beta}\left(|w(x)||g(x)|^2\right).
\end{aligned}$$

Analogously, we have

$$\begin{aligned}
&\mathfrak{I}_a^{\alpha}\left(f_1(x)|w(x)f(x)|\right)\mathfrak{I}_a^{\beta}\left(g_1(x)|w(x)g(x)|\right) \\
&\quad + \mathfrak{I}_a^{\alpha}\left(f_2(x)|w(x)f(x)|\right)\mathfrak{I}_a^{\beta}\left(g_2(x)|w(x)g(x)|\right) \\
&\quad \geq \frac{1}{\zeta}\left(\zeta\mathfrak{I}_a^{\alpha}\left(|w(x)||f(x)|^2\right)\mathfrak{I}_a^{\beta}\left(g_1(x)g_2(x)|w(x)|\right)\right) \\
&\quad\quad + \frac{1}{\eta}\left(\eta\mathfrak{I}_a^{\alpha}\left(f_1(x)f_2(x)|w(x)|\right)\mathfrak{I}_a^{\beta}\left(|w(x)||g(x)|^2\right)\right). \quad (14)
\end{aligned}$$

From the well-known Young's inequality $\xi\omega \leq \frac{1}{\zeta}\xi^{\zeta} + \frac{1}{\eta}\omega^{\eta}$, valid for nonnegative real numbers $\xi$ and $\omega$, inequality (14) takes the form

$$\begin{aligned}
&\mathcal{J}_a^{\alpha}\left(f_1(x)|w(x)f(x)|\right)\mathcal{J}_a^{\beta}\left(g_1(x)|w(x)g(x)|\right)\\
&\qquad\qquad + \mathcal{J}_a^{\alpha}\left(f_2(x)|w(x)f(x)|\right)\mathcal{J}_a^{\beta}\left(g_2(x)|w(x)g(x)|\right)\\
&\qquad\qquad \geq \left(\zeta\mathcal{J}_a^{\alpha}\left(|w(x)||f(x)|^2\right)\mathcal{J}_a^{\beta}\left(g_1(x)g_2(x)|w(x)|\right)\right)^{\frac{1}{\zeta}}\\
&\qquad\qquad\qquad\qquad \times\left(\eta\mathcal{J}_a^{\alpha}\left(f_1(x)f_2(x)|w(x)|\right)\mathcal{J}_a^{\beta}\left(|w(x)||g(x)|^2\right)\right)^{\frac{1}{\eta}}. \qquad (15)
\end{aligned}$$

Inequality (15) directly yields inequality (12). The proof of Theorem 1 is completed. □

Now, we give an extension of Schweitzer's and Kantorovich's inequalities on time scales by using the $\nabla$-Riemann–Liouville type fractional integral.

**Theorem 2.** *Let $w, f, g \in C_{ld}\left([a,b]_{\mathbb{T}}, \mathbb{R} - \{0\}\right)$ be $\nabla$-integrable functions. Assume that there exist four positive $\nabla$-integrable functions $f_1$, $f_2$, $g_1$ and $g_2$ such that:*

$$0 < f_1(y) \leq |f(y)| \leq f_2(y) < \infty \quad \textit{and} \quad 0 < g_1(y) \leq |g(y)| \leq g_2(y) < \infty,$$

*$y \in [a,x]_{\mathbb{T}}$, $\forall x \in [a,b]_{\mathbb{T}}$. Let $\alpha, \beta \geq 1$ and $\hat{h}_{\alpha-1}(.,.), \hat{h}_{\beta-1}(.,.) > 0$. If $\zeta > 1$ with $\frac{1}{\zeta} + \frac{1}{\eta} = 1$, then we have the following inequality*

$$\begin{aligned}
&\frac{\left(\mathfrak{J}_a^{\alpha}\left(|w(x)||f(x)|^2\right)\mathfrak{J}_a^{\beta}\left((g_1g_2)(x)|w(x)|\right)\right)^{\frac{1}{\zeta}}\left(\mathfrak{J}_a^{\alpha}\left((f_1f_2)(x)|w(x)|\right)\mathfrak{J}_a^{\beta}\left(|w(x)||g(x)|^2\right)\right)^{\frac{1}{\eta}}}{\mathfrak{J}_a^{\alpha}\left(f_1(x)|(wf)(x)|\right)\mathfrak{J}_a^{\beta}\left(g_1(x)|(wg)(x)|\right) + \mathfrak{J}_a^{\alpha}\left(f_2(x)|(wf)(x)|\right)\mathfrak{J}_a^{\beta}\left(g_2(x)|(wg)(x)|\right)}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \leq \frac{1}{\zeta^{\frac{1}{\zeta}}\eta^{\frac{1}{\eta}}}. \qquad (16)
\end{aligned}$$

*Proof.* Similar to the proof of Theorem 1. □

**Remark 1.** *We have the following:*

*(i)* *Let $\alpha = \beta = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $x = b = p+1$, $x_k > 0$, $w(k) = w_k = \frac{1}{x_k}$, $f(k) = x_k$ for $k = 1, \dots, p$, $f_1 = m$, $f_2 = M$, $g_1 = g = g_2 = 1$ and $\zeta = \eta = 2$. Then inequality* (12) *reduces to inequality* (1).

*(ii)* *Let $\alpha = \beta = 1$, $\mathbb{T} = \mathbb{R}$, $x = b$, $w(y) = \frac{1}{f(y)}$ on $[a,b]$, $0 < m \leq f(y) \leq M$ on $[a,b]$, $f_1 = m$, $f_2 = M$, $g_1 = g = g_2 = 1$ and $\zeta = \eta = 2$. Then inequality* (12) *reduces to inequality* (2).

*(iii)* *Let $\alpha = \beta = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $x = b = p+1$, $x_k > 0$, $y_k \in \mathbb{R}$, $w(k) = w_k = \frac{1}{x_k}y_k^2$, $f(k) = x_k$ for $k = 1, \dots, p$, $f_1 = m$, $f_2 = M$, $g_1 = g = g_2 = 1$ and $\zeta = \eta = 2$. Then inequality* (12) *reduces to inequality* (4).

Next, we give an extension of Pólya–Szegö's inequality by using the time scale $\Delta$-Riemann–Liouville type fractional integral.

**Theorem 3.** *Let $w, f, g \in C_{rd}([a,b]_{\mathbb{T}}, \mathbb{R} - \{0\})$ be $\Delta$-integrable functions such that $0 < m \leq |f(y)| \leq M < \infty$ and $0 < n \leq |g(y)| \leq N < \infty$ on the set $[a,x]_{\mathbb{T}}$, $\forall x \in [a,b]_{\mathbb{T}}$. If $\zeta > 1$ with $\frac{1}{\zeta} + \frac{1}{\eta} = 1$, then for $\alpha \geq 1$ and $h_{\alpha-1}(.,.) > 0$, we have the following inequality*

$$\left(\mathfrak{I}_a^{\alpha}(|w(x)||f(x)|^2)\right)^{\frac{1}{\zeta}} \left(\mathfrak{I}_a^{\alpha}(|w(x)||g(x)|^2)\right)^{\frac{1}{\eta}} \leq \frac{1}{\zeta^{\frac{1}{\zeta}} \eta^{\frac{1}{\eta}}} \frac{MN + mn}{(Mm)^{\frac{1}{\eta}} (Nn)^{\frac{1}{\zeta}}} \mathfrak{I}_a^{\alpha}(|w(x)||f(x)g(x)|). \quad (17)$$

*Proof.* Using the given conditions, for $y \in [a,x]_{\mathbb{T}}$, $\forall x \in [a,b]_{\mathbb{T}}$, we have

$$\frac{m}{N} \leq \frac{|f(y)|}{|g(y)|} \leq \frac{M}{n},$$

from which one has

$$0 \leq \left(\frac{|f(y)|}{|g(y)|} - \frac{m}{N}\right)\left(\frac{M}{n} - \frac{|f(y)|}{|g(y)|}\right).$$

Therefore,

$$\frac{|f(y)|^2}{|g(y)|^2} + \frac{Mm}{Nn} \leq \left(\frac{M}{n} + \frac{m}{N}\right)\frac{|f(y)|}{|g(y)|}.$$

Multiplying both sides by $h_{\alpha-1}(x, \sigma(y))|w(y)||g(y)|^2$ and integrating over $y$ from $a$ to $x$, we have

$$\mathfrak{I}_a^{\alpha}\left(|w(x)||f(x)|^2\right) + \frac{Mm}{Nn}\left(\mathfrak{I}_a^{\alpha}\left(|w(x)||g(x)|^2\right)\right) \leq \left(\frac{M}{n} + \frac{m}{N}\right)\mathfrak{I}_a^{\alpha}\left(|w(x)||f(x)g(x)|\right), \quad (18)$$

which leads to

$$\frac{1}{\zeta}\left(\zeta\mathfrak{I}_a^{\alpha}(|w(x)||f(x)|^2)\right) + \frac{1}{\eta}\left(\eta\frac{Mm}{Nn}\mathfrak{I}_a^{\alpha}\left(|w(x)||g(x)|^2\right)\right) \leq \left(\frac{M}{n} + \frac{m}{N}\right)\mathfrak{I}_a^{\alpha}\left(|w(x)||f(x)g(x)|\right). \quad (19)$$

From the well-known Young's inequality $\xi\omega \leq \frac{1}{\zeta}\xi^{\zeta} + \frac{1}{\eta}\omega^{\eta}$, valid for nonnegative real numbers $\xi$ and $\omega$, inequality (19) takes the form

$$\left(\zeta\mathfrak{I}_a^{\alpha}(|w(x)||f(x)|^2)\right)^{\frac{1}{\zeta}}\left(\eta\frac{Mm}{Nn}\mathfrak{I}_a^{\alpha}\left(|w(x)||g(x)|^2\right)\right)^{\frac{1}{\eta}} \leq \left(\frac{M}{n} + \frac{m}{N}\right)\mathfrak{I}_a^{\alpha}\left(|w(x)||f(x)g(x)|\right). \quad (20)$$

Inequality (20) directly yields (17). The proof of Theorem 3 is completed. □

Now, we give an extension of Pólya–Szegö's inequality by using the time scale $\nabla$-Riemann–Liouville type fractional integral.

**Theorem 4.** *Let $w, f, g \in C_{ld}\left([a,b]_{\mathbb{T}}, \mathbb{R}-\{0\}\right)$ be $\nabla$-integrable functions such that $0 < m \leq |f(y)| \leq M < \infty$ and $0 < n \leq |g(y)| \leq N < \infty$ on the set $[a,x]_{\mathbb{T}}$, $\forall x \in [a,b]_{\mathbb{T}}$. If $\zeta > 1$ with $\frac{1}{\zeta}+\frac{1}{\eta}=1$, then for $\alpha \geq 1$ and $\hat{h}_{\alpha-1}(.,.) > 0$, we have the following inequality*

$$\begin{aligned}\left(\mathcal{J}_a^{\alpha}(|w(x)||f(x)|^2)\right)^{\frac{1}{\zeta}}\left(\mathcal{J}_a^{\alpha}(|w(x)||g(x)|^2)\right)^{\frac{1}{\eta}}&\\ \leq \frac{1}{\zeta^{\frac{1}{\zeta}}\eta^{\frac{1}{\eta}}}\frac{MN+mn}{(Mm)^{\frac{1}{\eta}}(Nn)^{\frac{1}{\zeta}}}\mathcal{J}_a^{\alpha}(|w(x)||f(x)g(x)|).&\quad (21)\end{aligned}$$

*Proof.* Similar to the proof of Theorem 3. □

**Remark 2.** *Let $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $x = b = p+1$, $w \equiv 1$, $f(k) = x_k > 0$, $g(k) = y_k > 0$, $k = 1, \ldots, p$ and $\zeta = \eta = 2$. Then inequality* (17) *reduces to inequality* (3)*.*

Next, we give the following two additive versions of Cassels' inequality by using the time scale $\Delta$-Riemann–Liouville type fractional integral.

**Corollary 1.** *Let $w, f, g \in C_{rd}\left([a,b]_{\mathbb{T}}, \mathbb{R}-\{0\}\right)$ be $\Delta$-integrable functions such that $0 < \phi \leq \frac{|f(y)|}{|g(y)|} \leq \varphi < \infty$ on the set $[a,x]_{\mathbb{T}}$, $\forall x \in [a,b]_{\mathbb{T}}$. Then for $\alpha \geq 1$ and $h_{\alpha-1}(.,.) > 0$, we have the following inequalities*

$$\begin{aligned}0 \leq \left\{\left(\mathfrak{I}_a^{\alpha}(|w(x)||f(x)|^2)\right)\left(\mathfrak{I}_a^{\alpha}(|w(x)||g(x)|^2)\right)\right\}^{\frac{1}{2}} - \mathfrak{I}_a^{\alpha}(|w(x)||f(x)g(x)|)&\\ \leq \frac{(\sqrt{\varphi}-\sqrt{\phi})^2}{2\sqrt{\varphi\phi}}\mathfrak{I}_a^{\alpha}(|w(x)||f(x)g(x)|)&\quad (22)\end{aligned}$$

*and*

$$\begin{aligned}0 \leq \left(\mathfrak{I}_a^{\alpha}(|w(x)||f(x)|^2)\right)\left(\mathfrak{I}_a^{\alpha}(|w(x)||g(x)|^2)\right) - \left(\mathfrak{I}_a^{\alpha}(|w(x)||f(x)g(x)|)\right)^2&\\ \leq \frac{(\varphi-\phi)^2}{4\varphi\phi}\left(\mathfrak{I}_a^{\alpha}(|w(x)||f(x)g(x)|)\right)^2.&\quad (23)\end{aligned}$$

*Proof.* Let $\zeta = \eta = 2$, $\phi = \frac{m}{N}$ and $\varphi = \frac{M}{n}$. Subtracting $\mathfrak{I}_a^{\alpha}(|w(x)||f(x)g(x)|)$ on both sides of inequality (17), we get the desired inequality (22).

Further, if $\zeta = \eta = 2$, $\phi = \frac{m}{N}$ and $\varphi = \frac{M}{n}$, then inequality (17) reduces to

$$1 \leq \frac{\left(\mathfrak{I}_a^{\alpha}(|w(x)||f(x)|^2)\right)^{\frac{1}{2}}\left(\mathfrak{I}_a^{\alpha}(|w(x)||g(x)|^2)\right)^{\frac{1}{2}}}{\mathfrak{I}_a^{\alpha}(|w(x)||f(x)g(x)|)} \leq \frac{\varphi+\phi}{2\sqrt{\varphi\phi}}. \qquad (24)$$

By taking the square and subtracting 1 on both sides of the inequality (24), respectively, we get the desired inequality (23). □

Now, we give the following two additive versions of Cassels' inequality by using the time scale $\nabla$-Riemann–Liouville type fractional integral.

**Corollary 2.** *Let $w, f, g \in C_{ld}([a,b]_{\mathbb{T}}, \mathbb{R}-\{0\})$ be $\nabla$-integrable functions such that $0 < \phi \leq \frac{|f(y)|}{|g(y)|} \leq \varphi < \infty$ on the set $[a,x]_{\mathbb{T}}$, $\forall x \in [a,b]_{\mathbb{T}}$. Then for $\alpha \geq 1$ and $\hat{h}_{\alpha-1}(.,.) > 0$, we have the following inequalities*

$$\begin{aligned} 0 \leq \left\{ \left(\mathcal{J}_a^\alpha(|w(x)||f(x)|^2)\right)\left(\mathcal{J}_a^\alpha(|w(x)||g(x)|^2)\right)\right\}^{\frac{1}{2}} - \mathcal{J}_a^\alpha(|w(x)||f(x)g(x)|) \\ \leq \frac{(\sqrt{\varphi}-\sqrt{\phi})^2}{2\sqrt{\varphi\phi}}\mathcal{J}_a^\alpha(|w(x)||f(x)g(x)|) \end{aligned} \tag{25}$$

*and*

$$\begin{aligned} 0 \leq \left(\mathcal{J}_a^\alpha(|w(x)||f(x)|^2)\right)\left(\mathcal{J}_a^\alpha(|w(x)||g(x)|^2)\right) - \left(\mathcal{J}_a^\alpha(|w(x)||f(x)g(x)|)\right)^2 \\ \leq \frac{(\varphi-\phi)^2}{4\varphi\phi}\left(\mathcal{J}_a^\alpha(|w(x)||f(x)g(x)|)\right)^2. \end{aligned} \tag{26}$$

*Proof.* Similar to the proof of Corollary 1. □

**Remark 3.** *If we set $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $x = b = p+1$, $w(k) = w_k \geq 0$, $f(k) = x_k > 0$ and $g(k) = y_k > 0$, $k = 1, \ldots, p$, then* (22) *reduces to* (6) *and* (23) *reduces to* (7).

Next, we give the following two additive versions of the Pólya–Szegö inequality by using the time scale $\Delta$-Riemann–Liouville type fractional integral.

**Corollary 3.** *Let $w, f, g \in C_{rd}([a,b]_{\mathbb{T}}, \mathbb{R}-\{0\})$ be $\Delta$-integrable functions such that $0 < m \leq |f(y)| \leq M < \infty$ and $0 < n \leq |g(y)| \leq N < \infty$ on the set $[a,x]_{\mathbb{T}}$, $\forall x \in [a,b]_{\mathbb{T}}$. Then for $\alpha \geq 1$ and $h_{\alpha-1}(.,.) > 0$, we have the following inequalities*

$$\begin{aligned} 0 \leq \left\{ \left(\mathfrak{I}_a^\alpha(|w(x)||f(x)|^2)\right)\left(\mathfrak{I}_a^\alpha(|w(x)||g(x)|^2)\right)\right\}^{\frac{1}{2}} - \mathfrak{I}_a^\alpha(|w(x)||f(x)g(x)|) \\ \leq \frac{\left(\sqrt{MN}-\sqrt{mn}\right)^2}{2\sqrt{MNmn}}\mathfrak{I}_a^\alpha(|w(x)||f(x)g(x)|) \end{aligned} \tag{27}$$

*and*

$$\begin{aligned} 0 \leq \left(\mathfrak{I}_a^\alpha(|w(x)||f(x)|^2)\right)\left(\mathfrak{I}_a^\alpha(|w(x)||g(x)|^2)\right) - \left(\mathfrak{I}_a^\alpha(|w(x)||f(x)g(x)|)\right)^2 \\ \leq \frac{(MN-mn)^2}{4MNmn}\left(\mathfrak{I}_a^\alpha(|w(x)||f(x)g(x)|)\right)^2. \end{aligned} \tag{28}$$

*Proof.* Setting $\zeta = \eta = 2$ and subtracting $\mathfrak{I}_a^\alpha(|w(x)||f(x)g(x)|)$ on both sides of inequality (17), we get the desired inequality (27).

Further, if $\zeta = \eta = 2$, then inequality (17) reduces to

$$1 \leq \frac{\left(\mathfrak{I}_a^\alpha(|w(x)||f(x)|^2)\right)^{\frac{1}{2}}\left(\mathfrak{I}_a^\alpha(|w(x)||g(x)|^2)\right)^{\frac{1}{2}}}{\mathfrak{I}_a^\alpha(|w(x)||f(x)g(x)|)} \leq \frac{MN+mn}{2\sqrt{MNmn}}. \tag{29}$$

By taking the square and subtracting 1 on both sides of the inequality (29), respectively, we get the desired inequality (28). □

Now, we give the following two additive versions of the Pólya–Szegö inequality by using the time scale $\nabla$-Riemann–Liouville type fractional integral.

**Corollary 4.** *Let $w, f, g \in C_{ld}([a,b]_{\mathbb{T}}, \mathbb{R} - \{0\})$ be $\nabla$-integrable functions such that $0 < m \leq |f(y)| \leq M < \infty$ and $0 < n \leq |g(y)| \leq N < \infty$ on the set $[a,x]_{\mathbb{T}}$, $\forall x \in [a,b]_{\mathbb{T}}$. Then for $\alpha \geq 1$ and $\hat{h}_{\alpha-1}(.,.) > 0$, we have the following inequalities*

$$0 \leq \left\{ \left(\mathcal{J}_a^\alpha(|w(x)||f(x)|^2)\right)\left(\mathcal{J}_a^\alpha(|w(x)||g(x)|^2)\right)\right\}^{\frac{1}{2}} - \mathcal{J}_a^\alpha(|w(x)||f(x)g(x)|) \leq \frac{\left(\sqrt{MN} - \sqrt{mn}\right)^2}{2\sqrt{MNmn}} \mathcal{J}_a^\alpha(|w(x)||f(x)g(x)|) \quad (30)$$

*and*

$$0 \leq \left(\mathcal{J}_a^\alpha(|w(x)||f(x)|^2)\right)\left(\mathcal{J}_a^\alpha(|w(x)||g(x)|^2)\right) - \left(\mathcal{J}_a^\alpha(|w(x)||f(x)g(x)|)\right)^2 \leq \frac{(MN - mn)^2}{4MNmn} \left(\mathcal{J}_a^\alpha(|w(x)||f(x)g(x)|)\right)^2. \quad (31)$$

*Proof.* Similar to the proof of Corollary 3. □

**Remark 4.** *Let $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $x = b = p+1$, $w \equiv 1$, $f(k) = x_k > 0$ and $g(k) = y_k > 0$, $k = 1, \ldots, p$. Then inequality (27) reduces to*

$$0 \leq \left(\sum_{k=1}^{p} x_k^2 \sum_{k=1}^{p} y_k^2\right)^{\frac{1}{2}} - \sum_{k=1}^{p} x_k y_k \leq \frac{\left(\sqrt{MN} - \sqrt{mn}\right)^2}{2\sqrt{MNmn}} \sum_{k=1}^{p} x_k y_k \quad (32)$$

*and inequality (28) reduces to*

$$0 \leq \sum_{k=1}^{p} x_k^2 \sum_{k=1}^{p} y_k^2 - \left(\sum_{k=1}^{p} x_k y_k\right)^2 \leq \frac{(MN - mn)^2}{4MNmn} \left(\sum_{k=1}^{p} x_k y_k\right)^2. \quad (33)$$

*Inequalities (32) and (33) are given in [8].*

Next, we give an extension of Greub–Rheinboldt's inequality by using the time scale $\Delta$-Riemann–Liouville type fractional integral.

**Theorem 5.** *Let $w, f, g \in C_{rd}\left([a,b]_{\mathbb{T}}, \mathbb{R} - \{0\}\right)$ be $\Delta$-integrable functions. Assume that there exist four positive $\Delta$-integrable functions $f_1$, $f_2$, $g_1$ and $g_2$ such that:*

$$0 < f_1(y) \leq |f(y)| \leq f_2(y) < \infty \quad \text{and} \quad 0 < g_1(y) \leq |g(y)| \leq g_2(y) < \infty,$$

*$y \in [a,x]_{\mathbb{T}}$, $\forall x \in [a,b]_{\mathbb{T}}$. Let $\alpha \geq 1$ and $h_{\alpha-1}(.,.) > 0$. If $\zeta > 1$ with $\frac{1}{\zeta} + \frac{1}{\eta} = 1$, then we have the following inequality*

$$\frac{\left(\mathfrak{I}_a^\alpha\left(g_1(x)g_2(x)|w(x)||f(x)|^2\right)\right)^{\frac{1}{\zeta}}\left(\mathfrak{I}_a^\alpha\left(f_1(x)f_2(x)|w(x)||g(x)|^2\right)\right)^{\frac{1}{\eta}}}{\mathfrak{I}_a^\alpha\left((f_1(x)g_1(x) + f_2(x)g_2(x))|w(x)||f(x)g(x)|\right)} \leq \frac{1}{\zeta^{\frac{1}{\zeta}}\eta^{\frac{1}{\eta}}}. \quad (34)$$

*Proof.* Using the given conditions, for $y \in [a, x]_{\mathbb{T}}$, $\forall x \in [a, b]_{\mathbb{T}}$, we have

$$\left(\frac{f_2(y)}{g_1(y)} - \frac{|f(y)|}{|g(y)|}\right) \geq 0,$$

and

$$\left(\frac{|f(y)|}{|g(y)|} - \frac{f_1(y)}{g_2(y)}\right) \geq 0.$$

Multiplying the last two inequalities, we have

$$\left(\frac{f_2(y)}{g_1(y)} - \frac{|f(y)|}{|g(y)|}\right)\left(\frac{|f(y)|}{|g(y)|} - \frac{f_1(y)}{g_2(y)}\right) \geq 0,$$

which implies

$$\left(\frac{f_1(y)}{g_2(y)} + \frac{f_2(y)}{g_1(y)}\right)\frac{|f(y)|}{|g(y)|} \geq \frac{|f(y)|^2}{|g(y)|^2} + \frac{f_1(y)f_2(y)}{g_1(y)g_2(y)}.$$

Multiplying both sides by $g_1(y)g_2(y)|g(y)|^2$, we have

$$\begin{aligned} &f_1(y)g_1(y)|f(y)g(y)| + f_2(y)g_2(y)|f(y)g(y)| \\ &\qquad \geq g_1(y)g_2(y)|f(y)|^2 + f_1(y)f_2(y)|g(y)|^2. \end{aligned} \tag{35}$$

Multiplying both sides of (35) by $h_{\alpha-1}(x, \sigma(y))|w(y)|$ and integrating over $y$ from $a$ to $x$, we have

$$\begin{aligned} &\mathcal{I}_a^\alpha\left(\left(f_1(x)g_1(x) + f_2(x)g_2(x)\right)|w(x)||f(x)g(x)|\right) \\ &\qquad \geq \mathcal{I}_a^\alpha\left(g_1(x)g_2(x)|w(x)||f(x)|^2\right) + \mathcal{I}_a^\alpha\left(f_1(x)f_2(x)|w(x)||g(x)|^2\right), \end{aligned}$$

which leads to

$$\begin{aligned} &\mathcal{I}_a^\alpha\left(\left(f_1(x)g_1(x) + f_2(x)g_2(x)\right)|w(x)||f(x)g(x)|\right) \\ &\geq \frac{1}{\zeta}\left(\zeta\mathcal{I}_a^\alpha\left(g_1(x)g_2(x)|w(x)||f(x)|^2\right)\right) + \frac{1}{\eta}\left(\eta\mathcal{I}_a^\alpha\left(f_1(x)f_2(x)|w(x)||g(x)|^2\right)\right). \end{aligned} \tag{36}$$

From the well-known Young's inequality $\xi\omega \leq \frac{1}{\zeta}\xi^\zeta + \frac{1}{\eta}\omega^\eta$, valid for nonnegative real numbers $\xi$ and $\omega$, we get

$$\begin{aligned} &\mathcal{I}_a^\alpha\left(\left(f_1(x)g_1(x) + f_2(x)g_2(x)\right)|w(x)||f(x)g(x)|\right) \\ &\quad \geq \left(\zeta\mathcal{I}_a^\alpha\left(g_1(x)g_2(x)|w(x)||f(x)|^2\right)\right)^{\frac{1}{\zeta}}\left(\eta\mathcal{I}_a^\alpha\left(f_1(x)f_2(x)|w(x)||g(x)|^2\right)\right)^{\frac{1}{\eta}}. \end{aligned} \tag{37}$$

Analogously, we have that

$$\begin{aligned} &\left(\mathcal{I}_a^\alpha\left(g_1(x)g_2(x)|w(x)||f(x)|^2\right)\right)^{\frac{1}{\zeta}}\left(\mathcal{I}_a^\alpha\left(f_1(x)f_2(x)|w(x)||g(x)|^2\right)\right)^{\frac{1}{\eta}} \\ &\qquad \leq \frac{1}{\zeta^{\frac{1}{\zeta}}\eta^{\frac{1}{\eta}}}\left(\mathcal{I}_a^\alpha\left(\left(f_1(x)g_1(x) + f_2(x)g_2(x)\right)|w(x)||f(x)g(x)|\right)\right). \end{aligned} \tag{38}$$

Inequality (38) directly yields (34). The proof of Theorem 5 is completed. $\square$

Now, we give an extension of Greub–Rheinboldt's inequality by using the time scale $\nabla$-Riemann–Liouville type fractional integral.

**Theorem 6.** *Let $w, f, g \in C_{ld}([a,b]_{\mathbb{T}}, \mathbb{R}-\{0\})$ be $\nabla$-integrable functions. Assume that there exist four positive $\nabla$-integrable functions $f_1$, $f_2$, $g_1$ and $g_2$ such that:*

$$0 < f_1(y) \leq |f(y)| \leq f_2(y) < \infty \quad \text{and} \quad 0 < g_1(y) \leq |g(y)| \leq g_2(y) < \infty,$$

*$y \in [a,x]_{\mathbb{T}}$, $\forall x \in [a,b]_{\mathbb{T}}$. Let $\alpha \geq 1$ and $\hat{h}_{\alpha-1}(.,.) > 0$. If $\zeta > 1$ with $\frac{1}{\zeta} + \frac{1}{\eta} = 1$, then we have the following inequality*

$$\frac{\left(\mathcal{J}_a^{\alpha}\left(g_1(x)g_2(x)|w(x)||f(x)|^2\right)\right)^{\frac{1}{\zeta}}\left(\mathcal{J}_a^{\alpha}\left(f_1(x)f_2(x)|w(x)||g(x)|^2\right)\right)^{\frac{1}{\eta}}}{\mathcal{J}_a^{\alpha}\left((f_1(x)g_1(x)+f_2(x)g_2(x))|w(x)||f(x)g(x)|\right)} \leq \frac{1}{\zeta^{\frac{1}{\zeta}}\eta^{\frac{1}{\eta}}}. \tag{39}$$

*Proof.* Similar to the proof of Theorem 5. □

**Remark 5.** *Let $\alpha = 1$, $\mathbb{T} = \mathbb{Z}$, $a = 1$, $x = b = p+1$, $z_k \in \mathbb{R}$, $w(k) = w_k = z_k^2$, $f(k) = x_k > 0$, $g(k) = y_k > 0$, $k = 1, \ldots, p$, $f_1 = m$, $f_2 = M$, $g_1 = n$, $g_2 = N$ and $\zeta = \eta = 2$. Then inequality (34) reduces to inequality (5).*

In order to conclude this paper, we give an extension of reverse Minkowski's inequality by using the time scale $\Delta$-Riemann–Liouville type fractional integral.

**Theorem 7.** *Let $w, f, g \in C_{rd}([a,b]_{\mathbb{T}}, \mathbb{R}-\{0\})$ be $\Delta$-integrable functions such that $0 < m \leq |f(y)|^p, |g(y)|^p \leq M < \infty$ on the set $[a,x]_{\mathbb{T}}$, $\forall x \in [a,b]_{\mathbb{T}}$ for $p > 1$. Then for $\alpha \geq 1$ and $h_{\alpha-1}(.,.) > 0$, we have the following inequality*

$$\begin{aligned}(\mathcal{I}_a^{\alpha}(|w(x)||f(x)|^p))^{\frac{1}{p}} &+ (\mathcal{I}_a^{\alpha}(|w(x)||g(x)|^p))^{\frac{1}{p}} \\ &\leq 2\left(\frac{M}{m}\right)^{\frac{1}{p}}(\mathcal{I}_a^{\alpha}(|w(x)|(|f(x)|+|g(x)|)^p))^{\frac{1}{p}}.\end{aligned} \tag{40}$$

*Proof.* Using the given conditions, for $y \in [a,x]_{\mathbb{T}}$, $\forall x \in [a,b]_{\mathbb{T}}$, we have

$$m^{\frac{1}{p}} \leq |f(y)| \leq M^{\frac{1}{p}} \text{ and } m^{\frac{1}{p}} \leq |g(y)| \leq M^{\frac{1}{p}}.$$

Multiplying by $(\mathcal{I}_a^{\alpha}(|w(x)||f(x)|^p))^{\frac{1}{p}}$ and $(\mathcal{I}_a^{\alpha}(|w(x)||g(x)|^p))^{\frac{1}{p}}$, respectively, we get

$$\begin{aligned}m^{\frac{1}{p}}(\mathcal{I}_a^{\alpha}(|w(x)||f(x)|^p))^{\frac{1}{p}} &\leq M^{\frac{1}{p}}(\mathcal{I}_a^{\alpha}(|w(x)||f(x)|^p))^{\frac{1}{p}} \\ &\leq M^{\frac{1}{p}}(\mathcal{I}_a^{\alpha}(|w(x)|(|f(x)|+|g(x)|)^p))^{\frac{1}{p}}\end{aligned} \tag{41}$$

and

$$\begin{aligned}m^{\frac{1}{p}}(\mathcal{I}_a^{\alpha}(|w(x)||g(x)|^p))^{\frac{1}{p}} &\leq M^{\frac{1}{p}}(\mathcal{I}_a^{\alpha}(|w(x)||g(x)|^p))^{\frac{1}{p}} \\ &\leq M^{\frac{1}{p}}(\mathcal{I}_a^{\alpha}(|w(x)|(|f(x)|+|g(x)|)^p))^{\frac{1}{p}}.\end{aligned} \tag{42}$$

Adding (41) and (42), we get the desired inequality (40). □

Next, we give an extension of reverse Minkowski's inequality by using the time scale $\nabla$-Riemann–Liouville type fractional integral.

**Theorem 8.** *Let $w, f, g \in C_{ld}([a,b]_{\mathbb{T}}, \mathbb{R}-\{0\})$ be $\nabla$-integrable functions such that $0 < m \leq |f(y)|^p, |g(y)|^p \leq M < \infty$ on the set $[a,x]_{\mathbb{T}}$, $\forall x \in [a,b]_{\mathbb{T}}$ for $p > 1$. Then for $\alpha \geq 1$ and $\hat{h}_{\alpha-1}(.,.) > 0$, we have the following inequality*

$$\begin{aligned} &(\mathcal{J}_a^\alpha(|w(x)||f(x)|^p))^{\frac{1}{p}} + (\mathcal{J}_a^\alpha(|w(x)||g(x)|^p))^{\frac{1}{p}} \\ &\qquad \leq 2\left(\frac{M}{m}\right)^{\frac{1}{p}} (\mathcal{J}_a^\alpha(|w(x)|(|f(x)|+|g(x)|)^p))^{\frac{1}{p}}. \end{aligned} \tag{43}$$

*Proof.* Similar to the proof of Theorem 7. □

## References


[1] Agarwal, R.P., O'Regan, D. and Saker, S.H., *Dynamic Inequalities on Time Scales*, Springer International Publishing, Cham, Switzerland, 2014.

[2] Anastassiou, G.A., *Principles of delta fractional calculus on time scales and inequalities*, Math. Comput. Modelling **52** (2010), no. 3-4, 556–566.

[3] Anastassiou, G.A., *Foundations of nabla fractional calculus on time scales and inequalities*, Comput. Math. Appl. **59** (2010), no. 12, 3750–3762.

[4] Anastassiou, G.A., *Integral operator inequalities on time scales*, Int. J. Difference Equ., **7** (2012), no. 2, 111–137.

[5] Bohner, M. and Peterson, A., *Dynamic Equations on Time Scales*, Birkhäuser Boston, Inc., Boston, MA, 2001.

[6] Bohner, M. and Peterson, A., *Advances in Dynamic Equations on Time Scales*, Birkhäuser Boston, Boston, MA, 2003.

[7] Bohner, M. and Luo, H., *Singular second-order multipoint dynamic boundary value problems with mixed derivatives*, Adv. Difference Equ. (2006), 1–15. https://doi.org/10.1155/ade/2006/54989.

[8] Dragomir, S.S., *A survey on Cauchy–Bunyakovsky–Schwarz type discrete inequalities*, Journal of Inequalities in Pure and Applied Mathematics, **4** (2003), no. 3, Article 63.

[9] Greub, W. and Rheinboldt, W., *On a generalization of an inequality of L.V. Kantorovich*, Proc. Amer. Math. Soc., **10** (1959), 407–415.

[10] Hilger, S., *Ein Maβkettenkalkül mit Anwendung auf Zentrumsmannigfaltigkeiten*, Ph.D. Thesis, Universität Würzburg, 1988.

[11] Kantorovich, L.V., *Functional analysis and applied mathematics* (Russian), Uspehi Mat. Nauk (N.S.), **3** (1948), no. 6 (28), 89–185, (in particular, pp. 142–144) [also translated from Russian into English by C. D. Benster, Nat. Bur. Standards Rep. No. 1509. 1952, 202 pp. (in particular, pp. 106–109)].

[12] Mitrinović, D.S., *Analytic Inequalities*, Springer-Verlag, Berlin, 1970.

[13] Pólya, G. and Szegö, G., *Aufgaben und Lehrsätze aus der Analysis*, Berlin, **1**, p. 57 and pp. 213–214, 1925.

[14] Sahir, M.J.S., *Consonancy of dynamic inequalities correlated on time scale calculus*, Tamkang J. Math. **51** (2020), no. 3, 233–243.

[15] Sahir, M.J.S., *Homogeneity of classical and dynamic inequalities compatible on time scales*, Int. J. Difference Equ. **15** (2020), no. 1, 173–186.

[16] Sahir, M.J.S., *Analogy of classical and dynamic inequalities merging on time scales*, Journal of Mathematics and Applications, **43** (2020), 139–152.

[17] Sahir, M.J.S., *Integrity of variety of inequalities sketched on time scales*, Journal of Abstract and Computational Mathematics, **6** (2021), no. 2, 8–15.

[18] Schweitzer, P., *An inequality concerning the arithmetic mean* (Hungarian), Math. Phys. Lapok, **23** (1914), 257–261.