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A NEW CLASS OF METRICS ON THE COTANGENT BUNDLE

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Abstract

In this paper, we introduce a new class of metrics on the cotangent bundle T^*M over an m-dimensional Riemannian manifold (M, g) as a new natural metric with respect to g non-rigid on T^*M. First, we investigate the Levi-Civita connection, curvature and we characterize some geodesic properties for the new class of metrics on the cotangent bundle T^*M.

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1 Introduction

In the field, one of the first works which deal with the cotangent bundles of a manifold as a Riemannian manifold is that of Patterson, E.M. and Walker, A.G. [7], who constructed from an affine symmetric connection on a manifold a Riemannian metric on the cotangent bundle, which they call the Riemann extension of the connection. A generalization of this metric had been given by Sekizawa, M.[12] in his classification of natural transformations of affine connections on manifolds to metrics on their cotangent bundles, obtaining the class of natural Riemann extensions which is a 2-parameter family of metrics, and which had been intensively studied by many authors. On the other hand, inspired by the concept of g-natural metrics on tangent bundles of Riemannian manifolds, F. Ağca considered another class of metrics on cotangent bundles of Riemannian manifolds, that he callad g-natural metrics [1]. Also, there are studies by other authors, Salimov, A.A. and Ağca, F. [9, 10], Yano, K. and Ishihara, S.[13], Ocak, F. and Kazimova, S. [5], Gezer, A. and Altunbas, M.[3] etc...

The main idea in this note consists in the modification of the Sasaki metric. First, we introduce a new class of metrics, noted g^f on the cotangent bundle T^*M

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over an m-dimensional Riemannian manifold (M, g), where f is a strictly positive smooth function on M. Then, we establish the Levi-Civita connection (Theorem 1) and the curvature tensor (Theorem 2) of the metric g^f. We also gives some results on the geodesics on the cotangent bundle (Theorem 3 and Theorem 4). After that, we construct some examples of geodesics on the cotangent bundle with the metric g^f.

Let (M^m, g) be an m-dimensional Riemannian manifold, T^*M be its cotangent bundle and $\pi : T^*M \to M$ the natural projection. A local chart $(U, x^i)_{i=\overline{1,m}}$ on M induces a local chart $(\pi^{-1}(U), x^i, x^{\bar{i}} = p_i)_{i=\overline{1,m}, \bar{i}=m+i}$ on T^*M, where p_i is the component of covector p in each cotangent space $T^*_x M$, $x \in U$ with respect to the natural coframe dx^i. Let $C^\infty(M)$ (resp. $C^\infty(T^*M)$) be the ring of real-valued C^∞ functions on M(resp. T^*M) and $\Im^r_s(M)$ (resp. $\Im^r_s(T^*M)$) be the module over $C^\infty(M)$ (resp. $C^\infty(T^*M)$) of C^∞ tensor fields of type (r, s).

Denote by Γ^k_{ij} the Christoffel symbols of g and by ∇ the Levi-Civita connection of g.

Let $X = X^i \frac{\partial}{\partial x^i}$ and $\omega = \omega_i dx^i$ be local expressions in $U \subset M$ of a vector and covector (1-form) field $X \in \Im^1_0(M)$ and $\omega \in \Im^0_1(M)$, respectively. Then the complete and horizontal lifts X^C, $X^H \in \Im^1_0(T^*M)$ of $X \in \Im^1_0(M)$ and the vertical lift $\omega^V \in \Im^1_0(T^*M)$ of $\omega \in \Im^0_1(M)$ are defined, respectively by

$$X^C = X^i \frac{\partial}{\partial x^i} - p_h \frac{\partial X^h}{\partial x^i} \frac{\partial}{\partial x^{\bar{i}}}, \tag{1}$$

$$X^H = X^i \frac{\partial}{\partial x^i} + p_h \Gamma^h_{ij} X^j \frac{\partial}{\partial x^{\bar{i}}}, \tag{2}$$

$$\omega^V = \omega_i \frac{\partial}{\partial x^{\bar{i}}}, \tag{3}$$

with respect to the natural frame $\{\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^{\bar{i}}}\}$, where Γ^h_{ij} are components of the Levi-Civita connection ∇ on (M, g) (see [13] for more details).

From (2) and (3) we see that $(\frac{\partial}{\partial x^i})^H$ and $(dx^i)^V$ have respectively local expressions of the form

$$\tilde{e}_{(i)} = (\frac{\partial}{\partial x^i})^H = \frac{\partial}{\partial x^i} + p_a \Gamma^a_{hi} \frac{\partial}{\partial x^{\bar{h}}}, \tag{4}$$

$$\tilde{e}_{(\bar{i})} = (dx^i)^V = \frac{\partial}{\partial x^{\bar{i}}}. \tag{5}$$

The set $\{\tilde{e}_{(\alpha)}\} = \{\tilde{e}_{(i)}, \tilde{e}_{(\bar{i})}\}$ is called the frame adapted to Levi-Civita connection ∇ on (M, g). The indices $\alpha, \beta, \ldots = \overline{1, 2m}$ indicate the indices with respect to the adapted frame.

Using (2), (3) we have.

$$X^H = X^i \tilde{e}_{(i)}\ ,\ X^H = \begin{pmatrix} X^i \\ 0 \end{pmatrix}, \tag{6}$$

$$\omega^V = \omega_i \tilde{e}_{(\bar{i})}\ ,\ \omega^V = \begin{pmatrix} 0 \\ \omega_i \end{pmatrix}, \tag{7}$$

with respect to the adapted frame $\{\tilde{e}_{(\alpha)}\}_{\alpha=\overline{1,2m}}$, (see [13] for more details).

Lemma 1. *[13] Let (M,g) be a Riemannian manifold, ∇ be the Levi-Civita connection and R be the Riemannian curvature tensor. Then the Lie bracket of the cotangent bundle T^*M of M satisfies the following*

(1) $[\omega^V, \theta^V] = 0$,

(2) $[X^H, \theta^V] = (\nabla_X \theta)^V$,

(3) $[X^H, Y^H] = [X,Y]^H + (pR(X,Y))^V$,

for all vector fields $X, Y \in \Im^1_0(M)$ and $\omega, \theta \in \Im^0_1(M)$.

Let (M,g) be a Riemannian manifold, we define the map

$$\begin{array}{rcl} \sharp : \Im^0_1(M) & \to & \Im^1_0(M) \\ \omega & \mapsto & \sharp\omega \end{array}$$

for all $X \in \Im^1_0(M)$, $g(\sharp\omega, X) = \omega(X)$, the map $\sharp$ is $C^\infty(M)$-isomorphism.
Locally for all $\omega = \omega_i dx^i \in \Im^0_1(M)$, we have $\sharp\omega = g^{ij}\omega_i \frac{\partial}{\partial x^j}$, where (g^{ij}) is the inverse matrix of the matrix (g_{ij}).

For each $x \in M$ the scalar product $g^{-1} = (g^{ij})$ is defined on the cotangent space $T^*_x M$ by $g^{-1}(\omega,\theta) = g(\sharp\omega, \sharp\theta) = g^{ij}\omega_i\theta_j$.
If ∇ is the Levi-Civita connection of (M,g) we have

$$\nabla_X(\sharp\omega) = \sharp(\nabla_X \omega), \tag{8}$$

$$Xg^{-1}(\omega,\theta) = g^{-1}(\nabla_X\omega,\theta) + g^{-1}(\omega,\nabla_X\theta), \tag{9}$$

for all $X \in \Im^1_0(M)$ and $\omega, \theta \in \Im^0_1(M)$.

In the following, we noted $\sharp\omega$ by $\tilde{\omega}$ for all $\omega \in \Im^0_1(M)$.

2 New class of metrics g^f

Definition 1. *Let (M, g) be a Riemannian manifold and $f : M \to]0, +\infty[$ be a strictly positive smooth function on M. On the cotangent bundle T^*M, we define a new class of metrics noted g^f by*

$$g^f(X^H, Y^H) = g(X, Y)^V = g(X, Y) \circ \pi, \tag{10}$$
$$g^f(X^H, \theta^V) = 0, \tag{11}$$
$$g^f(\omega^V, \theta^V) = fg^{-1}(\omega, p)g^{-1}(\theta, p), \tag{12}$$

where $X, Y \in \Im^1_0(M)$, $\omega, \theta \in \Im^0_1(M)$.

Since any tensor field of type $(0, s)$ on T^*M where $s \geq 1$ is completely determined with the vector fields of type X^H and ω^V where $X \in \Im^1_0(M)$ and $\omega \in \Im^0_1(M)$ (see [13]). In the particular case the metric g^f is tensor field of type $(0, 2)$ on T^*M. It follows that g^f is completely determined by its formulas (10), (11) and (12).

By means of (1) and (2), the complete lift X^C of $X \in \Im^1_0(M)$ is given by

$$X^C = X^H - (p(\nabla X))^V \tag{13}$$

where $p(\nabla X) = p_h(\nabla_i X^h)dx^i = p_h(\dfrac{\partial X^h}{\partial x^i} + \Gamma^h_{ij}X^j)dx^i$.

Taking account of (10), (11), (12) and (13), we obtain

$$g^f(X^C, Y^C) = g(X, Y)^V + fg^{-1}(p(\nabla X), p)g^{-1}(p(\nabla Y), p). \tag{14}$$

Since the tensor field $g^f \in \Im^0_2(T^*M)$ is completely determined also by its action on vector fields of type X^C and Y^C (see [13]), we say that formula (14) is an alternative characterization of g^f.

Remark 1. *From formulas* (10)*,* (11)*,* (12) *we see that*

$$g^f_{ij} = g^f(\tilde{e}_{(i)}, \tilde{e}_{(j)}) = g(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})^V = g_{ij},$$
$$g^f_{i\bar{j}} = g^f(\tilde{e}_{(i)}, \tilde{e}_{(\bar{j})}) = 0,$$
$$g^f_{\bar{i}\bar{j}} = fg^{ih}g^{jk}p_hp_k.$$

Then the metric g^f has components with respect to the adapted frame $\{\tilde{e}_{(\alpha)}\}_{\alpha=\overline{1,2m}}$

$$g^f = \begin{pmatrix} g_{ij} & 0 \\ 0 & fg^{ih}g^{jk}p_hp_k \end{pmatrix} \tag{15}$$

Lemma 2.
Let (M, g) be a Riemannian manifold and $\rho : \mathbb{R} \to \mathbb{R}$ a smooth function.
For all $X, Y \in \Im^1_0(M)$ and $\omega, \theta \in \Im^0_1(M)$, we have:

1. $X^H(\rho(r^2))_{(x,p)} = 0,$
2. $\omega^V(\rho(r^2))_{(x,p)} = 2\rho'(r^2)g^{-1}(\omega, p)_x,$
3. $X^H(g^{-1}(\theta, p))_{(x,p)} = g^{-1}(\nabla_X\theta, p)_x,$
4. $\omega^V(g^{-1}(\theta, p))_{(x,p)} = g^{-1}(\omega, \theta)_x.$

where $r^2 = g^{-1}(p, p)$ *and* $(x, p) \in T^*M$.

Proof. Let $(x, p) \in T^*M$, If $\mathcal{P}$ be a local covector field constant on each fiber T^*_xM, such that $\mathcal{P}_x = p \in T^*_xM$, we have:

$$\begin{aligned}
1.\ X^H(\rho(r^2))_{(x,p)} &= \big[X^i \frac{\partial}{\partial x^i}(\rho(r^2)) + p_h\Gamma^h_{ij}X^j\frac{\partial}{\partial p_i}(\rho(r^2))\big]_{(x,p)} \\
&= \big[X^i\rho'(r^2)\frac{\partial}{\partial x^i}(r^2) + \rho'(r^2)p_h\Gamma^h_{ij}X^j\frac{\partial}{\partial p_i}(r^2)\big]_{(x,p)} \\
&= \rho'(r^2)\big[X^i\frac{\partial}{\partial x^i}(g^{st}p_sp_t) + p_h\Gamma^h_{ij}X^j\frac{\partial}{\partial p_i}(g^{st}p_sp_t)\big]_{(x,p)} \\
&= \rho'(r^2)\big[Xg^{-1}(\mathcal{P}, \mathcal{P}) + 2g^{ti}p_tp_h\Gamma^h_{ij}X^j\big]_x \\
&= \rho'(r^2)[Xg^{-1}(\mathcal{P}, \mathcal{P}) - 2g^{-1}(\mathcal{P}, \nabla_X\mathcal{P})]_x \\
&= 0,
\end{aligned}$$

where $\nabla_X\mathcal{P} = -p_h\Gamma^h_{ij}X^jdx_i$

$$\begin{aligned}
2.\ \omega^V(\rho(r^2))_{(x,p)} &= [\omega_i\rho'(r^2)\frac{\partial}{\partial p_i}(g^{st}p_sp_t)]_{(x,p)} \\
&= 2\rho'(r^2)\omega_ig^{it}p_t \\
&= 2\rho'(r^2)g^{-1}(\omega, p)_x. \\
3.\ X^H(g^{-1}(\theta, p))_{(x,p)} &= \big[X^i\frac{\partial}{\partial x^i}(g^{st}\theta_sp_t) + p_h\Gamma^h_{ij}X^j\frac{\partial}{\partial p_i}(g^{st}\theta_sp_t)\big]_p \\
&= Xg^{-1}(\theta, \mathcal{P})_x - (p_h\Gamma^h_{ij}X^jg^{si}\theta_s)_x \\
&= Xg^{-1}(\theta, \mathcal{P})_x - g^{-1}(\theta, \nabla_X\mathcal{P})_x \\
&= g^{-1}(\nabla_X\theta, \mathcal{P})_x. \\
4.\ \omega^V(g^{-1}(\theta, p))_{(x,p)} &= [\omega_i\frac{\partial}{\partial p_i}(g^{st}\theta_sp_t)]_{(x,p)} \\
&= \omega_ig^{si}\theta_s \\
&= g^{-1}(\omega, \theta)_x.
\end{aligned}$$

□

Lemma 3. *Let* (M, g) *be a Riemannian manifold and* (T^*M, g^f) *its cotangent bundle equipped with the metric* g^f *, for all* $X \in \Im^1_0(M)$ *and* $\omega, \theta \in \Im^0_1(M)$*, we* *have*

$$
\begin{aligned}
(1)\ \ X^H g^f(\theta^V,\eta^V) &= \frac{1}{f}X(f)g^f(\theta^V,\eta^V)+g^f((\nabla_X\theta)^V,\eta^V)\\
&\quad +g^f(\theta^V,(\nabla_X\eta)^V),\\
(2)\ \ \omega^V g^f(\theta^V,\eta^V) &= fg^{-1}(\omega,\theta)g^{-1}(\eta,p)+fg^{-1}(\omega,\eta)g^{-1}(\theta,p).
\end{aligned}
$$

Proof. The proof of Lemma 3 follows directly from Lemma 2.

$$
\begin{aligned}
(1)\ \ X^H g^f(\theta^V,\eta^V) &= X^H\big[fg^{-1}(\theta,p)g^{-1}(\eta,p)\big]\\
&= X(f)g^{-1}(\theta,p)g^{-1}(\eta,p)+fg^{-1}(\nabla_X\theta,p)g^{-1}(\eta,p)\\
&\quad +fg^{-1}(\theta,p)g^{-1}(\nabla_X\eta,p)\\
&= \frac{1}{f}X(f)g^f(\theta^V,\eta^V)+g^f((\nabla_X\theta)^V,\eta^V)\\
&\quad +g^f(\theta^V,(\nabla_X\eta)^V).\\
(2)\ \ \omega^V g^f(\theta^V,\eta^V) &= \omega^V\big[fg^{-1}(\theta,p)g^{-1}(\eta,p)\big]\\
&= \omega(f)g^{-1}(\theta,p)g^{-1}(\eta,p)+fg^{-1}(\omega,\theta)g^{-1}(\eta,p)\\
&\quad +fg^{-1}(\theta,p)g^{-1}(\omega,\eta)\\
&= fg^{-1}(\omega,\theta)g^{-1}(\eta,p)+fg^{-1}(\omega,\eta)g^{-1}(\theta,p).
\end{aligned}
$$

□

3 The Levi-Civita connection of g^f

We shall calculate the Levi-Civita connection ∇^f of the cotangent bundle T^*M equipped with the metric g^f. This connection is characterized by the Koszul formula:

$$
\begin{aligned}
2g^f(\nabla^f_{\widetilde{U}}\widetilde{V},\widetilde{W}) &= \widetilde{U}g^f(\widetilde{V},\widetilde{W})+\widetilde{V}g^f(\widetilde{W},\widetilde{U})-\widetilde{W}g^f(\widetilde{U},\widetilde{V})\\
&\quad +g^f(\widetilde{W},[\widetilde{U},\widetilde{V}])+g^f(\widetilde{V},[\widetilde{W},\widetilde{U}])-g^f(\widetilde{U},[\widetilde{V},\widetilde{W}]), \quad (16)
\end{aligned}
$$

for all $\widetilde{U},\widetilde{V},\widetilde{W}\in\Im^1_0(T^*M)$.

Lemma 4. *Let (M,g) be a Riemannian manifold and T^*M its cotangent bundle* *equipped with the metric g^f, then we have:*

$$\begin{aligned}
1)\ g^f(\nabla^f_{X^H}Y^H, Z^H) &= g^f((\nabla_X Y)^H, Z^H),\\
2)\ g^f(\nabla^f_{X^H}Y^H, \eta^V) &= 0,\\
3)\ g^f(\nabla^f_{X^H}\theta^V, Z^H) &= 0,\\
4)\ g^f(\nabla^f_{X^H}\theta^V, \eta^V) &= g^f((\nabla_X\theta)^V, \eta^V) + \frac{1}{2f}X(f)g^f(\theta^V, \eta^V),\\
5)\ g^f(\nabla^f_{\omega^V}Y^H, Z^H) &= 0,\\
6)\ g^f(\nabla^f_{\omega^V}Y^H, \eta^V) &= \frac{1}{2f}Y(f)g^f(\omega^V, \eta^V),\\
7)\ g^f(\nabla^f_{\omega^V}\theta^V, Z^H) &= \frac{-1}{2}g^{-1}(\omega, p)g^{-1}(\theta, p)g^f((grad\, f)^H, Z^H),\\
8)\ g^f(\nabla^f_{\omega^V}\theta^V, \eta^V) &= \frac{1}{r^2}g^{-1}(\omega, \theta)g^f(\mathcal{P}^V, \eta^V).
\end{aligned}$$

for all $X, Y, Z \in \Im^1_0(M)$ and $\omega, \theta, \eta \in \Im^0_1(M)$, where $r^2 = g^{-1}(p, p)$, $\mathcal{P} \in \Im^0_1(M)$ such that $\mathcal{P}_x = p \in T^*_x M$.($\mathcal{P}^V$ the canonical vertical or Liouville vector field on T^*M).

Proof.
The proof of Lemma 4 follows directly from Kozul formula (16), Lemma 1, Definition 1 and Lemma 3.
1) The statement is obtained as follows.

$$\begin{aligned}
2g^f(\nabla^f_{X^H}Y^H, Z^H) &= X^H g^f(Y^H, Z^H) + Y^H g^f(Z^H, X^H) - Z^H g^f(X^H, Y^H)\\
&\quad + g^f(Z^H, [X^H, Y^H]) + g^f(Y^H, [Z^H, X^H])\\
&\quad - g^f(X^H, [Y^H, Z^H])\\
&= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) + g^f(Z^H, [X, Y]^H)\\
&\quad + g^f(Y^H, [Z, X]^H) - g^f(X^H, [Y, Z]^H)\\
&= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) + g(Z, [X, Y])\\
&\quad + g(Y, [Z, X]) - g(X, [Y, Z])\\
&= 2g(\nabla_X Y, Z)\\
&= 2g^f((\nabla_X Y)^H, Z^H).
\end{aligned}$$

2) Direct calculations give

$$\begin{aligned}
2g^f(\nabla^f_{X^H}Y^H, \eta^V) &= X^H g^f(Y^H, \eta^V) + Y^H g^f(\eta^V, X^H) - \eta^V g^f(X^H, Y^H)\\
&\quad + g^f(\eta^V, [X^H, Y^H]) + g^f(Y^H, [\eta^V, X^H])\\
&\quad - g^f(X^H, [Y^H, \eta^V])\\
&= g^f(\eta^V, [X^H, Y^H])\\
&= g^f((pR(X, Y))^V, \eta^V)\\
&= fg^{-1}(pR(X, Y), p)g^{-1}(\eta, p)\\
&= 0.
\end{aligned}$$

Where

$$\begin{aligned} g^{-1}(pR(X,Y),p) &= g^{kl}(pR(X,Y))_k p_l = (pR(X,Y))_k \tilde{p}^k, \\ &= p_s R^s_{ijk} X^i Y^j \tilde{p}^k = g_{st}\tilde{p}^t R^s_{ijk} X^i Y^j \tilde{p}^k \\ &= R_{ijkt} X^i Y^j \tilde{p}^t \tilde{p}^k = g(R(X,Y)\tilde{p},\tilde{p}) \\ &= 0. \end{aligned}$$

3) Calculations similar to those in 2) give

$$\begin{aligned} 2g^f(\nabla^f_{X^H}\theta^V, Z^H) &= X^H g^f(\theta^V, Z^H) + \theta^V g^f(Z^H, X^H) - Z^H g^f(X^H, \theta^V) \\ &\quad + g^f(Z^H, [X^H, \theta^V]) + g^f(\theta^V, [Z^H, X^H]) \\ &\quad - g^f(X^H, [\theta^V, Z^H]) \\ &= g^f(\theta^V, [Z^H, X^H]) \\ &= g^f((pR(Z,X))^V, \theta^V) \\ &= f g^{-1}(pR(Z,X),p) g^{-1}(\theta, p) \\ &= 0. \end{aligned}$$

4) The statement is obtained as follows.

$$\begin{aligned} 2g^f(\nabla^f_{X^H}\theta^V, \eta^V) &= X^H g^f(\theta^V, \eta^V) + \theta^V g^f(\eta^V, X^H) - \eta^V g^f(X^H, \theta^V) \\ &\quad + g^f(\eta^V, [X^H, \theta^V]) + g^f(\theta^V, [\eta^V, X^H]) \\ &\quad - g^f(X^H, [\theta^V, \eta^V]) \\ &= X^H g^f(\theta^V, \eta^V) + g^f(\eta^V, [X^H, \theta^V]) + g^f(\theta^V, [\eta^V, X^H]) \end{aligned}$$

Using the first formula of Lemma 3 we have

$$\begin{aligned} 2g^f(\nabla^f_{X^H}\theta^V, \eta^V) &= \frac{1}{f} X(f) g^f(\theta^V, \eta^V) + g^f((\nabla_X\theta)^V, \eta^V) + g^f(\theta^V, (\nabla_X\eta)^V) \\ &\quad + g^f(\eta^V, (\nabla_X\theta)^V) - g^f(\theta^V, (\nabla_X\eta)^V) \\ &= 2g^f((\nabla_X\theta)^V, \eta^V) + \frac{1}{f} X(f) g^f(\theta^V, \eta^V). \end{aligned}$$

5) Calculations similar to those in 3) give the result.
6) Calculations similar to those in 4) give the result.
7) Direct calculations give

$$\begin{aligned} 2g^f(\nabla^f_{\omega^V}\theta^V, Z^H) &= \omega^V g^f(\theta^V, Z^H) + \theta^V g^f(Z^H, \omega^V) - Z^H g^f(\omega^V, \theta^V) \\ &\quad + g^f(Z^H, [\omega^V, \theta^V]) + g^f(\theta^V, [Z^H, \omega^V]) \\ &\quad - g^f(\omega^V, [\theta^V, Z^H]) \\ &= -Z^H g^f(\omega^V, \theta^V) + g^f(\theta^V, [Z^H, \omega^V]) - g^f(\omega^V, [\theta^V, Z^H]). \end{aligned}$$

Using the second formula of Lemma 3 we have

$$\begin{aligned} 2g^f(\nabla^f_{\omega^V}\theta^V, Z^H) &= \frac{-1}{f}Z(f)g^f(\omega^V,\theta^V) - g^f((\nabla_Z\omega)^V,\theta^V) \\ &\quad -g^f(\omega^V,(\nabla_Z\theta)^V) + g^f(\theta^V,(\nabla_Z\omega)^V) \\ &\quad +g^f(\omega^V,(\nabla_Z\theta)^V) \\ &= \frac{-1}{f}Z(f)g^f(\omega^V,\theta^V) \\ &= -Z(f)g^{-1}(\omega,p)g^{-1}(\theta,p) \\ &= -g^{-1}(\omega,p)g^{-1}(\theta,p)g^f((grad\, f)^H, Z^H). \end{aligned}$$

Where $g^f((grad\, f)^H, Z^H) = g(grad\, f, Z) = Z(f)$.

8) Direct calculations give

$$\begin{aligned} 2g^f(\nabla^f_{\omega^V}\theta^V, \eta^V) &= \omega^V g^f(\theta^V,\eta^V) + \theta^V g^f(\eta^V,\omega^V) - \eta^V g^f(\omega^V,\theta^V) \\ &\quad +g^f(\eta^V,[\omega^V,\theta^V]) + g^f(\theta^V,[\eta^V,\omega^V]) \\ &\quad -g^f(\omega^V,[\theta^V,\eta^V]) \\ &= \omega^V g^f(\theta^V,\eta^V) + \theta^V g^f(\eta^V,\omega^V) - \eta^V g^f(\omega^V,\theta^V) \\ &= fg^{-1}(\omega,\theta)g^{-1}(\eta,p) + fg^{-1}(\omega,\eta)g^{-1}(\theta,p) \\ &\quad +fg^{-1}(\theta,\eta)g^{-1}(\omega,p) + fg^{-1}(\theta,\omega)g^{-1}(\eta,p) \\ &\quad -fg^{-1}(\eta,\omega)g^{-1}(\theta,p) - fg^{-1}(\eta,\theta)g^{-1}(\omega,p) \\ &= 2fg^{-1}(\omega,\theta)g^{-1}(\eta,p) \\ &= \frac{2}{r^2}g^{-1}(\omega,\theta)g^f(\mathcal{P}^V,\eta^V). \end{aligned}$$

Where $g^f(\mathcal{P}^V,\eta^V) = fg^{-1}(p,p)g^{-1}(\eta,p) = fr^2g^{-1}(\eta,p)$. □

As a direct consequence of Lemma 4, we get the following theorem .

Theorem 1. *Let (M,g) be a Riemannian manifold and T^*M its cotangent bundle equipped with the metric g^f. Then the corresponding Levi-Civita connection ∇^f satisfies the followings:*

$$\begin{aligned} (1)\ \nabla^f_{X^H}Y^H &= (\nabla_X Y)^H, \\ (2)\ \nabla^f_{X^H}\theta^V &= (\nabla_X\theta)^V + \frac{1}{2f}X(f)\theta^V, \\ (3)\ \nabla^f_{\omega^V}Y^H &= \frac{1}{2f}Y(f)\omega^V, \\ (4)\ \nabla^f_{\omega^V}\theta^V &= \frac{-1}{2}g^{-1}(\omega,p)g^{-1}(\theta,p)(grad\, f)^H + \frac{1}{r^2}g^{-1}(\omega,\theta)\mathcal{P}^V, \end{aligned}$$

for all $X, Y \in \Im^1_0(M)$ and $\omega,\theta \in \Im^0_1(M)$, where $\mathcal{P}^V$ is the canonical vertical vector field on T^*M and R denotes the curvature tensor of (M,g).

Lemma 5. *Let (M, g) be a Riemannian manifold and T^*M its cotangent bundle equipped with the metric g^f, then we have:*

$$\begin{aligned} 1.\, (\nabla^f_{X^H} \mathcal{P}^V) &= \frac{1}{2f} X(f) \mathcal{P}^V, \\ 2.\, (\nabla^f_{\omega^V} \mathcal{P}^V) &= \omega^V - \frac{r^2}{2} g^{-1}(\omega, p)(grad\, f)^H + \frac{1}{r^2} g(\omega, p) \mathcal{P}^V, \end{aligned}$$

for all vector fields $X \in \Im^1_0(M)$ and $\omega \in \Im^0_1(M)$, where $\mathcal{P}^V$ is the canonical vertical vector field on T^*M.

Proof. By Theorem 1 we have:

$$\begin{aligned} 1.\, \nabla^f_{X^H} \mathcal{P}^V &= \nabla^f_{X^H} p_k (dx^k)^V \\ &= X^H(p_k)(dx^k)^V + p_k \nabla^f_{X^H} (dx^k)^V \\ &= p_h \Gamma^h_{kj} X^j (dx^k)^V + p_k (\nabla_X dx^k)^V + \frac{p_k}{2f} X(f)(dx^k)^V \\ &= -(\nabla_X \mathcal{P})^V + (\nabla_X \mathcal{P})^V + \frac{1}{2f} X(f) \mathcal{P}^V \\ &= \frac{1}{2f} X(f) \mathcal{P}^V. \end{aligned}$$

where $\nabla_X \mathcal{P} = -p_h \Gamma^h_{kj} X^j dx_k$.
The second formula is obtained by a similar calculation. □

4 Curvatures of g^f

We shall calculate the Riemannian curvature tensor R^f of the cotangent bundle T^*M equipped with the metric g^f. This curvature tensor is characterized by the formula:

$$R^f(\widetilde{U}, \widetilde{V})\widetilde{W} = \nabla^f_{\widetilde{U}} \nabla^f_{\widetilde{V}} \widetilde{W} - \nabla^f_{\widetilde{V}} \nabla^f_{\widetilde{U}} \widetilde{W} - \nabla^f_{[\widetilde{U}, \widetilde{V}]} \widetilde{W}, \tag{17}$$

for all $\widetilde{U}, \widetilde{V}, \widetilde{W} \in \Im^1_0(T^*M)$.

Theorem 2. *Let (M, g) be a Riemannian manifold and T^*M its cotangent bundle* *equipped with the metric g^f, then we have the following formulas*

$$\begin{aligned} R^f(X^H,Y^H)Z^H &= (R(X,Y)Z)^H - \frac{1}{2f}Z(f)(pR(X,Y))^V, \quad (18)\\ R^f(X^H,\theta^V)\eta^V &= \frac{-1}{2}g^{-1}(\theta,p)g^{-1}(\eta,p)(\nabla_X grad\, f)^H \\ &\quad + \frac{1}{4f}X(f)g^{-1}(\theta,p)g^{-1}(\eta,p)(grad\, f)^H, \quad (19)\\ R^f(\omega^V,\theta^V)Z^H &= 0, \quad (20)\\ R^f(X^H,\theta^V)Z^H &= \big(\frac{1}{2f}g(Z,\nabla_X grad\, f) - \frac{1}{4f^2}X(f)Z(f)\big)\theta^V, \quad (21)\\ R^f(X^H,Y^H)\eta^V &= 0, \quad (22)\\ R^f(\omega^V,\theta^V)\eta^V &= \frac{1}{r^2}\big[g^{-1}(\theta,\eta)\omega^V - g^{-1}(\omega,\eta)\theta^V\big] \\ &\quad - \frac{1}{4f}\|grad\, f\|^2 g^{-1}(\eta,p)\big[g^{-1}(\theta,p)\omega^V - g^{-1}(\omega,p)\theta^V\big], \quad (23) \end{aligned}$$

for all $X, Y, Z \in \Im^1_0(M)$ and $\omega, \theta, \omega \in \Im^0_1(M)$, where $\mathcal{P}^V$ is the canonical vertical vector field on T^*M and R denotes the Riemannian curvature tensor of (M, g).

Proof.
Let $X, Y, Z \in \Im^1_0(M)$, $\omega, \theta, \eta \in \Im^0_1(M)$ and $\mathcal{P}^V$ the canonical vertical vector field on T^*M. By applying Definition 1, Lemma 2, Lemma 3, Theorem 1 and Lemma 5 we have:

1)$R^f(X^H,Y^H)Z^H = \nabla^f_{X^H}\nabla^f_{Y^H}Z^H - \nabla^f_{Y^H}\nabla^f_{X^H}Z^H - \nabla^f_{[X^H,Y^H]}Z^H$

Direct calculations give

$$\nabla^f_{X^H}\nabla^f_{Y^H}Z^H = (\nabla_X\nabla_Y Z)^H,$$

and

$$\nabla^f_{Y^H}\nabla^f_{X^H}Z^H = (\nabla_Y\nabla_X Z)^H,$$

and

$$\begin{aligned} \nabla^f_{[X^H,Y^H]}Z^H &= \nabla^f_{[X,Y]^H}Z^H + \nabla^f_{(pR(X,Y))^V}Z^H \\ &= (\nabla_{[X,Y]}Z)^H + \frac{1}{2f}Z(f)(pR(X,Y)Z)^V. \end{aligned}$$

Hence, we have:

$$R^f(X^H,Y^H)Z^H = (R(X,Y)Z)^H - \frac{1}{2f}Z(f)(pR(X,Y))^V,$$

for all , $X, Y, Z \in \Im^1_0(M)$.
2)$R^f(X^H,\theta^V)\eta^V = \nabla^f_{X^H}\nabla^f_{\theta^V}\eta^V - \nabla^f_{\theta^V}\nabla^f_{X^H}\eta^V - \nabla^f_{[X^H,\theta^V]}\eta^V$

From direct calculation we get:

$$\begin{aligned}
\nabla^f_{X^H}\nabla^f_{\theta^V}\eta^V &= \nabla^f_{X^H}\big[\frac{-1}{2}g^{-1}(\theta,p)g^{-1}(\eta,p)(grad\, f)^H + \frac{1}{r^2}g^{-1}(\theta,\eta)\mathcal{P}^V\big] \\
&= -\frac{1}{2}\big[g^{-1}(\nabla_X\theta,p)g^{-1}(\eta,p) + g^{-1}(\theta,p)g^{-1}(\nabla_X\eta,p)\big](grad\, f)^H \\
&\quad -\frac{1}{2}g^{-1}(\theta,p)g(\eta,p)(\nabla_X grad\, f)^H + \frac{1}{2fr^2}X(f)g^{-1}(\theta,\eta)\mathcal{P}^V \\
&\quad +\frac{1}{r^2}\big[g^{-1}(\nabla_X\theta,\eta) + g^{-1}(\theta,\nabla_X\eta)\big]\mathcal{P}^V.
\end{aligned}$$

where $X^H g^{-1}(\theta,\eta) = g^{-1}(\nabla_X\theta,\eta) + g^{-1}(\theta,\nabla_X\eta)$.
and

$$\begin{aligned}
\nabla^f_{\theta^V}\nabla^f_{X^H}\eta^V &= \nabla^f_{\theta^V}\big[\big(\nabla_X\eta\big)^V + \frac{1}{2f}X(f)\eta^V\big] \\
&= -\frac{1}{2}g^{-1}(\theta,p)g^{-1}(\nabla_X\eta,p)(grad\, f)^H + \frac{1}{r^2}g^{-1}(\theta,\nabla_X\eta)\mathcal{P}^V \\
&\quad -\frac{1}{4f}X(f)g^{-1}(\theta,p)g^{-1}(\eta,p)(grad\, f)^H + \frac{1}{2fr^2}X(f)g^{-1}(\theta,\eta)\mathcal{P}^V.
\end{aligned}$$

and

$$\nabla^f_{[X^H,\theta^V]}\eta^V = -\frac{1}{2}g^{-1}(\nabla_X\theta,p)g^{-1}(\eta,p)(grad\, f)^H + \frac{1}{r^2}g^{-1}(\nabla_X\theta,\eta)\mathcal{P}^V,$$

which gives,

$$\begin{aligned}
R^f(X^H,\theta^V)\eta^V &= \frac{-1}{2}g^{-1}(\theta,p)g^{-1}(\eta,p)(\nabla_X grad\, f)^H \\
&\quad +\frac{1}{4f}X(f)g^{-1}(\theta,p)g^{-1}(\eta,p)(grad\, f)^H,
\end{aligned}$$

for all $X \in \Im^1_0(M)$ and $\theta,\eta \in \Im^0_1(M)$.
3) Applying formula (19) and 1^{st} Bianchi identity.

$$R^f(\omega^V,\theta^V)Z^H = R^f(Z^H,\theta^V)\omega^V - R^f(Z^H,\omega^V)\theta^V,$$

we get

$$\begin{aligned}
R^f(Z^H,\theta^V)\omega^V &= \frac{-1}{2}g^{-1}(\theta,p)g^{-1}(\omega,p)(\nabla_Z grad\, f)^H \\
&\quad +\frac{1}{4f}Z(f)g^{-1}(\theta,p)g^{-1}(\omega,p)(grad\, f)^H,
\end{aligned}$$

and

$$\begin{aligned}
R^f(Z^H,\omega^V)\theta^V &= \frac{-1}{2}g^{-1}(\omega,p)g^{-1}(\theta,p)(\nabla_Z grad\, f)^H \\
&\quad +\frac{1}{4f}Z(f)g^{-1}(\omega,p)g^{-1}(\theta,p)(grad\, f)^H,
\end{aligned}$$

which gives,

$$R^f(\omega^V,\theta^V)Z^H = 0,$$

for all $Z \in \Im^1_0(M)$ and $\omega,\theta \in \Im^0_1(M)$.
The other formulas are obtained by a similar calculation. □

5 Geodesics of g^f

Let (M, g) be a Riemannian manifold and $\gamma : I \to M$ be a curve on M ($I \subset \mathbb{R}$). We define on T^*M the curve $C : I \to T^*M$ by $C(t) = (\gamma(t), \vartheta(t))$, for all $t \in I$ where $\vartheta(t) \in T^*_{\gamma(t)}M$ i.e $\vartheta(t)$ is a covector field along $\gamma(t)$.

Definition 2. *Let (M, g) be a Riemannian manifold, $C(t) = (\gamma(t), \vartheta(t))$ be a curve on T^*M and ∇ denote the Levi-Civita connection of (M, g). If $\nabla_{\dot\gamma}\vartheta = 0$ the curve $C(t)$ is said to be a horizontal lift of the cure $\gamma(t)$, where $\dot\gamma$ the tangent field along $\gamma(t)$.*

Lemma 6. *Let (M, g) be a Riemannian manifold. If $\omega \in \Im^0_1(M)$ is a covector field on M and $(x, p) \in T^*M$ such that $\omega_x = p$, then we have:*

$$d_x\omega(X_x) = X^H_{(x,p)} + (\nabla_X\omega)^V_{(x,p)}.$$

for all $X \in \Im^1_0(M)$.

Proof.
Let (U, x^i) be a local chart on M in $x \in M$ and $(\pi^{-1}(U), x^i, p_j)$ be the induced chart on T^*M, if $X_x = X^i(x)\frac{\partial}{\partial x^i}|_x$ and $\omega_x = \omega_i(x)dx^i|_x = p$, then

$$\begin{aligned}
d_x\omega(X_x) &= X^i(x)\frac{\partial}{\partial x^i}|_{(x,p)} + X^i(x)\frac{\partial\omega_j}{\partial x^i}(x)\frac{\partial}{\partial p_j}|_{(x,p)} \\
&= X^i(x)\frac{\partial}{\partial x^i}|_{(x,p)} + \omega_k(x)\Gamma^k_{ji}(x)X^j(x)\frac{\partial}{\partial p_i}|_{(x,p)} \\
&\quad -\omega_k(x)\Gamma^k_{ji}(x)X^j(x)\frac{\partial}{\partial p_i}|_{(x,p)} + X^i(x)\frac{\partial\omega_j}{\partial x^i}(x)\frac{\partial}{\partial p_j}|_{(x,p)} \\
&= X^i(x)\frac{\partial}{\partial x^i}|_{(x,p)} + p_k\Gamma^k_{ji}(x)X^j(x)\frac{\partial}{\partial p_i}|_{(x,p)} \\
&\quad +X^i(x)\frac{\partial\omega_j}{\partial x^i}(x)\frac{\partial}{\partial p_j}|_{(x,p)} - \omega_k(x)\Gamma^k_{ij}(x)X^i(x)\frac{\partial}{\partial p_j}|_{(x,p)} \\
&= X^H_{(x,p)} + X^i(x)\big[\frac{\partial\omega_j}{\partial x^i}(x) - \omega_k(x)\Gamma^k_{ij}(x)X^i(x)\big](dx^i)^V_{(x,p)} \\
&= X^H_{(x,p)} + (\nabla_X\omega)^V_{(x,p)}.
\end{aligned}$$

□

Lemma 7. *Let (M, g) be a Riemannian manifold and ∇ denote the Levi-Civita connection of (M, g). If $\gamma(t)$ is a curve on M and $C(t) = (\gamma(t), \vartheta(t))$ is a curve on T^*M, then*

$$\dot C = \dot\gamma^H + (\nabla_{\dot\gamma}\vartheta)^V. \tag{24}$$

Proof. Locally, if $\omega \in \Im^0_1(M)$ is a covector field such $\omega(\gamma(t)) = \vartheta(t)$, then

$$\dot C(t) = dC(t) = d\omega(\gamma(t)) = d_{\gamma(t)}\omega(d_t\gamma) = d_{\gamma(t)}\omega(\dot\gamma).$$

Using Lemma 6 we obtain $\dot C(t) = \dot\gamma^H + (\nabla_{\dot\gamma}\vartheta)^V$. □

Theorem 3. *Let (M,g) be a Riemannian manifold and T^*M its cotangent bundle equipped with the metric g^f. If ∇ (resp. ∇^f) denote the Levi-Civita connection of (M,g) (resp. (T^*M,g^f)) and $C(t)=(\gamma(t),\vartheta(t))$ is the cure on T^*M such $\vartheta(t)$ is a covector field along $\gamma(t)$, then*

$$\begin{aligned}\nabla^f_{\dot{C}}\dot{C} &= \Big[\nabla_{\dot\gamma}\dot\gamma - \frac{1}{2}g^{-1}(\nabla_{\dot\gamma}\vartheta,\vartheta)^2 grad\, f\Big]^H \\ &\quad + \Big[\nabla_{\dot\gamma}\nabla_{\dot\gamma}\vartheta + \frac{1}{f}\dot\gamma(f)\nabla_{\dot\gamma}\vartheta + \frac{1}{r^2}g^{-1}(\nabla_{\dot\gamma}\vartheta,\nabla_{\dot\gamma}\vartheta)\vartheta\Big]^V. \end{aligned} \tag{25}$$

Proof. Using Lemma 6 we obtain

$$\begin{aligned}\nabla^f_{\dot{C}}\dot{C} &= \nabla^f_{[\dot\gamma^H+(\nabla_{\dot\gamma}\vartheta)^V]}[\dot\gamma^H+(\nabla_{\dot\gamma}\vartheta)^V] \\ &= \nabla^f_{\dot\gamma^H}\dot\gamma^H + \nabla^f_{\dot\gamma^H}(\nabla_{\dot\gamma}\vartheta)^V + \nabla^f_{(\nabla_{\dot\gamma}\vartheta)^V}\dot\gamma^H + \nabla^f_{(\nabla_{\dot\gamma}\vartheta)^V}(\nabla_{\dot\gamma}\vartheta)^V \\ &= (\nabla_{\dot\gamma}\dot\gamma)^H + (\nabla_{\dot\gamma}\nabla_{\dot\gamma}\vartheta)^V + \frac{1}{2f}\dot\gamma(f)(\nabla_{\dot\gamma}\vartheta)^V + \frac{1}{2f}\dot\gamma(f)(\nabla_{\dot\gamma}\vartheta)^V \\ &\quad -\frac{1}{2}g^{-1}(\nabla_{\dot\gamma}\vartheta,\vartheta)g^{-1}(\nabla_{\dot\gamma}\vartheta,\vartheta)(grad\, f)^H + \frac{1}{2}g^{-1}(\nabla_{\dot\gamma}\vartheta,\nabla_{\dot\gamma}\vartheta)\vartheta^V \\ &= (\nabla_{\dot\gamma}\dot\gamma)^H - \frac{1}{2}g^{-1}(\nabla_{\dot\gamma}\vartheta,\vartheta)^2(grad\, f)^H \\ &\quad +(\nabla_{\dot\gamma}\nabla_{\dot\gamma}\vartheta)^V + \frac{1}{f}\dot\gamma(f)(\nabla_{\dot\gamma}\vartheta)^V + \frac{1}{2}g^{-1}(\nabla_{\dot\gamma}\vartheta,\nabla_{\dot\gamma}\vartheta)\vartheta^V \\ &= \Big[\nabla_{\dot\gamma}\dot\gamma - \frac{1}{2}g^{-1}(\nabla_{\dot\gamma}\vartheta,\vartheta)^2 grad\, f\Big]^H \\ &\quad + \Big[\nabla_{\dot\gamma}\nabla_{\dot\gamma}\vartheta + \frac{1}{f}\dot\gamma(f)\nabla_{\dot\gamma}\vartheta + \frac{1}{r^2}g^{-1}(\nabla_{\dot\gamma}\vartheta,\nabla_{\dot\gamma}\vartheta)\vartheta\Big]^V. \end{aligned}$$

□

Theorem 4. *Let (M,g) be a Riemannian manifold, T^*M its cotangent bundle equipped with the metric g^f and $C(t)=(\gamma(t),\vartheta(t))$ a cure on T^*M such $\vartheta(t)$ is a covector field along $\gamma(t)$, then $C(t)$ is a geodesic on T^*M if and only if*

$$\begin{cases} \nabla_{\dot\gamma}\dot\gamma = \frac{1}{2}g^{-1}(\nabla_{\dot\gamma}\vartheta,\vartheta)^2 grad\, f, \\ \\ \nabla_{\dot\gamma}\nabla_{\dot\gamma}\vartheta = -\frac{1}{f}\dot\gamma(f)\nabla_{\dot\gamma}\vartheta - \frac{1}{r^2}g^{-1}(\nabla_{\dot\gamma}\vartheta,\nabla_{\dot\gamma}\vartheta)\vartheta. \end{cases} \tag{26}$$

Proof. The statement is a direct consequence of Theorem 2 and definition of geodesic. □

Corollary 1.
Let (M,g) be a Riemannian manifold and T^*M its cotangent bundle equipped with the metric g^f and $C(t)=(\gamma(t),\vartheta(t))$ be a horizontal lift of the curve $\gamma(t)$. Then $C(t)$ is a geodesic on T^*M if and only if $\gamma(t)$ is a geodesic on M.

Proof. Let $C(t) = (\gamma(t), \vartheta(t))$ be a horizontal lift of the curve $\gamma(t)$, then $\nabla_{\dot{\gamma}}\vartheta = 0$. Using Theorem 4 we deduce the result. □

Remark 2. *If $C(t) = (\gamma(t), \vartheta(t))$ horizontal lift of the curve $\gamma(t)$, locally we have:*

$$\begin{aligned} \nabla_{\dot{\gamma}}\vartheta = 0 \quad &\Leftrightarrow \quad \frac{d\vartheta_h}{dt} - \Gamma^i_{jh}\frac{d\gamma^j}{dt}\vartheta_i = 0 \\ &\Leftrightarrow \quad \vartheta(t) = \exp(A(t)).K \end{aligned}$$

where , $K \in \mathbb{R}^n$, $A(t) = [a_{hi}]$, $a_{hi} = \sum_{j=1}^{n} \Gamma^i_{jh}\frac{d\gamma^j}{dt}$.

Remark 3. *Using Remark 2 we can construct an infinity of examples of geodesics on (T^*M, g^f).*

Example 1. *Let $\mathbb{R}$ equipped with the Riemannian metric $g = e^x dx^2$.*
The Christoffel symbols of Riemannian connection are given by

$$\Gamma^1_{11} = \frac{1}{2}g^{11}\left(\frac{\partial g_{11}}{\partial x^1} + \frac{\partial g_{11}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^1}\right) = \frac{1}{2}$$

The geodesics $\gamma(t)$ such that $\gamma(0) = a \in \mathbb{R}$, $\gamma'(0) = v \in \mathbb{R}$ satisfy the equation,

$$\frac{d^2\gamma^k}{dt^2} + \sum_{i,j=1}^{n} \frac{d\gamma^i}{dt}\frac{d\gamma^j}{dt}\Gamma^k_{ij} = 0 \Leftrightarrow \gamma'' + \frac{1}{2}(\gamma')^2 = 0.$$

Hence $\gamma'(t) = \dfrac{2v}{2 + vt}$ *and therefore* $\gamma(t) = a + 2\ln(1 + \dfrac{vt}{2})$.
If $C(t) = (\gamma(t), \vartheta(t))$ is horizontal lift of the curve $\gamma(t)$ i.e $\nabla_{\dot{\gamma}}\vartheta = 0$ then,

$$\frac{d\vartheta_h}{dt} - \Gamma^i_{jh}\frac{d\gamma^j}{dt}\vartheta_i = 0 \Leftrightarrow \vartheta' - \frac{1}{2}\vartheta\gamma' = 0 \Leftrightarrow \vartheta(t) = k.\exp(\frac{1}{2}\gamma'(t)) = k.\exp(\frac{v}{2 + vt}).$$

Example 2. *Consider the upper half-plane*

$$\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2, y > 0\},$$

with the metric of Lobachevsky's non-euclidean geometry given by

$$g_{11} = g_{22} = \frac{1}{y^2} \ , \ g_{12} = g_{21} = 0.$$

The Christoffel symbols of the Riemannian connection are given by:

$$\Gamma^1_{11} = \Gamma^1_{22} = \Gamma^2_{12} = \Gamma^2_{21} = 0 \ , \ \Gamma^2_{11} = \frac{1}{y} \ , \ \Gamma^2_{22} = \Gamma^1_{12} = \Gamma^1_{21} = -\frac{1}{y}.$$

1) *If $C(t) = (\gamma(t), \vartheta(t))$ is a horizontal lift of the curve $\gamma(t) = (a, y(t))$, $a \in \mathbb{R}$ then the matrix $A(t)$ is given by*

$$A(t) = \frac{-1}{y(t)}\begin{pmatrix} y'(t) & 0 \\ 0 & y'(t) \end{pmatrix}$$

and

$$\vartheta(t) = \exp\Big(\frac{-1}{y(t)}\begin{pmatrix} y'(t) & 0 \\ 0 & y'(t) \end{pmatrix}\Big).K \ , \ K \in \mathbb{R}^2.$$

2) *If* $C(t) = (\gamma(t), \vartheta(t))$ *is a horizontal lift of the curve* $\gamma(t) = (x(t), y(t))$ *such* $y(t) = ax(t) + b$, $a, b \in \mathbb{R}$ *and* $x \neq 0$, *then the matrix* $A(t)$ *is given by*

$$A(t) = \frac{-x'(t)}{ax(t)+b}\begin{pmatrix} a & -1 \\ 1 & a \end{pmatrix},$$

and

$$\vartheta(t) = \exp\Big(\frac{-x'(t)}{ax(t)+b}\begin{pmatrix} a & -1 \\ 1 & a \end{pmatrix}\Big).K \ , \ K \in \mathbb{R}^2.$$

Theorem 5. *Let* (M, g) *be a Riemannian manifold,* T^*M *its cotangent bundle equipped with the metric* g^f *and* $\gamma(t)$ *be a geodesic on* M. *If* $C(t) = (\gamma(t), \vartheta(t))$ *is a geodesic on* T^*M *such that* $\nabla_{\dot\gamma}\vartheta \neq 0$, *then* f *is constant.*

Proof. Let $\gamma(t)$ be a geodesic on M , then $\nabla_{\dot\gamma}\dot\gamma = 0$. Using the first equation of formula (26) we obtain $grad\, f = 0$ i.e f is constant.

□

Corollary 2. *Let* (M, g) *be a Riemannian manifold,* T^*M *its cotangent bundle equipped with the metric* g^f *and* $\gamma(t)$ *be a curve on* M. *If* $C(t) = (\gamma(t), \vartheta(t))$ *is a geodesic on* T^*M *such that* $\|\vartheta\|$ *is constant, then* $\gamma(t)$ *is a geodesic on* M.

Proof. We have $0 = \dot\gamma g^{-1}(\vartheta, \vartheta) = 2g^{-1}(\nabla_{\dot\gamma}\vartheta, \vartheta)$ Using the first equation of formula (26) we obtain $\nabla_{\dot\gamma}\dot\gamma = 0$. □

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