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ON WARPED PRODUCT SEMI INVARIANT SUBMANIFOLDS OF NEARLY (ε, δ)-TRANS SASAKIAN MANIFOLD

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Abstract

In this paper, we have concentrated on the inquest of warped product semi-invariant submanifolds of a nearly (ε, δ)-trans-Sasakian manifold. Firstly, some properties of this structure are acquired. Further, we established the warped product of the type $E_\perp \times_y E_P$ is a usual Riemannian product of $E_\perp$ and E_P , where $E_\perp$ and E_P are anti-invariant and invariant submanifolds of a nearly (ε, δ)-trans-Sasakian manifold $\bar{M}$, respectively. Also, we explored the warped product of the type $E_P \times_y E_\perp$ and acquired a depiction for such type of warped product.

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1 Introduction

Bishop and Neill [10] in 1969 premeditated the concept of warped product manifolds. After that several papers appeared which dealt with various geometric aspects of warped product submanifolds [1, 4, 5, 9, 10]. Chen initiated the notion of warped product CR submanifolds and established there exists no warped product CR-submanifolds of the form $M = E_\perp \times_y E_P$ such that $E_\perp$ is a real submanifold and E_P is a holomorphic submanifold of a Kaehler manifold $\bar{M}$ so he

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termed it as warped product CR submanifolds in the form $M = E_P \times_y E_\perp$ where E_P and $E_\perp$ are holomorphic and totally real submanifolds of a Kaehler manifold $\bar{M}$ [6, 7]. In [13], some kinds of warped products were studied. Bejancu and Duggal [2] also used the idea of (ε)-Sasakian manifolds. Xufeng and Xiaoli premeditated that these manifolds are real hypersurfaces of indefinite Kahlerian manifolds [14]. Kumar et al. in [11] also premeditated the curvature conditions of these manifolds and Tripathi et al. in [13] investigated (ε)-almost para contact manifolds. De and Sarkar in [8] also initiated (ε)-Kenmotsu manifolds and premeditated conformally flat, Weyl semisymmetric, ϕ-recurrent (ε)-Kenmotsu manifolds. In [12], the authors initiated and premeditated CR submanifolds and CR structure of a CR-submanifold of nearly (ε, δ)- trans-Sasakian structures and, thus, those of Sasakian manifolds. In [2, 14, 16], some properties of semi-invariant submanifolds were studied.

The aim of the paper is to inquest the concept of warped product semi-invariant submanifolds of a nearly (ε, δ)-trans-Sasakian manifold. We have shown that the warped product in the form $M = E_\perp \times_y E_P$ is simply Riemannian product of $E_\perp$ and E_P where $E_\perp$ is an anti-invariant submanifold and E_P is an invariant submanifold of a nearly (ε, δ)-trans-Sasakian manifold $\bar{M}$. Thus we deliberate the warped product submanifold of the type $M = E_P \times_y E_\perp$ by transposing the two factors $E_\perp$ and E_P that will simply be called warped product semi-invariant submanifold. Thus, we deduce the integrability of the involved distributions in the warped product and acquire a depiction result.

2 Preliminaries

If $\bar{M}$ is an n-dimensional almost contact metric manifold with structure tensors (f, ξ, η, g) where f is a $(1,1)$ type tensor field, ξ is a vector field, η is dual of ξ and g is also Riemannian metric tensor on $\bar{M}$, then

$$f^2U = -U + \eta(U)\xi, \quad \eta(\xi) = 1, \quad f\xi = 0, \quad \eta(fU) = 0, \quad g(\xi, \xi) = \varepsilon \qquad (1)$$

and

$$\eta(U) = \varepsilon g(U, \xi), \quad g(fU, fV) = g(U, V) - \varepsilon\eta(U)\eta(V) \qquad (2)$$

where $\varepsilon = g(\xi, \xi) = \pm 1$, for any vector fields U, V on $\bar{M}$, then $\bar{M}$ is called (ε)-almost contact metric manifold. An (ε)-almost contact metric manifold is called (ε, δ)-trans-Sasakian manifold if

$$(\bar{\nabla}_U f)V = \alpha\{g(U, V)\xi - \varepsilon\eta(V)U\} + \beta\{g(fU, V)\xi - \delta\eta(V)fU\} \qquad (3)$$

$$\bar{\nabla}_U \xi = -\varepsilon\alpha fU - \beta\delta f^2U \qquad (4)$$

holds for some smooth functions α and β on $\bar{M}$ and $\varepsilon = \pm 1, \delta = \pm 1$. Further, an (ε)-almost contact metric manifold is called a nearly (ε, δ)-trans-Sasakian manifold if

$$(\bar{\nabla}_U f)V + (\bar{\nabla}_V f)U = \alpha\{2g(U, V)\xi - \varepsilon\eta(V)U - \varepsilon\eta(U)V\}$$

$$-\beta\delta\{\eta(V)fU + \eta(U)fV\} \tag{5}$$

The covariant derivative of the tensor filed f is defined as

$$(\bar{\nabla}_U \phi)V = \bar{\nabla}_U fV - f\bar{\nabla}_U V \tag{6}$$

for all $U, V \epsilon P\bar{M}$.

If M is a submanifold immersed in $\bar{M}$ and deliberate the induced metric on M also denoted by g, then the Gauss and Weingarten formulas for a warped product semi-invariant submanifolds of a nearly (ε, δ)-trans-Sasakian manifold are given by

$$\bar{\nabla}_U V = \nabla_U V + h(U, V) \tag{7}$$

$$\bar{\nabla}_U N = -A_N U + \nabla^{\perp}_U N \tag{8}$$

for any U, V in PM and N in $P^{\perp}M$, where PM is the Lie algebras of vector fields in M and $P^{\perp}M$ is the set of all vector fields normal to M. $\nabla^{\perp}$ is the connection on the normal bundle, h is the second fundamental form and A_N is the Weingarten map associated with N as,

$$g(A_N U, V) = g(h(U, V), N). \tag{9}$$

For any $U \epsilon PM$, we write

$$fU = PU + SU \tag{10}$$

where PU is the tangential component and SU is the normal component of fU. Similarly for any $N \epsilon P^{\perp}M$, we write

$$fN = BN + KN \tag{11}$$

where BN is the tangential component and KN is the normal component of fN. The covariant derivatives of the tensor fields P and S are defined as

$$(\nabla_U P)V = \nabla_U PV - P\nabla_U V \tag{12}$$

$$(\nabla_U S)V = \nabla^{\perp}_U SV - S\nabla_U V \tag{13}$$

for all $U, V \epsilon PM$. If M is a Riemannian manifold isometrically immersed in an almost contact metric manifold M, then for every $u \epsilon M$ there exist a maximal invariant subspace denoted by D_u of the tangent space $T_u M$ of M. If the dimension of D_u is the same for all values of $u \epsilon M$, then D_u gives an invariant distribution D on M.

A submanifold M of an almost contact metric manifold $\bar{M}$ with $\xi \epsilon PM$ is called a semi-invariant submanifold of $\bar{M}$ if there exists two differentiable distributions D and $D^{\perp}$ on M such that

(i) $PM = D \oplus D^{\perp} \oplus \langle \xi \rangle$,

(ii) $f(D_u) \subseteq D_u$

(iii) $f(D^{\perp}_u) \subset T^{\perp}_u M$.

for any $u \epsilon M$, where $P^{\perp}_u M$ denotes the orthogonal space of $P_u M$ in $P_u \bar{M}$. A

semi-invariant submanifold is called anti-invariant if $D_u = \{0\}$ and invariant if $D_u^{\perp} = \{0\}$, respectively, for any $u \epsilon M$. It is called the proper semi-invariant submanifold if neither $D_u = \{0\}$ nor $D_u^{\perp} = \{0\}$, for every $u \epsilon M$.
If M is a semi-invariant submanifold of an almost contact metric manifold $\bar{M}$, then, $S(P_u M)$ is a subspace of $P_u^{\perp} M$. Then for every $u \epsilon M$, there exists an invariant subspace x_u of $P_u \bar{M}$ such that

$$P_u^{\perp} M = S(P_u M) \oplus x_u \tag{14}$$

A semi-invariant submanifold M of an almost contact metric manifold $\bar{M}$ is called Riemannian product if the invariant distribution D and anti-invariant distribution $D^{\perp}$ are totally geodesic distributions in M.
If (E, g_E) and (F, g_F) are two Riemannian manifolds and y is a positive differentiable function on E, then the warped product of E and F is the Riemannian manifold $E \times_y F = (E \times F, g)$, where

$$g = g_E + y^2 g_F \tag{15}$$

A warped product manifold $E \times_y F$ is called trivial if the warping function y is constant. We recall.

Lemma 1. *If $M = E \times_y F$ is a warped product manifold with the warping function y, then*
(i) $\nabla_U V \epsilon \Gamma(PE)$, *for each* $U, V \epsilon PE$,
(ii) $\nabla_U W = \nabla_W U = (U \ln y) W$, *for each* $U \epsilon PE$ *and* $W \epsilon PF$,
(iii) $\nabla_W X = \nabla_W^F X - g(W, X)/y) grady$,
where ∇ *and* ∇^F *denote the Levi-Civita connections on* M *and* F *respectively.*

In the above lemma $grady$ is the gradient of function y defined by $g(grady, X) = Xy$, for each $X \epsilon PM$. From Lemma 1, the warped product manifold $M = E \times_y F$ are in the form
(i) E is totally geodesic in M;
(ii) F is totally geodesic in M;
Now, we denote by $\rho_U V$ and $Q_U V$ the tangential and normal parts of $(\bar{\nabla}_U f)V$, that is,

$$(\bar{\nabla}_U f)V = \rho_U V + Q_U V \tag{16}$$

for all $U, V \epsilon PM$. Making use of (7), (8), and (10) (2.13), the above equation yields,

$$\rho_U V = (\nabla_U P)V - A_{SV} U - Bh(U, V) \tag{17}$$

$$Q_U V = (\bar{\nabla}_U S)V + h(U, PV) - Kh(U, V) \tag{18}$$

It is quite simple to check the following properties of ρ and Q, which we write here for later use:

$$p_1(i) \quad \rho_{U+V} X = \rho_U X + \rho_V X \quad (ii) \quad Q_{U+V} X = Q_U X + Q_V X$$

$$p_2(i) \quad \rho_U(V+X) = \rho_U V + \rho_U X \quad (ii) \quad Q_U(V+X) = Q_U V + Q_U X$$

$$p_3(i) \quad g(\rho_U V, X) = -g(V, \rho_U X)$$

for all $U, V, X \epsilon PM$. On a submanifold M of a nearly (ε, δ)-trans-Sasakian manifold $\bar{M}$, we deduce from (6) and (16) that

$$(i) \quad \rho_U V + \rho_V U = \alpha\{2g(U,V)\xi - \varepsilon\eta(V)U - \varepsilon\eta(U)V\} \tag{19}$$

$$-\beta\delta\{\eta(V)PU + \eta(U)PV\}$$

$$(ii) \quad Q_U V + Q_V U = -\beta\delta\{\eta(V)SU + \eta(V)SU\}$$

for any $U, V \epsilon PM$.

3 Warped product semi-invariant submanifolds of nearly (ε, δ)-trans-Sasakian manifold

In this section we establish the warped product $M = E \times_y F$ is trivial when ξ is tangent to F, where E and F are the Riemannian submanifolds of a nearly (ε, δ)-trans-Sasakian manifold $\bar{M}$. Thus, we deliberate the warped product $M = E \times_y F$, when ξ is tangent to the submanifold E. We have the following non-existence theorem.

Theorem 1. *If $M = E \times_y F$ is a warped product semi invariant submanifold of a nearly (ε, δ)-trans-Sasakian manifold M such that E and F are the Riemannian submanifolds of $\bar{M}$ then M is a usual Riemannian product if the structure vector field ξ is tangent to F.*

Proof. Consider any $U \epsilon PE$ and ξ tangent to F, then we have

$$\bar{\nabla}_U \xi = \nabla_U \xi + h(U, \xi) \tag{20}$$

From (4) and Lemma 1 (ii), we have

$$-\varepsilon\alpha fU + \beta\delta U - \beta\delta\eta(U)\xi = (Ulny)\xi + h(U,\xi) \tag{21}$$

The tangential component of (21), we conclude that

$$(Ulny)\xi = -\varepsilon\alpha PU + \beta\delta U - \beta\delta\eta(U)\xi,$$

for all $U \epsilon PE$, that is, y is constant function on E. Thus, M is the Riemannian product.

□

Now, we will explore the other case of warped product $M = E\times_y F$ when $\xi \epsilon PE$, where E and F are the Riemannian submanifolds of $\bar{M}$. For any $U \epsilon PF$, we have

$$\bar{\nabla}_U \xi = \nabla_U \xi + h(U, \xi) \tag{22}$$

From (4) and Lemma 1 (ii), we get

$$(i) \quad \xi lny = -\varepsilon\alpha P - \beta\delta P^2, \quad (ii) \quad h(U,\xi) = \varepsilon\alpha SU - \beta\delta S^2 U \tag{23}$$

Here there are two subcases such as :

$$(i) \quad M = E_\perp \times_y E_P$$

$$(ii) \quad M = E_P \times_y E_\perp$$

where E_P and $E_\perp$ are invariant and anti-invariant submanifolds of M, respectively. In the following theorem we prove that the warped product semi-invariant submanifold of the type (i) is CR-product.

Theorem 2. *If $M = E_\perp \times_y E_P$ is a warped product semi invariant submanifold of a nearly (ε, δ)-trans-Sasakian manifold M such that $E_\perp$ is a anti-invariant and E_P is a invariant submanifolds of $\bar{M}$, then M is a usual Riemannian product.*

Proof. When $\xi \epsilon PE_P$, then by Theorem 1, M is a Riemannian product. Thus, we consider $\xi \epsilon PE_\perp$. Consider $U \epsilon PE_P$ and $W \epsilon PE_\perp$, then we have

$$g(h(U, fU), SW) = g(h(U, fU), fW) = g(\bar{\nabla}_U fU, fW)$$

$$g(h(U, fU), SW) = g(f\bar{\nabla}_U U, fW) + g((\bar{\nabla}_U f)U, fW) \tag{24}$$

From the structure equation of nearly (ε, δ)-trans-Sasakian manifold, the second term of right hand side vanishes identically. Thus from (2), we derive

$$g(h(U, fU), SW) = -g(U, \bar{\nabla}_U W) + \varepsilon\eta(W)g(U, \bar{\nabla}_U \xi)$$

$$-\alpha\varepsilon\eta(U)g(U, fW) - \beta\delta\eta(U)g(fU, fW) \tag{25}$$

Using then from (7), Lemma 1 (ii) , and (4), we obtain

$$g(h(U, \phi U), SW) = (\beta\delta\varepsilon\eta(W) - Wlny)||U||^2 - \beta\delta\varepsilon\eta(U)g(U, W) \tag{26}$$

Replacing U by fU in (26) and by use of the fact that $\xi \epsilon PE_\perp$, we obtain

$$g(h(U, fU), SW) = (\beta\delta\varepsilon\eta(W) - Wlny)||U||^2 \tag{27}$$

It follows from (26) and (27) that $Wlny = 0$, for all $W \epsilon PE_\perp$. Also, from (23) we have $\xi lny = -\varepsilon\alpha P - \beta\delta P^2$.
From the above theorem we have seen that the warped product of the type $M = E_\perp \times_y E_P$ is a usual Riemannian product of an anti-invariant submanifold $E_\perp$ and an invariant submanifold E_P of a nearly (ε, δ)-trans-Sasakian manifold $\bar{M}$. Since both $E_\perp$ and E_P are totally geodesic in M , then M is CR-product. Now, we study the warped product semi-invariant submanifold $M = E_\perp \times_y E_P$ of a nearly (ε, δ)-trans-Sasakian manifold $\bar{M}$. □

Theorem 3. *If $M = E_P \times_y E_\perp$ is a warped product semi-invariant submanifold of a nearly (ε,δ)-trans-Sasakian manifold $\bar{M}$, then the invariant distribution D and the anti-invariant distribution $D^\perp$ are always integrable.*

Proof. Consider $U, V \epsilon D$, then we have

$$S[U,V] = S\nabla_U V - S\nabla_V U \tag{28}$$

From (13), we have

$$S[U,V] = (\bar{\nabla}_U S)V - (\bar{\nabla}_V S)U \tag{29}$$

Using (18), we get

$$S[U,V] = Q_U V - h(U,PV) + Kh(U,V) - Q_V U + h(V,PU) - Kh(U,V) \tag{30}$$

Then from (19) (ii), we derive

$$S[U,V] = 2Q_U V + h(V,PU) - h(U,PV) + \beta\delta\{\eta(V)SU + \eta(U)SV\} \tag{31}$$

Now, analyse $U, V \epsilon D$, then we have

$$h(U,PV) + \nabla_U PV = \bar{\nabla}_U PV = \bar{\nabla}_U fV \tag{32}$$

By means of the covariant derivative property of $\bar{\nabla} f$, we acquire

$$h(U,PV) + \nabla_U PV = (\bar{\nabla}_U f)V + f\bar{\nabla}_U V \tag{33}$$

From (7) and (16), we have

$$h(U,PV) + \nabla_U PV = \rho_U V + Q_U V + f(\nabla_U V + h(U,V)) \tag{34}$$

Since E_P is totally geodesic in M see Lemma 1 (i), then from (10) and (11), we get

$$h(U,PV) + \nabla_U PV = \rho_U V + Q_U V + P\nabla_U V + Bh(U,V) + Kh(U,V) \tag{35}$$

Equating normal parts, we get

$$h(U,PV) = Q_U V + Kh(U,V) \tag{36}$$

Similarly,

$$h(V,PU) = Q_V U + Kh(U,V) \tag{37}$$

Using (36) and (38), we get

$$h(V,PU) - h(U,PV) = Q_U V - Q_V U \tag{38}$$

In view of (19) (ii), we have

$$h(V,PU) - h(U,PV) = -2Q_U V - \beta\delta\{\eta(V)SU + \eta(U)SV\} \tag{39}$$

Then, it shows from (31) and (39) that $S[U,V]=0$, for all $U,V\epsilon D$. This establishes the integrability of D. Now, for the integrability of $D^{\perp}$, we deliberate any $U\epsilon D$ and $W,X\epsilon D^{\perp}$, and we have

$$\begin{aligned} g([W,X],U) &= g(\bar{\nabla}_W X - \bar{\nabla}_X W, U) \\ &= -g(\nabla_W U, X) + g(\nabla_X U, W) \end{aligned} \tag{40}$$

From Lemma 1 (ii), we acquire

$$g([W,X],U) = -(Ulny)g(W,X) + (Ulny)g(W,X) = 0 \tag{41}$$

Then from (41), we conclude that $[W,X]\epsilon D^{\perp}$, for each $W,X\epsilon D^{\perp}$. □

Lemma 2. *If a nearly (ε,δ)-trans-Sasakian manifold $\bar{M}$ admits a warped product semi invariant submanifold $M=E_P\times_y E_{\perp}$, then*

$$(i) g(\rho_U V, W) = g(h(U,V), SW) = 0$$

$$(ii) g(\rho_U W, X) = g(h(U,W), SX) - g(A_{SW}U, X)$$
$$= -(fUlny)g(W,X) - g(h(U,W),SX) + 2\alpha g(U,W)\eta(X) - \alpha\varepsilon g(U,X)\eta(W)$$
$$-\alpha\varepsilon g(W,X)\eta(U) - \beta\delta g(fU,X)\eta(W) - \beta\delta g(fW,X)\eta(U)$$
$$(iii) g(h(fU,W),SZ) = (Ulny)||W||^2 + 2\alpha g(fU,W)\eta(W) + \alpha\varepsilon\eta(W)g(fU,W)$$
$$-\beta\delta\eta(W)g(U,W) + \beta\delta\eta(U)\eta(W)\eta(W)$$

for all $U,V\epsilon PE_P$ and $W,X\epsilon PE_{\perp}$.

Proof. Assume that $M=E_P\times_y E_{\perp}$, is warped product submanifold of a nearly (ε,δ)-trans-Sasakian manifold $\bar{M}$ such that E_P is totally geodesic in M. Then using (12) and (17) we get

$$g(\rho_U V, W) = g(Bh(U,V), W) = g(h(U,V), SW) \tag{42}$$

for any $U,V\epsilon PE_P$. The left-hand side of (42) is skew symmetric in U and V whereas the right hand side is and symmetric in U and V, which gives (i). Next by using (12) and (17), we have

$$\rho_U W = -P\nabla_U W - A_{SW}U - Bh(U,W) \tag{43}$$

for any $U\epsilon PE_P$ and $W\epsilon PE_{\perp}$. In view of Lemma 1 (ii), the first term of right-hand side is zero. Thus, taking the product with $X\epsilon PE_{\perp}$, we obtain

$$g(\rho_U W, X) = -g(A_{SW}U, X) - g(Bh(U,W), X) \tag{44}$$

Using (2) and (9), we get

$$g(\rho_U W, X) = -g(h(U,X), SW) + g(h(U,W), SX) \tag{45}$$

which gives the first equality of (ii). Again, from (12) and (17), we have

$$\rho_W U = \nabla_W PU - T\nabla_W U - Bh(U,W) \tag{46}$$

Then from Lemma 1(ii), we deduce

$$\rho_W U = (PUlny)W - Bh(U,W) \tag{47}$$

Taking inner product with $X\epsilon PE_\perp$ and using (2), we acquire

$$g(\rho_W U, X) = (fUlny)g(W,X) + g(h(U,W),SX) \tag{48}$$

Using (19) (i), we get

$$g(\rho_W U, X) = -(\phi Ulny)g(W,X) - g(h(U,W),SX) + 2\alpha g(U,W)\eta(X)$$

$$-\alpha\varepsilon g(U,X)\eta(W) - \alpha\varepsilon g(W,X)\eta(U) - \beta\delta g(fU,X)\eta(W) - \beta\delta g(fW,X)\eta(U) \tag{49}$$

which gives the second equality of (ii). Now, from (43) and (47), we have

$$\rho_U W + \rho_W U = -P\nabla_U W - A_{SW}U + (PUlny)W - 2Bh(U,W) \tag{50}$$

Using (19) and Lemma 1 (i), we get left-hand side and the first term of right-hand side are zero. Thus the above equation takes the form

$$(PUlny)W = \alpha\{2g(U,W)\xi - \varepsilon\eta(W)U - \varepsilon\eta(U)W\}$$

$$-\beta\delta\{\eta(W)PU + \eta(U)PW\} + A_{SW}U + 2Bh(U,W) \tag{51}$$

Taking the product with W and using (2) and (9), we get

$$(\phi Ulny)||W||^2 = -g(h(U,W),SW) + (2-\varepsilon)\alpha g(U,W)\eta(W) - \alpha\varepsilon\eta(U)||W||^2$$

$$-\beta\delta\eta(W)g(fU,W) - \beta\delta\eta(U)g(fW,W) \tag{52}$$

Replacing U by fU and using (1), we acqire

$$\{-U + \eta(U)\xi\}lny||W||^2 = -g(h(fU,W),SW) + 2\alpha g(fU,W)\eta(W) \tag{53}$$

$$-\alpha\varepsilon\eta(W)g(fU,W) + \beta\delta\eta(W)g(U,W) - \beta\delta\eta(U)\eta(W)\eta(W)$$

Then from (23) (i), the above equation reduces to

$$g(h(fU,W),SW) = (Ulny)||W||^2 + 2\alpha g(fU,W)\eta(W) + \alpha\varepsilon\eta(W)g(fU,W)$$

$$-\beta\delta\eta(W)g(U,W) + \beta\delta\eta(U)\eta(W)\eta(W)$$

□

Theorem 4. *If M is a proper semi-invariant submanifold M of a nearly (ε,δ)-trans-Sasakian manifold $\bar{M}$, then M is locally a semi-invariant warped product if and only if some function μ on M satisfying $Y(\mu) = 0$ for each $Y\varepsilon D^\perp$, then*

$$A_{fW}U = -(fUlny)W + 2\alpha g(U,W)\xi - \alpha(2+\varepsilon)\eta(U)\eta(W)\xi$$

$$+\alpha\varepsilon\eta(W)U + \beta\delta\eta(W)fU \tag{54}$$

Proof. Direct part shows from Lemma 2 (iii). For the converse, assume that M is a semi-invariant submanifold of a nearly (ε, δ)-trans-Sasakian manifold $\bar{M}$ satisfying (54) then we have

$$h(U,V),fW) = g(A_{fW}U,V) = -(fU\mu)g(V,W) + 2\alpha\eta(V)g(U,W)$$
$$-\alpha(2+\varepsilon)\eta(U)\eta(V)\eta(W) + \alpha\varepsilon\eta(W)g(U,V) + \beta\delta\eta(W)g(fU,V) \qquad (55)$$

Now, from (7) and the property of covariant derivative of $\bar{\nabla}$, we have

$$h(U,V),fW) = g(\bar{\nabla}_U V, fW) = -g(f\bar{\nabla}_U V, W)$$
$$= -g(\bar{\nabla}_U fV, W) + g((\bar{\nabla}_U f)V, W) \qquad (56)$$

Using (7), (16), and (55), we get

$$g(\nabla_U PV, W) = g(\rho_U V, W) - 2\alpha\eta(V)g(U,W) + \alpha(2+\varepsilon)\eta(U)\eta(V)\eta(W)$$
$$-\alpha\varepsilon\eta(W)g(U,V) - \beta\delta\eta(W)g(fU,V) \qquad (57)$$

Using (12) and (17), we acquire

$$g(\nabla_U PV, W) = g(\nabla_U PV, W) - g(P\nabla_U V, W) - g(Bh(U,V), W) - 2\alpha\eta(V)g(U,W)$$
$$+\alpha(2+\varepsilon)\eta(U)\eta(V)\eta(W) - \alpha\varepsilon\eta(W)g(U,V) - \beta\delta\eta(W)g(fU,V) \qquad (58)$$

Then from (2), the above equation reduces to

$$g(T\nabla_U V, W) = g(h(U,V), fW) - 2\alpha\eta(V)g(U,W)$$
$$+\alpha(2+\varepsilon)\eta(U)\eta(V)\eta(W) - \alpha\varepsilon\eta(W)g(U,V) - \beta\delta\eta(W)g(fU,V) \qquad (59)$$

Hence using (9) and (54), we get

$$g(P\nabla_U V, W) = g(A_{fW}U, V) \qquad (60)$$

which indicates $\nabla_U V \epsilon D \oplus \{\xi\}$, that is, $D \oplus \{\xi\}$ is integrable and its leaves are totally geodesic in M. Now, for any $W, X \epsilon D^{\perp}$ and $U \epsilon D \oplus \{\xi\}$, we have

$$g(\nabla_W X, fU) = g(\bar{\nabla}_W X, fU) = -g(f\bar{\nabla}_W X, U)$$
$$= g((\bar{\nabla}_W f)X, U) - g(\bar{\nabla}_W fX, U) \qquad (61)$$

Using (8) and (16), we acquire

$$g(\nabla_W X, fU) = g(\rho_W X, U) + g(A_{fX}W, U) \qquad (62)$$

Then from (9) and the property p_3, we arrive at

$$g(\nabla_W X, fU) = -g(X, \rho_W U) + g(h(W,U), fX) \qquad (63)$$

Again from (9) and (19) (i), we get

$$g(\nabla_W X, fU) = g(\rho_U W, X) - 2\alpha g(U,W)\eta(X) + \alpha\varepsilon\eta(W)g(U,X)$$

$$+\alpha\varepsilon\eta(U)g(W,X)+\beta\delta\eta(W)g(PU,X)+\beta\delta\eta(U)g(PW,X)+g(A_{fX}U,W) \quad (64)$$

On the other hand, from (12) and (17), we get

$$\rho_U W=-P\nabla_U W-A_{SW}U-Bh(U,W) \quad (65)$$

Taking the product with $X\epsilon D^{\perp}$ and using (54), we acquire

$$g(\rho_U W,X)=-g(P\nabla_U W,X)+(fU\mu)g(W,X)+\alpha(2+\varepsilon)\eta(U)\eta(W)\eta(X)$$

$$-\beta\delta\eta(W)g(fU,X)-2\alpha g(U,W)\eta(X)-\alpha\varepsilon\eta(W)g(U,X)+g(A_{fX}U,W) \quad (66)$$

The first term of right-hand side of the above equation is zero using the fact that $PX=0$, for any $X\epsilon D^{\perp}$. Again using (9), we get

$$g(\rho_U W,X)=(fU\mu)g(W,X)+\alpha(2+\varepsilon)\eta(U)\eta(W)\eta(X)$$

$$-2\alpha g(U,W)\eta(X)-\alpha\varepsilon\eta(W)g(U,X)+g(A_{fX}U,W) \quad (67)$$

Then from (54), we deduce

$$g(\rho_U W,X)=0 \quad (68)$$

Using (54), (64), and (68), we get

$$g(\nabla_W X,fU)=3\alpha\varepsilon\eta(U)g(W,X)+3\beta\delta\eta(W)g(PU,X)$$

$$-\alpha(2+\varepsilon)\eta(U)\eta(X)\eta(W)-(fU\mu)g(X,W) \quad (69)$$

If $M^{\perp}$ is a leaf of $D^{\perp}$, and let $h^{\perp}$ be the second fundamental form of the immersion of $M^{\perp}$ into M, then for any $W,X\epsilon D^{\perp}$, we have

$$g(h^{\perp}(W,X),fU)=g(\nabla_W X,fU) \quad (70)$$

Thus, from (69) and (70), we conclude that

$$g(h^{\perp}(W,X),fU)=3\alpha\varepsilon\eta(U)g(W,X)+3\beta\delta\eta(W)g(PU,X)$$

$$-\alpha(2+\varepsilon)\eta(U)\eta(X)\eta(W)-(fU\mu)g(X,W) \quad (71)$$

The above relation shows that integral manifold $M_{\perp}$ of $D^{\perp}$ is totally umbilical in M . Since the anti-invariant distribution $D^{\perp}$ of a semi-invariant submanifold M is always integrable Theorem 3 and $Y\mu=0$ for each $Y\epsilon D^{\perp}$, which indicates that the integral manifold of $D^{\perp}$ is an extrinsic sphere in M ; that is, it is totally umbilical and its mean curvature vector field is nonzero and parallel along $M_{\perp}$. Hence by virtue of results acquired in [9] , M is locally a warped product $E_P\times_y E_{\perp}$, where E_P and $E_{\perp}$ denote the integral manifolds of the distributions $D\oplus\langle\xi\rangle$ and $D^{\perp}$, respectively and y is the warping function. □

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