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MULTISET IDEAL TOPOLOGICAL SPACES AND KURATOWSKI CLOSURE OPERATOR

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Abstract

In this article we have introduced the notion of multiset local function on an ideal topological space using the concept of q-neighbourhood in a multiset topological space. The Kuratowski closure operator has also been extended to multiset topological space with the aim to generate new topologies as well as multiset ideal topology.

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1 Introduction

Multisets, like sets, are collections of elements but with the underlying assumption that its elements can be repeated. The number of times an element repeats is called its multiplicity. In the real world, one can speculate enormous repetition. For instance, there are many hydrogen atoms, repeated statistical data, many strands of DNA etc. This has motivated many researchers to develop the theory of multisets which claims to have three possible relations among any two physical entities; they are different, they are same but separate, or they are coinciding and identical.

N.G. de Bruijn introduced the notion of multiset, which is a generalisation of a set. Besides its applications in the field computer science, physics, logic, many fields of mathematics have been explored in the context of multisets. The topological structures of multisets have been introduced by Girish et.al. [5]. Girish et.al.

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[6] studied in detail some basic notions and properties like basis, limit points, closure, interior and continuity on multiset topological space. After the introduction of a topology on multisets many researchers have put their effort into studying many topological properties of multisets (one may refer to [**?**, 4, 5, 10]).

The notion of ideal topology was introduced by Kuratowski [9]. Afterwards Vaidyanathaswamy [20] studied the concept of ideals in point set topology. It was followed by the investigations on ideal topological and ideal fuzzy topological spaces by Tripathy and Shravan [11, 12], Tripathy and Acharjee [13], Triapthy and Ray [19]. Jankovic and Hamlett [7] investigated further properties of ideal topological spaces where they studied that ideal is a generalisation, or unification, of the concept of closure point, w-accumulation point, condensation point and the point of second category. Also, on considering the ideals of subsets on N, the set of natural numbers, the notion of I-convergent sequences have been studied by Tripathy and Dutta [14], Tripathy and Hazarika [15, 16, 17], Tripathy and Mahanta [18] and others. The concept of local function of an ideal in general topology was further introduced by Kuratowski [9]. Zakaria et al [21] have given a brief introduction to ideals in a multiset topological space.

In this article our aim is to extend Kuratowski closure operator to multiset topological space with the aim to generate new topologies as well as multiset ideal topology.

2 Definitions and Preliminaries

Here $C_M(x)$ is the number of occurrences of the element x in the mset M drawn from the set $X = \{x_1, x_2, ..., x_n\}$ as $M = \{m_1/x_1, m_2/x_2, ...m_n/x_n\}$ where m_i is the number of occurrences of the element x_i, $i = 1, 2, ...n$ in the mset M. The elements which are not included in the mset M have zero count, by convention. Clearly a set is a particular case of an mset, in which every element has multilpicity 1.

Consider two msets M and N drawn from a set X. The following arithmetic operations on the msets are defined, those will be used throughout this article.

1. M= N if $C_M(x) = C_N(x)$ for all $x \in X$.
2. $M \subseteq N$ if $C_M(x) = C_N(x)$ for all $x \in X$.
3. $P = M \cup N$ if $C_P(x)$= $max\{C_M(x), C_N(x)\}$ for all $x \in X$.
4. $P = M \cap N$ if $C_P(x) = min\{C_M(x), C_N(x)\}$.
5. $P = M \oplus N$ if $C_P(x) = C_M(x) + C_N(x)$ for all $x \in X$.

6. $M^c = Z - M = \{C_{M^c}(x)/x : C_{M^c}(x) = C_Z(x) - C_M(x); x \in X\}, Z$ be a mset with maximum multiplicity in the multiset space.

7. $P = M \ominus N$ if $C_P(x) = max\{C_M(x) - C_N(x), 0\}$ for all $x \in X$,

   where $\oplus$ and $\ominus$ represent mset addition and mset subtraction respectively.

Let $[X]^w$ be an mset space and Z be a subset of $[X]^w$ with $C_Z(x)$ as the multiplicities of $x \in X$ in Z. Let $\{M_1, M_2, ...\}$ be a collection of msets drawn from $[X]^w$. Then the following operations are possible under an arbitrary collection $\{M_i : i \in \Delta\}$ of msets, where $\Delta = \{1, 2, ...\}$ is the support set.

1. The Union
   $\cup_{i\in\Delta} M_i = \{C_{M_i}(x)/x : C_{M_i}(x) = max\{C_{M_i}(x) : x \in X\}\}$.

2. The intersection
   $\cap_{i\in\Delta} M_i = \{C_{\cap M_i}(x)/x : C_{\cap M_i}(x) = min\{C_{M_i}(x) : x \in X\}$.

3. The mset complement
   $M^c = Z \ominus M = \{C_{M^c}(x)/x : C_{M^c}(x) = C_Z(x) - C_M(x), x \in X\}$.

Definition 1. *Let M be an mset drawn from a set X. The support set of M denoted by M^* is a subset of X and $M^* = \{x \in X : C_M(x) > 0\}$.*

Definition 2. *An mset M is said to be an empty set if for all $x \in X$, $C_M(x) = 0$.*

Definition 3. *Let X be a support set and $[X]^w$ be the mset space defined over X. Then for any mset $M \in [X]^w$, the complement M^c of M in $[X]^w$ is an element of $[X]^w$ such that $C_{M^c} = w - C_M(x)$ for all $x \in X$.*

Since Cantor's power set theorem fails for msets, the following is the reasonable definition of a power mset of M for finite mset M that preserves Cantor's power set theorem.

Definition 4. *Let $M \in [X]^w$ be an mset. The power mset $P(M)$ of M is the set of all submsets of M. We have $N \in P(M)$ if and only if $N \subseteq M$. If $N = \emptyset$, then $N \in^1 P(M)$ and if $N \neq \emptyset$, then $N \in^k P(M)$, where $k = \prod_z \binom{[M]_z}{[N]_z}$, the product $\prod_z$ is taken over by distinct elements z of the mset N and $|[M]_z| = m$ if and only if $z \in^m M$ and $|[N]_z| = n$ if and only if $z \in^n N$ then,*

$$\binom{[M]_z}{[N]_z} = \binom{m}{n} = \frac{m!}{n!(m-n)!}.$$

The power set of an mset is the support set of the power mset and is denoted by $P^*(M)$.

Remark 1. *Power mset is an mset but its support set is an ordinary set whose elements are msets.*

Multiset Topology

Girish and John [5] introduced the concept of Multiset topology and defined as follows:

Let $M \in [X]^w$ and $\tau \subseteq P^*(M)$. Then τ is called a multiset topology of M if τ satisfies the following properties,

1. The mset M and the empty mset $\emptyset$ are in τ.

2. The mset union of elements of any subcollection of τ is in τ.

3. The mset intersection of the elements of any finite subcollection of τ is in τ.

A multiset topological space is an ordered pair (M, τ) consisting of an mset M extracted from $[X]^w$ and a multiset topology $\tau \subseteq P^*(M)$ on M. Multiset topology is abbreviated as M-Topology. The elements of τ are called open mset. The complement of an open mset in an M-Topological space is said to be closed mset.

Definition 5. *Given a submset A of M-topological space M in $[X]^w$, the interior of A is denoted by $int(A)$ and is defined as the mset union of all open msets contained in A i.e $C_{int(A)}(x) = max\{C_G(x) : G \subseteq A\}$.*

Definition 6. *Given a submset A of an M-topological space M in $[X]^w$, the closure of A is defined by the mset intersection of all closed msets containing A and is denoted by $cl(A)$, i.e $C_{Cl(A)}(x) = min\{C_K(x) : A \subseteq K\}$.*

The notion of ideal is defined as follows,

Definition 7. *A nonempty collection of subsets I of a nonempty set X is called a ideal on X if the following postulates are satisfied,*
(i) if $A \in I$ and $B \subseteq A$, then $B \in I$ (heredity).
(ii) if $A, B \in I$, then $A \cup B \in I$ (finite additivity).

The triplet (X, τ, I) is called ideal topological space with the ideal I and topology τ.

The notion of mset ideal is defined in Zakaria et.al.[20] as follows,

Definition 8. *A non-empty collection I of submsets of a non-empty mset M is said to be an mset ideal on M, if it satisfies the following conditions:*

1. $N_1 \in I$ *and* $C_{N_2}(x) \leq C_{N_1}(x)$*for all* $x \in X \Rightarrow N_2 \in I$.

2. $N_1 \in I$, $N_2 \in I \Rightarrow N_1 \cup N_2 \in I$.

The mset ideal is abbreviated as M-ideal.

Let M be an infinite set. Then consider the family of subsets by M, defined by $I = \{N \subseteq M : N \text{ is finite }\}$. It can be easily verified that I is an mset ideal of M. For further examples of mset ideals, one may refer to Zakaria et.al.][20].

Zakaria et.al.[20] have established that the union and intersection of two mset ideals is an mset ideal.

3 Main results

Definition 9. *Let $[X]^w$ be a space of multisets. A multipoint is a multiset M in X such that*

$C_M(x) = k$, *for* $x \in M$;

$= 0$, *otherwise,*

where $k \in \{1, 2, 3, ...w\}$ *and* $C_M(x)$ *is the multiplicity of* x *in* X.

Remark 2. *A multipoint, denoted by $\{k/x\}$ is a subset of a multiset M or $\{k/x\} \in M$ if $k \leq C_M(x)$.*

Quasi-coincidence in Multisets

Definition 10. *Let M be any multiset in the space $[X]^w$. If $N \subseteq M$, then $k/x \in M$ is said to be quasi-coincident with N if and only if $k > C_{N^c}(x)$.*

Definition 11. *A multiset M is said to be quasi-coincident with N i.e. MqN at x if and only if $C_M(x) > C_{N^c}$.*

Remark 3. *It can be easily verified, if M and N are quasi-coincident at x then both $C_M(x)$ and $C_N(x)$ are non-zero and hence M and N intersect at x.*

Definition 12. *A multiset N in an M-Topological space (M, τ) is said to be a q-neighborhood(Q-nbhd) of k/x if and only if there exists an open mset P such that,*

$$k/xqP \subset N.$$

Multiset Local Function

Let (M, τ) be a M-topological space and $\mathcal{I}$ be an M-ideal on M. If N is any submset of M, then the local function denoted by $N^*(I, \tau)$ is defined by,

$$N^*(\mathcal{I}, \tau)$$
$$= \cup\{m_i/x_i \in M : C_U(x_j) - C_{N^c}(x_j) > C_I(x_j), I \in \mathcal{I} \text{ for every } U \in N_q(m_i/x_i)$$

and at least one $x_j \in X\}$,
where $N_q(m_i/x_i)$ is the set of q-nbhd of m_i/x_i.

We will write $N^*(\mathcal{I})$ or N^* in place of $N^*(\mathcal{I}, \tau)$.

Kuratowski closure operator on multisets

A multiset closure operator is a mapping $Cl_M : P^*(M) \to P^*(M)$, which associates each $N \in P^*(M)$ with its closure i.e. $Cl_M(N)$.

The multiset closure operator satisfies the following closure axioms, similar to the Kuratowski closure axioms of general topology.

1. $Cl_M(\emptyset) = \emptyset$.
2. $N \in P^*(M) \Rightarrow N \subseteq Cl_M(N)$.
3. $N, P \in P^*(M) \Rightarrow Cl_M(N \cup P) = Cl_M(N) \cup Cl_M(P)$.
4. $N \in P^*(M) \Rightarrow Cl_M(Cl_M(N)) = Cl_M(N)$.

(1) Since $\emptyset$ is an mset and the closed for any closed mset is itself, so we have $Cl_M(\emptyset) = \emptyset$.
(2) By definition of closure, it is obvious.

(3) It is obvious that

$$Cl_M(N) \cup Cl_M(P) \subseteq Cl_M(N \cup P). \tag{1}$$

By Kuratowski closure axiom 2, we have

$$Cl_M(N \cup P) \subseteq Cl_M(N) \cup Cl_M(P). \tag{2}$$

Therefore, $Cl_M(Cl(N \cup P)) \subseteq Cl_M(Cl_M(N) \cup Cl_M(P))$.

But, $Cl_M(Cl_M(N) \cup Cl_M(P))$ is closed and so $Cl_M(Cl_M(N) \cup Cl_M(P)) = Cl_M(N) \cup Cl_M(P)$.

So,

$$Cl_M(N \cup P) \subseteq Cl_M(N) \cup Cl_M(P). \tag{3}$$

The result follows from equation (2) and (3),

(4) Since Cl_M is the smallest closed mset containing N and closure of a closed mset is equal to the mset itself. So $Cl_M(Cl_M(N)) = Cl_M(N)$.

M-topology generated by Kuratowski closure operator

Theorem 1. *Let Cl_M be the closure operator. Then for every $N \in P^*(M)$ $\tau_M = \{N \subseteq M : Cl_M(N^c) = N^c\}$ is the generated M-topology.*

Proof. (1) Since M and $\emptyset$ are both closed msets. So, $Cl_M(M) = M$ and so $M^c = \emptyset \in \tau_M$ and $Cl_M(\emptyset) = \emptyset$ and so $\emptyset^c = M \in \tau_M$.

(2) Let $N_\lambda \in \tau$. Then,

$$Cl_M(N_\lambda^c) = N_\lambda^c. \tag{4}$$

By Kuratowski closure axiom 2,

$$(\cup_\lambda N_\lambda)^c \subseteq Cl_M((\cup_\lambda N_\lambda)^c). \tag{5}$$

By De-Morgan's law, we have

$(\cup_\lambda N_\lambda)^c = \cap_\lambda N_\lambda^c$ and $\cap_\lambda N_\lambda^c \subseteq N_\lambda^c$ for all λ.

$Cl_M(\cap_\lambda N_\lambda^c) \subseteq Cl_M(N_\lambda^c)$ for all λ.

Therefore $Cl_M(\cap_\lambda N_\lambda^c) \subseteq \cap_\lambda Cl_M(N_\lambda^c) = \cap_\lambda N_\lambda^c$ or,

$$Cl_M((\cup_\lambda N_\lambda)^c) \subseteq (\cup_\lambda N_\lambda)^c. \tag{6}$$

From equations (5) and (6),
$Cl_M((\cup_\lambda N_\lambda)^c = (\cup_\lambda N_\lambda)^c$.

Hence, $\cup_\lambda N_\lambda \in \tau_M$ whenever $N_\lambda \in \tau_N$.

(3) Let $N_1, N_2 \in \tau_M$. Then, $Cl_M(N_1^c) = N_1^c$ and $Cl_M(N_2^c) = N_2^c$.

Now, $Cl_M((N_1 \cap N_2)^c) = Cl_M(N_1^c \cup N_2^c) = Cl_M(N_1^c) \cup Cl_M(N_2^c) = N_1^c \cup N_2^c = (N_1 \cap N_2)^c$.

Hence, $N_1 \cap N_2 \in \tau_M$.

□

Proposition 1. *Let us define $d_M : P^*(M) \to P^*(M)$ such that,*

1. *$d_M(\emptyset) = \emptyset$.*
2. *$d_M(A \cup B) = d_M(A) \cup d_M(B)$.*
3. *$d_M(d_M(A)) = d_M(A)$.*

Define $Cl_M : P^*(M) \to P^*(M)$ by,

$$Cl_M(N) = N \cup d_M(N).$$

Then Cl_M is a Kuratowski closure operator on the mset M.

Proof. The first two axioms can be established directly from the definition of d_M and Cl_M.

(3) Let $N, P \in P^*(M)$, then we have $Cl_M(N \cup P) = (N \cup P) \cup d_M(N \cup P) = (N \cup P) \cup d_M(N) \cup d_M(P) = (N \cup d_M(N)) \cup (P \cup d_M(P)) = Cl_M(N) \cup Cl_M(P)$.

(4) Let $N \in P^*(M)$, then we have $Cl_M(Cl_M N) = Cl_M N \cup d_M(Cl_M N) = Cl_M N \cup d_M(N \cup d_M(N)) = Cl_M N \cup d_M(N) \cup d_M(d_M(N)) = Cl_M N \cup d_M(N) \cup d_M(N) = N \cup d_M(N) \cup d_M(N) = N \cup d_M(N) = Cl_M(N)$.

Thus, Cl_M defined as above is a closure operator.

□

Remark 4. *The local function operator $^* : P^*(M) \to P^*(M)$ satisfies all conditions of d_M, so $Cl_M^*(N) = N \cup N^*$ is also a closure operator.*

Topology generated by Cl_M^*

$\tau_M^* = \{N \subseteq M : Cl_M^*(N^c) = N^c\}$.

Remark 5. *We have established, if $\mathfrak{I} = \{\emptyset\}$, then $N^*(\mathfrak{I}) = ClN$ and for $\mathfrak{I} = P^*(M)$, $N^*(\mathfrak{I}) = \emptyset$. So, for $\mathfrak{I} = \{\emptyset\}$, $Cl_M^*(N) = Cl_M(N)$ i.e $\tau_M^* = \tau_M$. Also for $\mathfrak{I} = P^*(M)$, τ_M^* is the discrete M-topology. Thus $\{\emptyset\} \subseteq \mathfrak{I} \subseteq P^*(M)$ implies $\tau^*(\{\emptyset\}) \subseteq \tau^*(\mathfrak{I}) \subseteq \tau^*(P^*(M))$ i.e. $\tau \subseteq \tau^*(\mathfrak{I}) \subseteq$ discrete M-topology.*

Theorem. 3.3. If A^{d^*} is the derived mset of τ^* M-topology and A^d is the derived mset of τ M-topology, then $A^{d^*} \subseteq A^d$.

Proof. Let $k/x \in A^{d^*}$, then by the definition of limit point every q-nbhd of k/x in τ_M is a M-topology and A are quasi-coincident. As $\tau_M \subseteq \tau_M^*$, every q-nbhd of k/x in τ_M M-topology and A are quasicoincident i.e $k/x \in A^d$. So $A^{d^*} \subseteq A^d$.

Remark 6. *It can be easily verified that an mset A is closed and discrete if and only if $A^d = \emptyset$.*

Lemma 1. *Let (M, τ_M) be an M-topological space and $\mathfrak{I}$ be an ideal on M. If $I \in \mathfrak{I}$, then I is closed and discrete in (M, τ_M^*).*

Proof. We have, $I \in \mathfrak{I} \Rightarrow I^* = \emptyset$. Thus, $I^{d^*} \subseteq I^* = \emptyset$. The result follows by **Remark 6**

□

Example 1 Let (M, τ) be an M-topological space. Then $\mathfrak{I} = \{N \subseteq M : N^d = \emptyset\}$ is an M-ideal. We have,

(1) Let $N_1 \in \mathfrak{I}$. Then $N_1^d = \emptyset$. If $N_2 \subseteq N_1$ such that $C_{N_2}(x) \leq C_{N_1}(x)$ for all $x \in X$. By definition of limit points it is obvious that $N_2^d = \emptyset$.

(2) Let $N_1, N_2 \in \mathfrak{I}$. Then we have $(N_1 \cup N_2)^d = N_1^d \cup N_2^d = \emptyset$ $[N_1, N_2 \in \mathfrak{I}]$.

Thus, $N_1, N_2 \in \mathfrak{I} \Rightarrow N_1 \cup N_2 \in \mathfrak{I}$. Hence, $\mathfrak{I}$ is an ideal.

Theorem 2. *Let (M, τ) be an M-topological space and $\mathfrak{I}$ be M-ideal in M. Then for any mset N,*
$N^* = Cl(N - I), I \in \mathfrak{I}$, where $C_{(N-I)}(x) = max\{C_N(x) - C_I(x), 0\}$ $\forall x \in X$.

Proof. We have, $N^* = \cup\{m_i/x_i \in M : C_U(y) - C_{N^c}(y) > C_I(y)\} = \cup\{m_i/x_i \in M : C_U(y) > C_{N^c}(y) + C_I(y)\} = \cup\{m_i/x_i \in M : C_U(y) > C_{(N^c+I)}(y)\} = \cup\{m_i/x_i \in M : C_U(y) > C_{(N-I)^c}(y)\} = Cl(N - I)$, where U is a q-nbhd of m_i/x_i. □

Theorem 3. *Let $\mathcal{B}(\mathfrak{I}, \tau) = \{N - I : N \in \tau, I \in \mathfrak{I}\}$. Then $\mathcal{B}$ forms an M-basis for τ^*.*

Proof. Let m/x be any m-point in (M, τ) where $\mathfrak{I}$ is any M-ideal in M. If U is any q-nbhd of m/x in τ^* M-topology, then there exists an mset $P \in \tau^*$ such that,

$$m/xqP \subseteq U. \tag{7}$$

Since, $P \in \tau^* \Rightarrow P^c$ is τ^* closed, therefore $Cl^*(P^c) = P^c \Rightarrow P^c \cup (P^c)^* = P^c$.

Hence,

$$(P^c)^* \subseteq P^c \Rightarrow P \subseteq ((P^c)^*)^c. \tag{8}$$

From(7), we have

$$\begin{aligned}
&m > C_{P^c}(x) \Rightarrow m > C_M(x) - C_P(x)\\
&\Rightarrow m + C_P(x) > C_M(x)\\
&\Rightarrow m + C_{((P^c)^*)^c} > C_M(x)\\
&\Rightarrow m + C_M(x) - C_{(P^c)^*}(x) > C_M(x)\\
&\Rightarrow m > C_{(P^c)^*}(x), \text{ i.e } m/x \notin (P^c)^*.
\end{aligned}$$

So, for every $y \in X$, there exists at least one q-nbhd V of m/x in τ such that, $C_V(y) - C_{(P^c)^c}(y) \leq C_{I_1}(x)$ for some $I_1 \in \mathcal{I}$.

$$C_V(y) - C_{I_1}(y) \leq C_P(y) \forall y \in X. \tag{9}$$

Since V is a q-nbhd of $m/x \in \tau$, there exists an open mset $N \in \tau$ such that $m/xqN \subseteq V$. By heredity property of M-ideal, for any M-ideal $I \in \mathcal{I}$, we have by using (9) we have, $m/xq(N - I) \subseteq P$ i.e. $P = \cup(N - I)$, which shows that every τ^* open mset can be expressed as the union of members of $\mathcal{B}$.

Hence, $\mathcal{B} = \{N - I : N \in \tau, I \in \mathcal{I}\}$ is an M-basis for the generated M-topology. □

Theorem 4. *Let (M, τ) be an M-topological space, If $\mathcal{I}$ and $\mathcal{J}$ are two M-ideals of M and $N \subseteq M$, then,*

1. $N^*(\mathcal{I} \cap \mathcal{J}, \tau) = N^*(\mathcal{I}, \tau) \cup N^*(\mathcal{J}, \tau)$.
2. $N^*(\mathcal{I} \cup \mathcal{J}, \tau) = N^*(\mathcal{I}, \tau^*(\mathcal{I})) \cap N^*(\mathcal{J}, \tau^*(\mathcal{J}))$.

Proof. (1) Clearly we have $N^*(\mathcal{I}) \subseteq N^*(\mathcal{I} \cap \mathcal{J})$ and $N^*(\mathcal{J}) \subseteq N^*(\mathcal{I} \cap \mathcal{J})$.

Hence,

$$N^*(\mathcal{I}) \cup N^*(\mathcal{J}) \subseteq N^*(\mathcal{I}) \cap N^*(\mathcal{J}). \tag{10}$$

Next, let $m/x \notin N^*(\mathcal{I}) \cup N^*(\mathcal{J})$. Then $m/x \notin N^*(\mathcal{I})$ and $m/x \notin N^*(\mathcal{J})$. $m/x \notin N^*(\mathcal{I})$ implies that there exists at least one q-nbhd U of m/x such that for all $y \in X$, $C_U(y) - C_{N^c}(y) \leq C_I(y)$, $I \in \mathcal{I}$.

Similarly, there exists at least one q-nbhd V of m/x such that for all $y \in X$, $C_V(y) - C_{N^c}(y) \leq C_J(y)$, $J \in \mathcal{J}$.

Thus, we have, $C_{U \cap V}(y) - C_{N^c}(y) \leq C_{\mathcal{I} \cap \mathcal{J}}(y)$. Since $U \cap V$ is also a q-nbhd of m/x(in τ) and $I \cap J \in \mathcal{I} \cap \mathcal{J}$, $m/x \notin N^*(\mathcal{I} \cap \mathcal{J})$ i.e.

$$N^*(\mathcal{I} \cap \mathcal{J}) \subseteq N^*(\mathcal{I}) \cup N^*(\mathcal{J}). \tag{11}$$

From (10) and (11), we have,

$N^*(\mathfrak{I} \cap \mathfrak{J}, \tau) = N^*(\mathfrak{I}, \tau) \cup N^*(\mathfrak{J}, \tau)$.

(2) Let $m/x \notin N^*(\mathfrak{I} \vee \mathfrak{J}, \tau)$. So there exists a q-nbhd U(in τ) such that for $y \in X$, $C_U(y) - C_{N^c}(y) \leq C_I(y)$, $I \in \mathfrak{I} \vee \mathfrak{J}$.

By hereditary property of M-ideals and by using the structure of open msets in the generated M-topology, we can find q-nbhds V and W of m/x in $\tau^*(\mathfrak{I})$ and $\tau^*(\mathfrak{J})$ respectively such that for every $y \in X$, $C_U(y) - C_{N^c}(y) \leq C_{I_1}(y)$, $I_1 \in \mathfrak{I}$ and $C_W(y) - C_{N^c}(y) \leq C_{J_1}(y)$, $J_1 \in \mathfrak{J}$.

So $m/x \notin N^*(\mathfrak{I}, \tau^*(\mathfrak{I}))$ and $m/x \notin N^*(\mathfrak{J}, \tau^*(\mathfrak{J}))$ i.e. $m/x \notin N^*(\mathfrak{I}, \tau^*(\mathfrak{I})) \cap N^*(\mathfrak{J}, \tau^*(\mathfrak{J}))$.
Therefore,

$$N^*(\mathfrak{I}, \tau^*(\mathfrak{I})) \cap N^*(\mathfrak{J}, \tau^*(\mathfrak{J})) \subseteq N^*(\mathfrak{I} \vee \mathfrak{J}, \tau). \tag{12}$$

Next, let $m/x \notin N^*(\mathfrak{I}, \tau^*(\mathfrak{I}))$, so there exists at least one q-nbhd U(in τ^*) such that for all $y \in X$, $C_U(y) - C_{N^c}(y) \leq C_I(y), I \in \mathfrak{I}$.

By hereditary property of M-ideals for a q-nbhd U in τ^*, there exists a q-nbhd V in τ such that,

$C_V(y) - C_N^c(y) \leq C_{I \cap J}(y)$, $I \in \mathfrak{I}$ and $J \in \mathfrak{J}$.

Therefore, $m/x \notin N^*(\mathfrak{I} \vee \mathfrak{J}, \tau)$. Thus we have

$$N^*(\mathfrak{I} \vee \mathfrak{J}, \tau) \subseteq N^*(\mathfrak{I}, \tau^*(\mathfrak{I})) \cap N^*(\mathfrak{J}, \tau^*(\mathfrak{J})). \tag{13}$$

From (12) and (13), we get

$N^*(\mathfrak{I} \vee \mathfrak{J}, \tau) = N^*(\mathfrak{I}, \tau^*(\mathfrak{I})) \cap N^*(\mathfrak{J}, \tau^*(\mathfrak{J}))$. □

Declaration. The authors declare that the article is free from conflict of interest.

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