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# FIXED POINT THEOREMS ON PRODUCT OF $b$-METRIC SPACES

**Izabella ABRAHAM** [1]

**Abstract**

The aim of this paper is to extend some fixed point results from [Şerban, M. A., *Teoria punctului fix pentru operatori definiţi pe produs cartezian*, Presa Universitară Clujeană, Cluj–Napoca, 2002] and [Prešić, S. B., *Sur une classe d' inéquations aux différences finite et sur la convergence de certaines suites*, Publ. Inst. Math. (Beograd) (N. S.). **5(19)** (1965), 75-78] in the framework of $b$-metric spaces.

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## 1 Introduction

One can observe, in the last decades, a considerable interest for generalizations of the notion of a metric space and the development of fixed point theory in such structures.

The notion of $b$-metric space was introduced by I. A. Bakhtin (1989) [2] and S. Czerwik (1998) [6], [7].

**Definition 1.** *Given a nonempty set $X$ and a real number $s \geq 1$, a function $d : X \times X \to [0, \infty)$ is called a $b$-metric if it satisfies the following properties:*

1. $d(x, y) = 0$ *if and only if* $x = y$,
2. $d(x, y) = d(y, x)$,
3. $d(x, y) \leq s[d(x, z) + d(z, y)]$,

*for all $x, y, z \in X$. The triplet $(X, d, s)$ is called a $b$-metric space.*

Inequality *3.* is called the *s-relaxed triangle inequality*.

[1]Faculty of Mathematics and Informatics, Transilvania University of Braşov, Romania, e-mail: izabella.abraham@student.unitbv.ro

**Remark 1.** *Every metric space is a b-metric space (with $s = 1$), but there exist b-metric spaces which are not metric spaces (see [3], [11]).*

Note that in a $b$-metric space $(X, d, s)$, the $s$-relaxed triangle inequality implies that (see [5]):

$$\begin{aligned} d(x_0, x_n) &\le sd(x_0, x_1) + sd(x_1, x_2) \\ &\le sd(x_0, x_1) + s^2 d(x_1, x_2) + s^2 d(x_2, x_n) \le ... \\ &\le sd(x_0, x_1) + ... + s^{n-1} d(x_{n-2}, x_{n-1}) + s^{n-1} d(x_{n-1}, x_n), \end{aligned} \tag{1}$$

for all $x_0, ..., x_n \in X$ and $n \in \mathbb{N}$.

**Definition 2.** *Let $(X, d, s)$ be a b-metric space. A sequence $(x_n)_n \subseteq X$ is called:*

- *convergent if there exists $x \in X$ such that $\lim_{n\to\infty} d(x_n, x) = 0$;*
- *Cauchy if $\lim_{n,m\to\infty} d(x_n, x_m) = 0$, i.e. for every $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$, for all $n, m \in \mathbb{N}$, $n, m \ge N_\varepsilon$.*

*The space $(X, d, s)$ is said to be complete if every Cauchy sequence of elements from $(X, d, s)$ is convergent.*

**Remark 2.** *Let $(X, d, s)$ be a b-metric space, $k \in \mathbb{N}^*$ and $d_{max} : X^k \times X^k \to [0, \infty)$ given by*

$$d_{max}((x_1, ..., x_k), (y_1, ..., y_k)) = \max\{d(x_1, y_1), ..., d(x_k, y_k)\},$$

*for all $(x_1, ..., x_k), (y_1, ..., y_k) \in X^k$. Then $(X^k, d_{max}, s)$ is a b-metric space.*
Indeed, we only need to check the $s$-relaxed triangle inequality, since the other conditions are trivially satisfied. Let $(x_1, ..., x_k), (y_1, ..., y_k), (z_1, ..., z_k) \in X^k$. Then, it follows that

$$\begin{aligned} &d_{max}((x_1, ..., x_k), (y_1, ..., y_k)) = \max\{d(x_1, y_1), ..., d(x_k, y_k)\} \le \\ &\le \max\{sd(x_1, z_1) + sd(z_1, y_1), ..., sd(x_k, z_k) + sd(z_k, y_k)\} \le \\ &\le s\max\{d(x_1, z_1), ..., d(x_k, z_k)\} + s\max\{d(z_1, y_1), ..., d(z_k, y_k)\} = \\ &= sd_{max}((x_1, ..., x_k), (z_1, ..., z_k)) + sd_{max}((z_1, ..., z_k), (y_1, ..., y_k)). \end{aligned}$$

In contrast to a metric space, the distance function in a $b$-metric space need not be continuous (see, for example, [1]).

If $(y_n)_n$ is a sequence of elements from $(X, d, s)$ such that $\lim_{n\to\infty} y_n = y$, the following chain of inequalities holds (see [8]):

$$\frac{1}{s} d(x, y) \le \liminf_{n\to\infty} d(x, y_n) \le \limsup_{n\to\infty} d(x, y_n) \le sd(x, y), \tag{2}$$

for all $x \in X$.

**Definition 3.** *Let $(X,d,s)$ and $(Y,\rho,r)$ be two b-metric spaces. A function $f: X\to Y$ is said to be continuous if for every $(x_n)_n \subseteq X$ and $x\in X$ such that $\lim\limits_{n\to\infty} x_n = x$ we have $\lim\limits_{n\to\infty} f(x_n) = f(x)$.*

Let $(X,d,s)$ be a $b$-metric space, $k\in\mathbb{N}^*$ and $f: X^k \to X$.
Inspired by the results from [10], we consider the sequence $(x_n)_n \subseteq X$ described as follows:

$$x_{k+n} = f(x_n, ..., x_{n+k-1}), n\in\mathbb{N}^* \tag{3}$$

with initial values $x_1, ..., x_k \in X$.
Denote by $F_f$ the set of fixed points of $f$, that is

$$F_f = \left\{x^* \in X : x^* = f(x^*, ..., x^*)\right\},$$

and define $\widetilde{f}: X\to X$ as follows:

$$\widetilde{f}(x) = f(x, ..., x),$$

for all $x\in X$.

**Definition 4.** *A mapping $\psi: \mathbb{R}_+^k \to \mathbb{R}_+$ is said to be*

- *monotonically increasing if:*

  $$\psi(e_1, ..., e_k) \le \psi(f_1, ..., f_k),$$

  *for all $e_1, ..., e_k, f_1, ..., f_k \in \mathbb{R}_+$ such that $e_i \le f_i$ for every $i\in\{1, ..., k\}$;*

- *positively semihomogenous if:*

  $$\psi(\lambda e_1, ..., \lambda e_k) \le \lambda\psi(e_1, ..., e_k),$$

  *for all $e_1, ..., e_k \in \mathbb{R}_+$ and $\lambda \ge 0$.*

For the proof of the main results we need the following two lemmas given by M. R. Tasković and T. Suzuki.

**Lemma 1** (see Proposition 2 from [13])**.** *Let $\psi: \mathbb{R}_+^k \to \mathbb{R}_+$ be a monotonically increasing, positively semihomogenous and continuous mapping and let $(\alpha_n)_n$ be a sequence of positive real numbers satisfying the following conditions:*

$$\alpha_{n+k} \le \psi(a_1\alpha_n, ..., a_k\alpha_{n+k-1}),$$

*and*

$$\psi(a_1, ..., a_k) < 1,$$

*for all $n\in\mathbb{N}^*$, where $a_1, ..., a_k$ are fixed positive real constants.*
*Then there exists $\theta\in(0,1)$ such that*

$$\alpha_n \le L\theta^n,$$

*for all $n\in\mathbb{N}^*$, where $L = \max\left\{\frac{\alpha_1}{\theta}, ..., \frac{\alpha_k}{\theta^k}\right\}\in\mathbb{R}_+$.*

**Lemma 2** (see Lemma 5 from [12])**.** *Let $(X, d, s)$ be a b-metric space and $g : \mathbb{N}^* \to \mathbb{N}$ given by*

$$g(n) = -\left[-\log_2 n\right],$$

*for all $n \in \mathbb{N}^*$. Then*

$$d(x_n, x_m) \leq s^{g(m-n)} \sum_{i=n}^{m-1} d(x_i, x_{i+1}),$$

*for all $x_n, ..., x_m \in X$, with $n, m \in \mathbb{N}^*$, $n \leq m$.*

## 2 Main results

In this section we extend some known results regarding the existence and uniqueness of fixed points for mappings defined on Cartesian products of $b$-metric spaces.

**Lemma 3.** *Let $(X, d, s)$ be a b-metric space, $k \in \mathbb{N}^*$, $f : X^k \to X$ and let $(x_n)_n$ be the sequence given by (3). Suppose that there exists a continuous, monotonically increasing and positively semihomogenous mapping $\psi : \mathbb{R}_+^k \to \mathbb{R}_+$ such that*

$$d(x_{n+k}, x_{n+k+1}) \leq \psi(a_1 d(x_n, x_{n+1}), ..., a_k d(x_{n+k-1}, x_{n+k})) \tag{4}$$

*for all $n \in \mathbb{N}^*$, where $a_1, ..., a_k$ are fixed positive real constants such that*

$$\psi(a_1, ..., a_k) < 1. \tag{5}$$

*Then the sequence $(x_n)_n$ is Cauchy.*

*Proof.* We apply Lemma 1 for the sequence $(\alpha_n)_n$, where $\alpha_n = d(x_n, x_{n+1})$, $n \in \mathbb{N}^*$. In view of (4) and (5), there exists $\theta \in (0, 1)$ such that

$$d(x_n, x_{n+1}) \leq L\theta^n,$$

for all $n \in \mathbb{N}^*$, where $L = \max\left\{\frac{\alpha_1}{\theta}, ..., \frac{\alpha_k}{\theta^k}\right\} \in \mathbb{R}_+$.
Define $g : \mathbb{N}^* \to \mathbb{N}$ by $g(n) = -\left[-\log_2 n\right]$, for all $n \in \mathbb{N}^*$ and choose $n_0 \in \mathbb{N}$ satisfying $s\theta^{2^{n_0}} < 1$.
In order to prove that $(x_n)_n$ is a Cauchy sequence, let $n, m \in \mathbb{N}$, $n < m$.
**Case 1.** If $n < m \leq n + 2^{n_0}$, then, making use of Lemma 2, we see that

$$\begin{aligned} d(x_n, x_m) &\leq s^{g(m-n)} \sum_{i=n}^{m-1} d(x_i, x_{i+1}) \leq s^{n_0} \sum_{i=n}^{m-1} L\theta^i \leq s^{n_0} L\theta^n \sum_{i=0}^{\infty} \theta^i \\ &= s^{n_0} L\theta^n \frac{1}{1-\theta}, \end{aligned} \tag{6}$$

where we used the fact that

$$n < m \leq n + 2^{n_0} \Rightarrow g(m-n) \leq n_0,$$

since $g$ is an increasing function.
**Case 2.** If $n + 2^{n_0} < m$, denote $\mu := \left[\frac{m-n}{2^{n_0}}\right] > 1$. We apply Lemma 2 again, as well as (1), to obtain

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{i=0}^{\mu-1} s^{i+1} d(x_{n+i2^{n_0}}, x_{n+(i+1)2^{n_0}}) + s^{\mu} d(x_{n+\mu 2^{n_0}}, x_m) \\ &\leq \sum_{i=0}^{\mu-1} s^{i+1} s^{n_0} \left[d(x_{n+i2^{n_0}}, x_{n+i2^{n_0}+1}) + ... + d(x_{n+(i+1)2^{n_0}-1}, x_{n+(i+1)2^{n_0}})\right] \\ &\quad + s^{\mu} s^{g(m-n-\mu 2^{n_0})} \left[d(x_{n+\mu 2^{n_0}}, x_{n+\mu 2^{n_0}+1}) + ... + d(x_{m-1}, x_m)\right]. \end{aligned}$$

Since $g$ is an increasing function, we can assert that

$$g\left(m - n - \mu 2^{n_0}\right) = g\left(2^{n_0}\left(\frac{m-n}{2^{n_0}} - \left[\frac{m-n}{2^{n_0}}\right]\right)\right) < g(2^{n_0}) = n_0.$$

Consequently,

$$\begin{aligned} d(x_n, x_m) &\leq \sum_{i=0}^{\mu-1} s^{n_0+i+1} L \left[\theta^{n+i2^{n_0}} + ... + \theta^{n+(i+1)2^{n_0}-1}\right] + \\ &\quad + s^{\mu+n_0} L \left[\theta^{n+\mu 2^{n_0}} + ... + \theta^{m-1}\right] \\ &\leq \sum_{i=0}^{\mu-1} s^{n_0+i+1} L\theta^{n+i2^{n_0}} \frac{1}{1-\theta} + s^{\mu+n_0+1} L\theta^{n+\mu 2^{n_0}} \frac{1}{1-\theta} \\ &= \frac{\theta^n}{1-\theta} L s^{n_0+1} \sum_{i=0}^{\mu} s^i \theta^{i2^{n_0}} \\ &\leq \frac{\theta^n}{1-\theta} L s^{n_0+1} \sum_{i=0}^{\infty} \left(s\theta^{2^{n_0}}\right)^i, \end{aligned}$$

hence

$$d(x_n, x_m) \leq \frac{\theta^n}{1-\theta} L s^{n_0+1} \frac{1}{1 - s\theta^{2^{n_0}}}, \tag{7}$$

since $n_0$ has been chosen such that $s\theta^{2^{n_0}} < 1$.
In view of the fact that $\lim_{n\to\infty} \theta^n = 0$, from (6) and (7) we deduce that $(x_n)_n$ is Cauchy. □

**Theorem 1.** *Let $(X, d, s)$ be a complete b-metric space, $k \in \mathbb{N}^*$ and $f : X^k \to X$. Suppose that:*

1. *there exists $\psi : \mathbb{R}_+^k \to \mathbb{R}_+$, a continuous, monotonically increasing and positively semihomogenous function that satisfies the conditions stated in Lemma 3;*

*2. there exists $\varphi : \mathbb{R}_+^{5k} \to \mathbb{R}_+$, a continuous and monotonically increasing function, such that*

$$\begin{aligned} d(f(\bar{x}), f(\bar{y})) \le \varphi(& d(x_1, f(\bar{x})), ..., d(x_k, f(\bar{x})),\\ & d(y_1, f(\bar{y})), ..., d(y_k, f(\bar{y})),\\ & d(x_1, f(\bar{y})), ..., d(x_k, f(\bar{y})),\\ & d(y_1, f(\bar{x})), ..., d(y_k, f(\bar{x})),\\ & d(x_1, y_1), ..., d(x_k, y_k)), \end{aligned}$$

*for all $\bar{x} = (x_1, ..., x_k), \bar{y} = (y_1, ..., y_k) \in X^k$;*

*1. for any $r \in \mathbb{R}_+$, the following implication holds:*

$$r \le s \cdot \varphi(\underbrace{0, ..., 0}_{k}, \underbrace{r, ..., r}_{k}, \underbrace{sr, ..., sr}_{k}, \underbrace{0, ..., 0}_{2k}) \Rightarrow r = 0.$$

*Then $f$ has a fixed point.*

*Proof.* On account of Lemma 3, *1.* implies that the sequence $(x_n)_n$ given by (3) is Cauchy. $X$ being complete, there is $x^* \in X$ with

$$\lim_{n\to\infty} d(x_n, x^*) = 0.$$

By the $s$-relaxed triangle inequality, we find that

$$d(x^*, f(x^*, ..., x^*)) \le sd(x^*, x_{n+k}) + sd(x_{n+k}, f(x^*, ..., x^*)),$$

for all $n \in \mathbb{N}$.
According to *2.*, we have

$$\begin{aligned} d(x_{n+k}, f(x^*, ..., x^*)) &= d(f(x_n, ..., x_{n+k-1}), f(x^*, ..., x^*)) \le\\ &\le \varphi(d(x_n, x_{n+k}), ..., d(x_{n+k-1}, x_{n+k})),\\ &\quad \underbrace{d(x^*, \widetilde{f}(x^*)), ..., d(x^*, \widetilde{f}(x^*))}_{k},\\ &\quad d(x_n, \widetilde{f}(x^*)), ..., d(x_{n+k-1}, \widetilde{f}(x^*)),\\ &\quad \underbrace{d(x^*, x_{n+k}), ..., d(x^*, x_{n+k})}_{k},\\ &\quad d(x_n, x^*), ..., d(x_{n+k-1}, x^*)), \end{aligned}$$

for all $n \in \mathbb{N}$.
Taking into account the properties of $\varphi$, from the previous relations we get

$$\begin{aligned} d(x^*, \widetilde{f}(x^*)) \le s\varphi(&\underbrace{0, ..., 0}_{k}, \underbrace{d(x^*, \widetilde{f}(x^*)), ..., d(x^*, \widetilde{f}(x^*))}_{k},\\ &\limsup_{n\to\infty} d(x_n, \widetilde{f}(x^*)), ..., \limsup_{n\to\infty} d(x_{n+k-1}, \widetilde{f}(x^*)), \underbrace{0, ..., 0}_{2k}), \end{aligned}$$

in view of the fact that

$$\limsup_{n\to\infty} d(x_n, \widetilde{f}(x^*)) \in \mathbb{R}_+, ..., \limsup_{n\to\infty} d(x_{n+k-1}, \widetilde{f}(x^*)) \in \mathbb{R}_+,$$

which follows from (2).
Since $\varphi$ is an increasing function, using (2), we have the following inequalities:

$$\limsup_{n\to\infty} d(x_n, \widetilde{f}(x^*)) \le sd(x^*, \widetilde{f}(x^*)), ..., \limsup_{n\to\infty} d(x_{n+k-1}, \widetilde{f}(x^*)) \le sd(x^*, \widetilde{f}(x^*)).$$

Therefore, we deduce that

$$d(x^*, \widetilde{f}(x^*)) \le s\varphi(\underbrace{0, ..., 0}_{k}, \underbrace{d(x^*, \widetilde{f}(x^*)), ..., d(x^*, \widetilde{f}(x^*))}_{k}, \underbrace{sd(x^*, \widetilde{f}(x^*)), ..., sd(x^*, \widetilde{f}(x^*))}_{k}, \underbrace{0, ..., 0}_{2k}).$$

Condition *3.* now yields

$$d(x^*, \widetilde{f}(x^*)) = 0,$$

which completes the proof. □

**Remark 3.** *If we take $s = 1$ in the previous theorem, we get Theorem 4.2.4 from [10].*

**Theorem 2.** *If, in addition to the conditions of Theorem 1, we suppose that:*

*4. for any $r \in \mathbb{R}_+$, the following implication holds*

$$r \le \varphi(\underbrace{0, ..., 0}_{2k}, \underbrace{r, ..., r}_{3k}) \Rightarrow r = 0, \tag{8}$$

*then $f$ has a unique fixed point.*

*Proof.* The existence of a fixed point follows from Theorem 1. Let us prove that it is unique. Suppose $x^*, y^* \in F_f$. Then

$$\begin{aligned} d(x^*, y^*) = d(\widetilde{f}(x^*), \widetilde{f}(y^*)) \le \varphi( & \underbrace{d(x^*, \widetilde{f}(x^*)), ..., d(x^*, \widetilde{f}(x^*))}_{k}, \\ & \underbrace{d(y^*, \widetilde{f}(y^*)), ..., d(y^*, \widetilde{f}(y^*))}_{k}, \\ & \underbrace{d(x^*, \widetilde{f}(y^*)), ..., d(x^*, \widetilde{f}(y^*))}_{k}, \\ & \underbrace{d(y^*, \widetilde{f}(x^*)), ..., d(y^*, \widetilde{f}(x^*))}_{k}, \\ & \underbrace{d(x^*, y^*), ..., d(x^*, y^*)}_{k}), \end{aligned}$$

hence

$$d(x^*,y^*) \le \varphi(\underbrace{0,...,0}_{2k},\underbrace{d(x^*,y^*),...,d(x^*,y^*)}_{3k}).$$

As $d(x^*,y^*) \ge 0$, by (8) we infer that

$$d(x^*,y^*) = 0,$$

hence $f$ has a unique fixed point.

□

**Remark 4.** *If we consider $s = 1$ in the previous theorem, we obtain Theorem 4.2.5 from [10].*

**Theorem 3.** *Let $(X,d,s)$ be a complete b-metric space, $k \in \mathbb{N}^*$, $f : X^k \to X$ and $\varphi : \mathbb{R}_+^k \to \mathbb{R}_+$ a mapping satisfying the following conditions:*

1. *$\varphi$ is continuous, monotonically increasing and positively semihomogenous;*
2. *the following inequality holds:*
$$d(f(x_1,...,x_k), f(x_2,...,x_{k+1})) \le \varphi\left(d(x_1,x_2),...,d(x_k,x_{k+1})\right), \qquad (9)$$
*for all $x_1,...,x_{k+1} \in X$;*
3. *$\varphi(1,...,1) < 1$;*
4. *$\varphi(0,...,0) = 0$.*

*Under the above assumptions, $f$ has a fixed point.*

*Proof.* Let $x_1,...,x_k \in X$ and $(x_n)_n$ be the sequence given by (3). Condition (9) implies that

$$d(f(x_n,...,x_{n+k-1}), f(x_{n+1},...,x_{n+k})) \le \varphi(d(x_n,x_{n+1}),...,d(x_{n+k-1},x_{n+k})),$$

for all $n \in \mathbb{N}^*$. Thus, in view of conditions *1.–3.*, we can apply Lemma 3 to infer that $(x_n)_n$ is Cauchy. Since $X$ is complete, there exists $x^* \in X$ so that

$$\lim_{n\to\infty} d(x_n,x^*) = 0.$$

We will show that $x^* \in F_f$. By the $s$-relaxed triangle inequality, we have

$$d(x^*, f(x^*,...,x^*)) \le sd(x^*,x_{n+k}) + sd(x_{n+k}, f(x^*,...,x^*)), \qquad (10)$$

for all $n \in \mathbb{N}$.
From (9) and (1) we deduce that

$$\begin{aligned} &d(x_{n+k}, f(x^*,...,x^*)) = d(f(x_n,...,x_{n+k-1}), f(x^*,...,x^*)) \le \\ &\le sd(f(x_n,...,x_{n+k-1}), f(x_{n+1},...,x_{n+k-1},x^*) + ... + s^k d(f(x_{n+k-1},x^*,...,x^*), \\ &\quad f(x^*,...,x^*)) \\ &\le s\varphi(d(x_n,x_{n+1}),...,d(x_{n+k-1},x^*)) + ... + s^k\varphi(d(x_{n+k-1},x^*),0,...,0), \qquad (11)\end{aligned}$$

for all $n \in \mathbb{N}$. But, since $\varphi$ is continuous and $\varphi(0,...,0) = 0$,

$$\lim_{n\to\infty}\Big(s\varphi(d(x_n, x_{n+1}), ..., d(x_{n+k-1}, x^*)) + ... + s^k\varphi(d(x_{n+k-1}, x^*), 0, ..., 0)\Big) = 0.$$

Consequently, via (11), we conclude that

$$\lim_{n\to\infty} d(x_{n+k}, f(x^*, ..., x^*)) = 0.$$

Using (10), it follows that $x^* \in F_f$. □

**Remark 5.** *If we consider $s = 1$ in the previous theorem, we get Theorem 4.2.6 from [10].*

**Theorem 4.** *Let $(X, d, s)$ be a complete b-metric space, $k \in \mathbb{N}^*$, $f : X^k \to X$ and suppose that there exists a function $\varphi : \mathbb{R}_+^k \to \mathbb{R}_+$ satisfying the following conditions:*

1. *$\varphi$ is continuous, monotonically increasing and positively semihomogenous;*
2. *inequality (9) holds:*
$$d(f(x_1, ..., x_k), f(x_2, ..., x_{k+1})) \leq \varphi\left(d(x_1, x_2), ..., d(x_k, x_{k+1})\right),$$
*for all $x_1, ..., x_{k+1} \in X$;*
3. *$\varphi(1, ..., 1) < 1$;*
4. *for any $r \in \mathbb{R}_+^*$, at least one of the following conditions is valid:*
   - $s\varphi(0, ..., 0, r) + ... + s^{k-1}\varphi(0, r, 0, ..., 0) + s^{k-1}\varphi(r, 0, ..., 0) < r$
   - $s\varphi(r, 0, ..., 0) + ... + s^{k-1}\varphi(0, ..., 0, r, 0) + s^{k-1}\varphi(0, ..., 0, r) < r.$

*Then $f$ has a unique fixed point.*

*Proof.* Note that *4.* assures that $\varphi(0, ..., 0) = 0$. Therefore, we are in the conditions of Theorem 3, thus $f$ has a fixed point. Let us prove its uniqueness. If it were true that there exist $x^*, y^* \in F_f$ and $x^* \neq y^*$, then, based on (9) and *4.*, we have

$$\begin{aligned}
& d(x^*, y^*) = d(f(x^*, ..., x^*), f(y^*, ..., y^*)) \\
\leq\ & sd(f(x^*, .., x^*), f(x^*, ..., x^*, y^*)) + ... + \\
& +s^{k-1}d(f(x^*, x^*, y^*, ..., y^*), f(x^*, y^*, ..., y^*)) \\
& +s^{k-1}d(f(x^*, y^*, ..., y^*), f(y^*, ..., y^*)) \\
\leq\ & s\varphi(0, ..., 0, d(x^*, y^*)) + ... + \\
& +s^{k-1}\varphi(0, d(x^*, y^*), 0, ..., 0) + s^{k-1}\varphi(d(x^*, y^*), 0, ..., 0) \\
<\ & d(x^*, y^*),
\end{aligned}$$

and

$$\begin{aligned} d(x^*,y^*) &= d(f(x^*,...,x^*), f(y^*,...,y^*)) \le \\ &\le sd(f(x^*,..,x^*), f(y^*,x^*...,x^*)) + ...+ \\ &\quad + s^{k-1}d(f(y^*,...,y^*,x^*,x^*), f(y^*,...,y^*,x^*)) \\ &\quad + s^{k-1}d(f(y^*,...,y^*,x^*), f(y^*,...,y^*)) \\ &\le s\varphi(d(x^*,y^*),0,...,0) + ... + s^{k-1}\varphi(0,...,0,d(x^*,y^*),0) \\ &\quad + s^{k-1}\varphi(0,...,0,d(x^*,y^*)) \\ &< d(x^*,y^*). \end{aligned}$$

We have arrived at a contradiction, hence $f$ has a unique fixed point. □

**Remark 6.** *If we consider $s = 1$ in the previous theorem, we obtain Theorem 4.2.7 from [10].*

If the function $\varphi : \mathbb{R}_+^k \to \mathbb{R}_+$ from Theorem 4 is given by

$$\varphi(r_1,...,r_k) = q_1r_1 + ... + q_kr_k,$$

for all $r_1,...,r_k \in \mathbb{R}_+$, where $q_1,...,q_k \in \mathbb{R}_+$ with $q_1 + ... + q_k < 1$, we obtain the following consequence of Theorem 4.

**Theorem 5.** *Let $(X,d,s)$ be a complete b-metric space, $k \in \mathbb{N}^*$ and $f : X^k \to X$ such that*

$$d(f(x_1,...,x_k), f(x_2,...,x_{k+1})) \le q_1d(x_1,x_2) + ... + q_kd(x_k,x_{k+1}), \tag{12}$$

*for all $x_1,...,x_{k+1} \in X$, where $q_1,...,q_k \in \mathbb{R}_+$ with $q_1 + ... + q_k < 1$. Then $f$ has a fixed point.*
*If, in addition, at least one of the following conditions is also satisfied*

- $sq_k + ... + s^{k-1}q_2 + s^{k-1}q_1 < 1$
- $sq_1 + ... + s^{k-1}q_{k-1} + s^{k-1}q_k < 1$

*then $f$ has a unique fixed point.*

**Remark 7.** *If $s = 1$, the previous theorem yields the generalization of the Banach-Picard-Caccioppoli Contraction Mapping Principle for mappings defined on products of metric spaces, obtained by S. B. Prešić in [9].*

If, for $\lambda \in (0,1)$, we consider $\varphi : \mathbb{R}_+^k \to \mathbb{R}_+$ given by

$$\varphi(r_1,...,r_k) = \lambda \cdot \max\{r_1,...,r_k\},$$

for all $r_1,...,r_k \in \mathbb{R}_+$, then we obtain the following consequence of Theorem 3.

**Theorem 6.** *Let $(X, d, s)$ be a complete b-metric space, $k \in \mathbb{N}^*$, $\lambda \in (0,1)$ and $f : X^k \to X$ such that*

$$d(f(x_1, ..., x_k), f(x_2, ..., x_{k+1})) \leq \lambda \cdot \max\{d(x_1, x_2), ..., d(x_k, x_{k+1})\}, \qquad (13)$$

*for all $x_1, ..., x_{k+1} \in X$. Then $f$ has a fixed point.*
*If the following condition is also satisfied*

$$d(f(x, ..., x), f(y, ..., y)) < d(x, y), \qquad (14)$$

*for all $x, y \in X$, with $x \neq y$, then $f$ has a unique fixed point.*

**Example 1.** *Let $X = [0,1]$, $k = 2$ and $d : [0,1] \times [0,1] \to [0,1]$, $d(x,y) = (x-y)^2$, for all $x, y \in [0,1]$. Then $(X, d, 2)$ is a complete b-metric space. Consider $f : [0,1] \times [0,1] \to [0,1]$ a mapping such that $f(x,y) = \frac{x}{2}$, for all $x, y \in [0,1]$. We have*

$$\begin{aligned} d\left(f(x_1, x_2), f(x_2, x_3)\right) &= d\left(\frac{x_1}{2}, \frac{x_2}{2}\right) = \frac{1}{4}(x_1 - x_2)^2 \leq \\ &\leq \frac{1}{4} max\left\{(x_1 - x_2)^2, (x_2 - x_3)^2\right\} = \frac{1}{4} max\left\{d(x_1, x_2), d(x_2, x_3)\right\}, \end{aligned}$$

*for all $x_1, x_2, x_3 \in [0,1]$. Therefore, (13) holds for $\lambda = \frac{1}{4} \in (0,1)$. Since*

$$d(f(x,x), f(y,y)) = d\left(\frac{x}{2}, \frac{y}{2}\right) = \frac{1}{4}(x-y)^2 < (x-y)^2 = d(x,y),$$

*for all $x, y \in [0,1]$, with $x \neq y$, it follows that $f$ admits a unique fixed point $x^* = 0 \in [0,1]$.*

**Remark 8.** *Theorem 6 is a generalization of Theorem 5.*

Indeed, let $x_1, ..., x_{k+1} \in X$ be arbitrary points. From (12) we see that

$$\begin{aligned} d(f(x_1, ..., x_k), f(x_2, ..., x_{k+1})) &\leq q_1 d(x_1, x_2) + ... + q_k d(x_k, x_{k+1}) \\ &\leq (q_1 + ... + q_k) \cdot \max\{d(x_1, x_2), ..., d(x_k, x_{k+1})\}. \end{aligned}$$

If we denote $q_1 + ... + q_k =: \lambda$, then $\lambda \in (0,1)$ and consequently, (13) is fulfilled. As for (14),

$$\begin{aligned} d(f(x, ..., x), f(y, ..., y)) &\leq s d(f(x, .., x), f(x, ..., x, y)) + ... + \\ &\quad + s^{k-1} d(f(x, x, y, ..., y), f(x, y, ..., y)) + \\ &\quad + s^{k-1} d(f(x, y, ..., y), f(y, ..., y)) \\ &\leq s q_k d(x,y) + ... + s^{k-1} q_2 d(x,y) + s^{k-1} q_1 d(x,y) \\ &= \left(s q_k + ... + s^{k-1} q_2 + s^{k-1} q_1\right) d(x,y) \\ &< d(x,y), \end{aligned}$$

for all $x, y \in X$ with $x \neq y$.

**Remark 9.** *Theorem 6 for $s = 1$ yields Theorem 2 from [4].*

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