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CERTAIN SUBCLASS OF POLYLOGARITHM FUNCTIONS DEFINED BY A GENERALIZED DIFFERENTIAL OPERATOR

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Abstract

In this paper, we define a new subclass of polylogarithm functions and obtained coefficient estimates, growth and distortion theorems, extreme points, radii of starlikeness, convexity and close to convexity for the class $TS_{\lambda}^{m}(\gamma, \alpha, k, \beta)$. Furthermore, we obtained the Fekete-Szego problem for the class also.

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1 Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

which are analytic and univalent in the open disc $U = \{z \in \mathbb{C} : |z| < 1\}$. A function f in the class of A is said to be in the class $S^*(\beta)$ of starlike functions of order β in E, if it satisfy the inequality

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \beta, \ (z \in E, \ 0 \leq \beta < 1). \tag{2}$$

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Note that $S^*(0) = S^*$ is the class of starlike functions. Denote by T the subclass of A consisting of functions f of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \ (a_n \geq 0). \tag{3}$$

This subclass was introduced and studied by Silverman [12]. For function $f \in A$ given by (1) and $g(z) \in A$ given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n,$$

we define the Hadmard product (or convolution) of f and g by

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \ z \in E. \tag{4}$$

Let $f \in A$. Denote by $\mathfrak{D}^\lambda : A \to A$ the operator defined by

$$\mathfrak{D}^\lambda = \frac{z}{(1-z)^{\lambda+1}} * f(z) \ (\lambda > -1).$$

It is obivious that $\mathfrak{D}^0 f(z) = f(z)$, $\mathfrak{D}^1 f(z) = zf'(z)$ and

$$\mathfrak{D}^\lambda f(z) = \frac{z(z^{\lambda-1} f(z))}{\lambda}, \ (\lambda \in N_0 = N \cup 0).$$

$$\text{Note that } \mathfrak{D}^\lambda f(z) = z + \sum_{n=2}^{\infty} c(n,\lambda) a_n z^n$$

$$\text{where } c(n,\lambda) = \begin{pmatrix} n+\lambda-1 \\ \lambda \end{pmatrix} \text{ and } \lambda \in N_0.$$

The operator $\mathfrak{D}^\lambda f$ is called the Ruscheweyh derivative operator [10]. We recall here the definition of well known generalization of the polylogarithm function $G(m,z)$ given by

$$G(m,z) = \sum_{n=1}^{\infty} \frac{z^n}{n^m}, \ (m \in \mathbb{C}, z \in E). \tag{5}$$

We note that $G(-1,z) = \frac{z}{(1-z)^2}$ is Koebe function.

For more about polylogarithms in the theory of univalent functions see [1, 7, 8] and [13].

We now introduce a function $(G(m,z))^{-1}$ given by

$$G(m,z) * (G(m,z))^{-1} = \frac{z}{(1-z)^{\lambda+1}}, \ \lambda > -1, m \in \mathbb{C} \tag{6}$$

and obtain the following linear operator

$$\mathfrak{D}^{\lambda} f(z) = (G(m,z))^{-1} * f(z). \tag{7}$$

Now we find the explicit form of the function $(G(m,z))^{-1}$, it is well known that $\lambda > -1$.

$$\frac{z}{(1-z)^{\lambda+1}} = \sum_{n=0}^{\infty} \frac{(\lambda+n)}{n!} z^{n+1}, \ (z \in E). \tag{8}$$

Putting (6) and (8) in (7), we get

$$\sum_{n=1}^{\infty} \frac{z^n}{n^m} * (G(m,z))^{-1} = \sum_{n=1}^{\infty} \frac{(n+\lambda-1)}{\lambda(n-1)} z^n.$$

Therefore the function $(G(m,z))^{-1}$ has the following form

$$(G(m,z))^{-1} = \sum_{n=1}^{\infty} \frac{(n+\lambda-1)!}{\lambda!(n-1)!} z^n, \ (z \in \mathbb{C}).$$

For $m, \lambda \in N_0$, we note that

$$\mathfrak{D}_{\lambda}^{m} f(z) = z + \sum_{n=2}^{\infty} n^m \frac{(n+\lambda-1)!}{\lambda!(n-1)!} a_n z^n, \ (z \in \mathbb{C}). \tag{9}$$

Note that $\mathfrak{D}_0^m \cong \mathfrak{D}^m$ and $\mathfrak{D}_{\lambda}^0 \cong \mathfrak{D}^{\lambda}$ which were Salagean [11] and Ruschewey [10] derivative operators respectively. It is clear that the operator $\mathfrak{D}_{\lambda}^m$ included two known derivative operators. Also note that $\mathfrak{D}_0^0 f(z) = f(z)$ and $\mathfrak{D}_0^1 f(z) = \mathfrak{D}_1^0 f(z) = zf'(z)$. If $f \in T$ is given by (3) then we have

$$\begin{aligned} \mathfrak{D}_{\lambda}^{m} f(z) &= z - \sum_{n=2}^{\infty} n^m \frac{(n+\lambda-1)!}{\lambda!(n-1)!} a_n z^n, \ (z \in \mathbb{C}) \\ &= z + \sum_{n=2}^{\infty} n^m c(n,\lambda) a_n z^n \end{aligned} \tag{10}$$

where $c(n,\lambda) = \begin{pmatrix} n+\lambda-1 \\ \lambda \end{pmatrix}$.

Using the differential operator (9), we define the following a new subclass of the class A.

Definition 1. *For $0 \le \gamma \le 1$, $\alpha \ge 1, k \ge 0$ and $0 \le \beta < 1$, a function $f \in A$ is said to be in the class $S_{\lambda}^{m}(\gamma,\alpha,k,\beta)$ if it satisfy the condition*

$$\mathbb{R}\Big\{\alpha \frac{zG'(z)}{G(z)} - (\alpha-1)\Big\} > k\Big|\alpha \frac{zG'(z)}{G(z)} - \alpha\Big| + \beta, \tag{11}$$

where

$$G(z) = (1-\gamma)\mathfrak{D}_{\lambda}^{m} f(z) + \gamma z(\mathfrak{D}_{\lambda}^{m} f(z))'. \tag{12}$$

We also define $TS_{\lambda}^{m}(\gamma,\alpha,k,\beta) = S_{\lambda}^{m}(\gamma,\alpha,k,\beta) \cap T$.

By suitably specializing the parameters involved, the class $S_\lambda^m(\gamma,\alpha,k,\beta)$ and if it satisfy the condition $TS_\lambda^m(\gamma,\alpha,k,\beta)$ can be reduced to new or to known much simpler classes of functions which were studied in earlier works (see [2, 3, 4, 5, 9]). The object of this paper is to study various properties for functions belonging to the class $S_\lambda^m(\gamma,\alpha,k,\beta)$ and $TS_\lambda^m(\gamma,\alpha,k,\beta)$ respectively.

2 Coefficient estimates

In order to prove our results from this section we need the following lemma.

Lemma 1. *Let β be a real number and w be a complex number. Then $\mathbb{R}(w) \geq \beta$ if and only if*

$$|w+(1-\beta)| - |w-(1+\beta)| \geq 0.$$

First we give a sufficient coefficient bound for functions in the class $S_\lambda^m(\gamma,\alpha,k,\beta)$.

Theorem 1. *Let $f \in A$ given by (1). If*

$$\sum_{n=2}^{\infty}[1-\beta+\alpha(n-1)(1+k)]\ A_n(\lambda,\gamma,m)|a_n| \leq 1-\beta \tag{13}$$

where

$$A_n(\lambda,\gamma,m) = [1+\gamma(m-1)]\ n^m c(n,\lambda). \tag{14}$$

Then $f \in S_\lambda^m(\gamma,\alpha,k,\beta)$.

Proof. In virtue of Definition 1 and Lemma 1, it is sufficient to show that

$$\begin{aligned} &\Big|\alpha\frac{zG'(z)}{G(z)} - (\alpha-1) - k\Big|\alpha\frac{zG'(z)}{G(z)} - \alpha| - (1+\beta)| \\ \leq\ &\Big|\alpha\frac{zG'(z)}{G(z)} - (\alpha-1) - k\Big|\alpha\frac{zG'(z)}{G(z)} - \alpha| + (1+\beta)|. \end{aligned} \tag{15}$$

For the right hand and left hand side of (15) we may respectively, write

$$\begin{aligned} R &= \Big|\alpha\frac{zG'(z)}{G(z)} - (\alpha-1) - k\Big|\alpha\frac{zG'(z)}{G(z)} - \alpha| + (1-\beta)| \\ &= \frac{1}{|G(z)|}\Big|\alpha zG'(z) - (\alpha-1)G(z) - ke^{i\theta}\Big|\alpha zG'(z) - \alpha G(z)| + (1-\beta)G(z)| \\ &> \frac{|z|}{|G(z)|}\Big[2-\beta-\sum_{n=2}^{\infty}2-\beta+\alpha(n-1)(k+1)\Big]A_n(\lambda,\gamma,m)|a_n| \end{aligned}$$

and similarly

$$\begin{aligned} L &= \Big|\alpha\frac{zG'(z)}{G(z)} - (\alpha-1) - k\Big|\alpha\frac{zG'(z)}{G(z)} - \alpha| - (1+\beta)| \\ &= \frac{1}{|G(z)|}\Big|\alpha zG'(z) - (\alpha-1)G(z) - ke^{i\theta}\Big|\alpha zG'(z) - \alpha G(z)| - (1+\beta)G(z)| \\ &< \frac{|z|}{|G(z)|}\Big[\beta+\sum_{n=2}^{\infty}\Big|\alpha(n-1)(1+k)-\beta\Big|A_n(\lambda,\gamma,m)|a_n|\Big] \end{aligned}$$

since

$$R - L > \frac{|z|}{|G(z)|}\left[2(1-\beta) - 2\sum_{n=2}^{\infty}[1-\beta+\alpha(n-1)(1+k)]A_n(\lambda,\gamma,m)|a_n|\right] \geq 0,$$

the required condition (13) is satisfied.

In the next theorem we obtain a necessary and sufficient condition for a function $f \in T$ to be in the class $TS_{\lambda}^{m}(\gamma,\alpha,k,\beta)$. □

Theorem 2. *Let* $f \in T$ *given by* (3). *Then* $f \in TS_{\lambda}^{m}(\gamma,\alpha,k,\beta)$ *if and only if*

$$\sum_{n=2}^{\infty}[1-\beta+\alpha(n-1)(1+k)]A_n(\lambda,\gamma,m)a_n \leq 1-\beta \tag{16}$$

where $A_n(\lambda,\gamma,m)$ *is defined by* (14). *The result is sharp.*

Proof. Assume that inequality (16) holds true. In virtue of Theorem 1 and the definition of $TS_{\lambda}^{m}(\gamma,\alpha,k,\beta)$. Choosing the values of z on the positive real axis the inequality (11) reduces to

$$\frac{1-\sum\limits_{n=2}^{\infty}[1+\alpha(n-1)]A_n(\lambda,\gamma,m)a_n z^{n-1}}{1-\sum\limits_{n=2}^{\infty}A_n(\lambda,\gamma,m)a_n z^{n-1}} - \beta > k\left|\frac{\sum\limits_{n=2}^{\infty}\alpha(n-1)A_n(\lambda,\gamma,m)a_n z^{n-1}}{1-\sum\limits_{n=2}^{\infty}A_n(\lambda,\gamma,m)a_n z^{n-1}}\right|. \tag{17}$$

Letting $z \to 1^-$, we obtain the desired inequality. Finally equality holds for the function f defined by

$$f(z) = z - \frac{1-\beta}{[1-\beta+\alpha(n-1)(1+k)]A_n(\lambda,\gamma,m)}z^n \ (n \geq 2). \tag{18}$$

□

Corollary 1. *If* $f \in TS_{\lambda}^{m}(\gamma,\alpha,k,\beta)$, *then*

$$a_n \leq \frac{1-\beta}{[1-\beta+\alpha(n-1)(1+k)]A_n(\lambda,\gamma,m)} \ (n \geq 2). \tag{19}$$

Equality is obtained for the function f *given by* (18).

3 Growth and Distortion theorem

Theorem 3. *Let* $f \in TS_{\lambda}^{m}(\gamma,\alpha,k,\beta)$. *Then for* $|z| = r < 1$

$$r - \frac{(1-\beta)}{B_2(\lambda,\gamma,m,\alpha,k,\beta)}r^2 \leq |f(z)| \leq r + \frac{(1-\beta)}{B_2(\lambda,\gamma,m,\alpha,k,\beta)}r^2 \tag{20}$$

and

$$1 - \frac{2(1-\beta)}{B_2(\lambda,\gamma,m,\alpha,k,\beta)}r^2 \leq |f'(z)| \leq 1 + \frac{2(1-\beta)}{B_2(\lambda,\gamma,m,\alpha,k,\beta)}r \tag{21}$$

where

$$B_n(\lambda,\gamma,m,\alpha,k,\beta)=[1-\beta+\alpha(n-1)(1+k)]A_n(\lambda,\gamma,m)\ (n\geq 2). \tag{22}$$

The inequalities (20) *and* (21) *are sharp for the function* f *given by*

$$f(z)=z-\frac{(1-\beta)}{B_2(\lambda,\gamma,m,\alpha,k,\beta)}z^2.$$

Proof. Since $f\in TS_\lambda^m(\gamma,\alpha,k,\beta)$ and from Theorem 2, it follows $\sum_{n=2}^{\infty}B_n(\lambda,\gamma,m,\alpha,k,\beta)a_n\leq(1-\beta)$, where $B_n(\lambda,\gamma,m,\alpha,k,\beta)$ is given by (22), we have

$$\begin{aligned}B_2(\lambda,\gamma,m,\alpha,k,\beta)\sum_{n=2}^{\infty}a_n=&\sum_{n=2}^{\infty}B_2(\lambda,\gamma,m,\alpha,k,\beta)a_n\\ &\leq\sum_{n=2}^{\infty}B_n(\lambda,\gamma,m,\alpha,k,\beta)a_n\\ &\leq\ 1-\beta\end{aligned}$$

and therefore

$$\sum_{n=2}^{\infty}a_n\leq\frac{(1-\beta)}{B_2(\lambda,\gamma,m,\alpha,k,\beta)}. \tag{23}$$

Since f is given by (3), we obtain

$$\begin{aligned}|f(z)|&\leq|z|+|z|^2\sum_{n=2}^{\infty}a_n|z|^{n-2}\\ &\leq r+r^2\sum_{n=2}^{\infty}a_n\\ &\leq r+\frac{(1-\beta)}{B_2(\lambda,\gamma,m,\alpha,k,\beta)}r^2\\ \text{and }|f(z)|&\geq|z|-|z|^2\sum_{n=2}^{\infty}a_n|z|^{n-2}\\ &\geq r-r^2\sum_{n=2}^{\infty}a_n\\ &\geq r-\frac{(1-\beta)}{B_2(\lambda,\gamma,m,\alpha,k,\beta)}r^2.\end{aligned}$$

In view of Theorem (15), we also have

$$\begin{aligned}
\frac{B_2(\lambda,\gamma,m,\alpha,k,\beta)}{2}\sum_{n=2}^{\infty} na_n &= \sum_{n=2}^{\infty}\frac{B_2(\lambda,\gamma,m,\alpha,k,\beta)}{2}\; na_n \\
&\le \sum_{n=2}^{\infty}(B_n,\lambda,\gamma,m,\alpha,k,\beta)a_n \le (1-\beta) \\
\text{which yields } \sum_{n=2}^{\infty} na_n &\le \frac{2(1-\beta)}{B_2(\lambda,\gamma,m,\alpha,k,\beta)}. \\
\text{Thus, } |f'(z)| &\le 1+\sum_{n=2}^{\infty} na_n|z|^{n-1} \\
&\le 1+r\sum_{n=2}^{\infty} na_n \\
&\le 1+\frac{2(1-\beta)}{B_2(\lambda,\gamma,m,\alpha,k,\beta)}r \\
\text{and } |f'(z)| &\ge 1-\sum_{n=2}^{\infty} na_n|z|^{n-1} \\
&\ge 1-r\sum_{n=2}^{\infty} na_n \\
&\ge 1-\frac{2(1-\beta)}{B_2(\lambda,\gamma,m,\alpha,k,\beta)}r.
\end{aligned}$$

Now, the proof of our theorem is completed. □

4 Extreme points

Next, we examine the extreme points for the function class $TS_\lambda^m(\gamma,\alpha,k,\beta)$.

Theorem 4. *Let the functions* $f_1(z)=z$ *and*

$$f_n(z)=z-\frac{(1-\beta)}{B_n(\lambda,\gamma,m,\alpha,k,\beta)}z^n \tag{24}$$
$$(0\le\lambda\le 1,\ 0\le\gamma\le 1,\ m\in N,\ \alpha\ge 1,\ k\ge 0,\ 0\le\beta<1,\ n\ge 2).$$

Then $f\in TS_\lambda^m(\gamma,\alpha,k,\beta)$ *if and only if*

$$f(z)=\sum_{n=2}^{\infty}\lambda_n f_n(z)\ (z\in E),\ \textit{where } \lambda_n\ge 0\ (n\ge 1)\ \textit{and } \sum_{n=1}^{\infty}\lambda_n=1. \tag{25}$$

Proof. Assume that f can be written as in (25). Then

$$\begin{aligned} f(z) &= \lambda_1 z + \sum_{n=2}^{\infty} \lambda_n \Big[z - \frac{(1-\beta)}{B_n(\lambda,\gamma,m,\alpha,k,\beta)} z^n \Big] \\ &= z - \sum_{n=2}^{\infty} \lambda_n \frac{(1-\beta)}{B_n(\lambda,\gamma,m,\alpha,k,\beta)} z^n. \\ \text{since } & \sum_{n=2}^{\infty} B_n(\lambda,\gamma,m,\alpha,k,\beta)\lambda_n \frac{(1-\beta)}{B_n(\lambda,\gamma,m,\alpha,k,\beta)} \\ &= (1-\beta)\sum_{n=2}^{\infty}\lambda_n \\ &= (1-\beta)(1-\lambda_1) \le (1-\beta) \end{aligned}$$

it follows in virtue of Theorem 2 that $f \in TS_{\lambda}^{m}(\gamma,\alpha,k,\beta)$.
Conversely, suppose $f \in TS_{\lambda}^{m}(\gamma,\alpha,k,\beta)$ and consider

$$\lambda_n = \frac{B_n(\lambda,\gamma,m,\alpha,k,\beta)}{(1-\beta)} a_n, \; n \ge 2$$

$$\text{and } \lambda_1 = 1 - \sum_{n=2}^{\infty} \lambda_n.$$

$$\text{Then } f(z) = \sum_{n=1}^{\infty} \lambda_n f_n(z).$$

Hence the proof is completed. □

5 Radii of Starlikeness, Convexity and close to Convexity

We begin this section with the following Theorem.

Theorem 5. *Let the function f given by (3) be in the class $TS_{\lambda}^{m}(\gamma,\alpha,k,\beta)$. Then f is starlike of order ρ $(0 \le \rho < 1)$ in $|z| < r_1(\lambda,\gamma,m,\alpha,k,\beta)$, where*

$$r_1(\lambda,\gamma,m,\alpha,k,\beta) = \inf_{n\ge 2} \Big[\frac{(1-\rho)B_n(\lambda,\gamma,m,\alpha,k,\beta)}{(n-\rho)(1-\beta)} \Big]^{\frac{1}{n-1}}.$$

Proof. To prove the theorem we must show that $|\frac{zf'(z)}{f(z)} - 1| \le 1-\rho, \quad 0 \le \rho < 1$

for $z \in E$ with $|z| < r_1(\lambda, \gamma, m, \alpha, k, \beta)$. We have

$$|\frac{zf'(z)}{f(z)} - 1| = \left| \frac{-\sum_{n=2}^{\infty}(n-1)a_n z^{n-1}}{1 - \sum_{n=2}^{\infty} a_n z^{n-1}} \right|$$

$$\leq \frac{\sum_{n=2}^{\infty}(n-1)a_n |z|^{n-1}}{1 - \sum_{n=2}^{\infty} a_n |z|^{n-1}}.$$

$$\text{Thus} \quad \left| \frac{zf'(z)}{f(z)} - 1 \right| \leq 1 - \rho \text{ if } \sum_{n=2}^{\infty} \frac{(n-\rho)}{(1-\rho)} a_n |z|^{n-1} \leq 1. \tag{26}$$

In virtue of (16), we have $\frac{\sum_{n=2}^{\infty} B_n(\lambda,\gamma,m,\alpha,k,\beta)}{1-\beta} a_n \leq 1$.

Hence, the inequality (26) will be true if

$$\frac{(n-\rho)}{(1-\rho)} |z|^{n-1} \leq \frac{B_n(\lambda, \gamma, m, \alpha, k, \beta)}{1-\beta} \ (n \geq 2).$$

$$\text{or if } |z| \leq \left[\frac{(1-\rho)B_n(\lambda, \gamma, m, \alpha, k, \beta)}{(n-\rho)(1-\beta)} \right]^{\frac{1}{n-1}} \ (n \geq 2).$$

Thus the proof of the theorem is completed. □

Theorem 6. *Let the function* f *is given by* (3) *be in the class* $TS_\lambda^m(\gamma, \alpha, k, \beta)$. *Then* f *is convex of order* ρ $(0 \leq \rho < 1)$ *in* $|z| < r_2(\lambda, \gamma, m, \alpha, k, \beta)$, *where*

$$r_2(\lambda, \gamma, m, \alpha, k, \beta) = \inf_{n \geq 2} \left[\frac{(1-\rho)B_n(\lambda, \gamma, m, \alpha, k, \beta)}{n(n-\rho)(1-\beta)} \right]^{\frac{1}{n-1}}.$$

Theorem 7. *Let the function* f *given by* (3) *be in the class* $TS_\lambda^m(\gamma, \alpha, k, \beta)$. *Then* f *in close-to-convex of order* ρ $(0 \leq \rho < 1)$ *in* $|z| < r_3(\lambda, \gamma, m, \alpha, k, \beta)$, *where*

$$r_3(\lambda, \gamma, m, \alpha, k, \beta) = \inf_{n \geq 2} \left[\frac{(1-\rho)B_n(\lambda, \gamma, m, \alpha, k, \beta)}{n(1-\beta)} \right]^{\frac{1}{n-1}}.$$

Proof. The proof of Theorem 6 and Theorem 7 is analogous to that of Theorem 5, so we omit the details □

6 The Fekete-Szego problem for the function class $S_\lambda^m(\gamma, \alpha, k, \beta)$

In this section we obtain the Fekete-Szego inequality for the functions in the class $S_\lambda^m(\gamma, \alpha, k, \beta)$. In the order to prove our main result we need the following lemma.

Lemma 2. *[6] If $p(z) = 1 + c_1 z + c_2 z + c_3 z^2 + \cdots$ is an analytic function with positive real part in E then* $|c_2 - \nu c_1^2| = \begin{cases} -4\nu + 2 & \nu \leq 0 \\ 2, & 0 \leq \nu \leq 1 \\ 4\nu - 2, & \nu \geq 1 \end{cases}$

when $\nu < 0$ or $\nu > 1$ the inequality holds if and only if $p(z) = \frac{1+z}{1-z}$ or one of its rotations. If $0 < \nu < 1$ then the equality holds if and only if

$$p(z) = \frac{1+z^2}{1-z^2}$$

or one of rotations. If $\nu = 0$ the equality holds if and only if

$$p(z) = \Big(\frac{1+\delta}{2}\Big)\frac{1+z}{1-z} + \Big(\frac{1-\delta}{2}\Big)\frac{1-z}{1+z} \quad (0 \leq \delta \leq 1) \text{ or one of its rotations.}$$

If $\nu = 1$, the equality holds if and only if $p(z)$ is the reciprocal of one of the functions such that the equality holds in the case of $\nu = 0$.

Theorem 8. *Let $\alpha \geq 1$, $0 \leq k \leq \beta < 1$. If $f \in S_\lambda^m(\gamma, \alpha, k, \beta)$ is given by (1) then*

$$|a_3 - \mu a_2^2| = \begin{cases} \frac{(1-\beta)}{\alpha^2(1-k)^2 A_3(\lambda,\gamma,m)}\Big[\alpha(1-k) + 2(1-\beta) - 4\mu(1-\beta)\frac{A_3(\lambda,\gamma,m)}{A_2^2(\lambda,\gamma,m)}\Big], & \mu \leq \sigma_1 \\ \frac{(1-\beta)}{\alpha(1-k)A_3(\lambda,\gamma,m)}, & \sigma_1 \leq \mu \leq \sigma_2 \\ \frac{-(1-\beta)}{\alpha^2(1-k)^2 A_3(\lambda,\gamma,m)}\Big[\alpha(1-k) + 2(1-\beta) - 4\mu(1-\beta)\frac{A_3(\lambda,\gamma,m)}{A_2^2(\lambda,\gamma,m)}\Big], & \mu \geq \sigma_2, \end{cases}$$

where

$$\sigma_1 = \frac{A_2^2(\lambda,\gamma,m)}{2A_3(\lambda,\gamma,m)} \text{ and } \sigma_2 = \frac{A_2^2(\lambda,\gamma,m)[1-\beta+\alpha(1-k)]}{2A_3(\lambda,\gamma,m)(1-\beta)}.$$

The result is sharp.

Proof. Since $\mathbb{R}(w) \leq |w|$ for any complex numbers, $f \in S_\lambda^m(\gamma, \alpha, k, \beta)$ implies that

$$\mathbb{R}\ [\alpha\frac{zG'(z)}{G(z)} - (\alpha-1)] > k\mathbb{R}\Big[\alpha\frac{zG'(z)}{G(z)} - \alpha\Big] + \beta$$

$$\text{or that } \mathbb{R}\ \frac{zG'(z)}{G(z)} > \frac{\beta - 1 + \alpha(1-k)}{\alpha(1-k)}.$$

$$\text{Hence} \quad G \in S^*\Big(\frac{\beta - 1 + \alpha(1-k)}{\alpha(1-k)}\Big).$$

$$\text{Let} \quad p(z) = \frac{\frac{zG'(z)}{G(z)} - \frac{\beta-1+\alpha(1-k)}{\alpha(1-k)}}{\frac{1-\beta}{\alpha(1-k)}} = 1 + c_1 z + c_2 z^2 + \cdots.$$

Then by virtue of (10) and (12), we have

$$a_2 = \frac{(1-\beta)}{\alpha(1-k)A_2(\lambda,\gamma,m)}c_1 \text{ and } a_3 = \frac{(1-\beta)}{2\alpha(1-k)A_2(\lambda,\gamma,m)}\Big[c_2 + \frac{1-\beta}{\alpha(1-k)}c_1^2\Big].$$

Therefore we obtain

$$a_3 - \mu a_2^2 = \frac{(1-\beta)}{2\alpha(1-k)A_3(\lambda,\gamma,m)}[c_2 - \frac{1-\beta}{\alpha(1-k)}c_1^2] - \mu\frac{(1-\beta)^2}{\alpha^2(1-k)^2A_2^2(\lambda,\gamma,m)}c_1^2$$

$$= \frac{(1-\beta)}{2\alpha(1-k)A_3(\lambda,\gamma,m)}\Big[c_2 - \frac{1-\beta}{\alpha(1-k)}c_1^2\Big(2\mu\frac{A_3\big(\lambda,\gamma,m\big)}{A_1^2(\lambda,\gamma,m)} - 1\Big)\Big].$$

We write

$$a_3 - \mu a_2^2 = \frac{(1-\beta)}{2\alpha(1-k)A_3(\lambda,\gamma,m)}(c_2 - \rho c_1^2),$$

where

$$\rho = \frac{(1-\beta)}{\alpha(1-k)}\Big[2\mu\frac{A_3(\lambda,\gamma,m)}{A_2^2(\lambda,\gamma,m)} - 1\Big].$$

Our result follows by the application of the above lemma. Denote

$$\xi = \frac{\beta - 1 + \alpha(1-k)}{\alpha(1-k)}.$$

If $\mu < \sigma_1$ or $\mu > \sigma_2$, then the equality holds true if and only if

$$G(z) = \frac{z}{(1-e^{i\theta}z)^{2(1-\xi)}} \ (\theta \in R).$$

When $\sigma_1 < \mu < \sigma_2$, the equality holds true if and only if

$$G(z) = \frac{z}{(1-e^{i\theta}z^2)^{(1-\xi)}} \ (\theta \in R).$$

If $\mu = \sigma_1$ then the equality holds true if and only if

$$G(z) = \Big[\frac{z}{(1-e^{i\theta}z)^{2(1-\xi)}}\Big]^{\frac{1+\delta}{2}}\Big[\frac{z}{(1+e^{i\theta}z)^{2(1-\xi)}}\Big]^{\frac{1-\delta}{2}}$$

$$= \frac{z}{[(1-e^{i\theta}z)^{1+\delta}(1+e^{i\theta}z)^{1-\delta}]^{1-\xi}}, \quad (0 \le \delta \le 1,\ \theta \in R).$$

Finally, when $\mu = \sigma_2$, the equality holds true if and only if $p(z)$ is the reciprocal of one of the functions such that equality and holds true in the case of $\mu = \sigma_2$. □

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