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ON A TYPE OF GENERALIZED (k, μ)-PARACONTACT METRIC MANIFOLDS

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Abstract

In this paper we investigate generalized (k, μ)-paracontact metric manifolds satisfying the curvature conditions $R \cdot P = 0$ and $P \cdot S = 0$, where R, P and S are the Riemannian curvature tensor, the projective curvature tensor and the Ricci tensor, respectively. Next, we study ξ-projectively flat generalized (k, μ)-paracontact metric manifolds. Further, we study generalized (k, μ)-paracontact metric manifolds satisfying the curvature condition $P(X, Y) \cdot \phi = 0$. Finally, we have cited an example of a generalized (k, μ)-paracontact metric manifold.

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1 Introduction

The notion of paracontact geometry was introduced by Kaneyuki and Williams [10] in 1985. A systematic investigation on paracontact metric manifold done by Zamkovoy [16]. Recently, Cappelletti-Montano et al [3] introduced a new type of paracontact geomerty so-called paracontact metric (k, μ) space, where k and μ are constant. It is known [1] that in contact case $k \leq 1$, but in paracontact case there is no restriction for k.

The conformal curvature tensor C is invariant under conformal transformation and vanishes identically for 3-dimensional manifolds. Using this result, several authors studied different types of 3-dimensional manifolds [5, 6, 7, 15].

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A Riemannian manifold (M, g) is called locally symmetric if its curvature tensor R is parallel (that is $\nabla R = 0$). A Riemannian manifold is said to be semi-symmetric if its curvature tensor R satisfies the condition

$$R(X,Y) \cdot R = 0, \tag{1}$$

where R is the Riemannian curvature tensor and $R(X,Y)$ is considered as a derivation of the tensor algebra at each point of the manifold for tangent vector fields X, Y. A complete intrinsic classification of these manifolds was given by Szabo in [12].

Let us consider a $(2n+1)$-dimensional Riemannian manifold M. If there exists a one-to-one correspondence between each coordinate neighborhood of M and a domain in Euclidean space such that any geodesic of the Riemannian manifold corresponds to a straight line in the Euclidean space, then M is said to be locally projectively flat. For $n \geq 1$, M is locally projectively flat if and only if the well-known Weyl projective curvature tensor P vanishes. Here P is defined by [11]

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{2n}[S(Y,Z)X - S(X,Z)Y], \tag{2}$$

for all $X, Y, Z \in T_pM$, where R is the curvature tensor and S is the Ricci tensor of type $(0, 2)$ of M. In fact, M is Weyl projectively flat if and only if the manifold is of constant curvature [13]. Thus, the Weyl projective curvature tensor is the measure of the failure of a Riemannian manifold to be of constant curvature. A Riemannian manifold is said to be Weyl projective semisymmetric if the curvature tensor P satisfies $R(X,Y) \cdot P = 0$.

A Riemannian or, pseudo-Riemannian manifold M is said to be ϕ-projectively semisymmetric [14] if $P(X,Y) \cdot \phi = 0$. In [14], Yildiz and De studied ϕ-projectively semisymmetric non-Sasakian (k, μ)-contact metric manifolds.

A (k, μ)-paracontact metric manifold is called an Einstein manifold if the Ricci tensor satisfies the condition $S = \lambda g$, where λ is some constant.
Recently, 3-dimensional generalized (k, μ)-paracontact metric manifolds studied by Kupeli Erken et al [8, 9].

The paper is organized in the following way. In Section 2, we discuss about some basic results of paracontact metric manifolds. Further, we characterize generalized (k, μ)-paracontact metric manifolds satisfying the curvature conditions $R \cdot P = 0$ and $P \cdot S = 0$. In Section 5, we investigate ξ-projectively flat generalized (k, μ)-paracontact metric manifolds. In the next section, we study generalized (k, μ)-paracontact metric manifolds satisfying the curvature condition $P(X,Y) \cdot \phi = 0$. Finally, we give an example of a generalized (k, μ)-paracontact metric manifold.

2 Preliminaries

A $(2n+1)$-dimensional smooth manifold M is said to be an almost paracontact structure if it carries a (1,1)-tensor ϕ, a vector field ξ and a 1-form η satisfying

[10]:
(i) $\phi^2X = X - \eta(X)\xi$, for all $X \in \chi(M)$, $\eta(\xi) = 1$,
(ii) the tensor field ϕ induces an almost paracomplex structure on each fibre of $D = ker(\eta)$, that is, the eigendistributions D^+_ϕ and D^-_ϕ of ϕ corresponding to the eigenvalues 1 and -1, respectively, have equal dimension n.
From the above conditions it follows that $\phi(\xi) = 0$, $\eta \circ \phi = 0$.

An almost paracontact structure is said to be normal [10] if and only if the (1,2) type torsion tensor $N_\phi = [\phi, \phi] - 2d\eta \otimes \xi$ vanishes identically, where $[\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]$. If an almost paracontact manifold admits a pseudo-Riemannian metric g such that

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y), \tag{3}$$

for $X, Y \in \chi(M)$, then we say that (M, ϕ, ξ, η, g) is an almost paracontact metric manifold. Any such pseudo-Riemannian metric manifold is of signature (n+1,n). An almost paracontact structure is said to be a paracontact structure if $g(X, \phi Y) = d\eta(X, Y)$ [16]. In a paracontact metric manifold we define (1,1)-type tensor fields h by $h = \frac{1}{2}\pounds_\xi\phi$, where $\pounds_\xi\phi$ is the Lie derivative of ϕ along the vector field ξ. Then, we observe that h is symmetric and anti-commutes with ϕ. Also, h satisfies the following conditions [16]:

$$h\xi = 0,\ tr(h) = tr(\phi h) = 0, \tag{4}$$

$$\nabla_X\xi = -\phi X + \phi hX. \tag{5}$$

for all $X \in \chi(M)$, where ∇ denotes the Levi-Civita connection of the pseudo-Riemannian manifold.
Moreover, h vanishes identically if and only if ξ is a Killing vector field and then (M, ϕ, ξ, η, g) is said to be a K-paracontact manifold.

A generalized (k,μ)-paracontact metric manifold means a 3-dimensional paracontact metric manifold which satisfies the nullity condition

$$R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY). \tag{6}$$

In a generalized $(k \neq -1, \mu)$-paracontact manifold the following results hold [2, 8]:

$$h^2 = (1 + k)\phi^2, \tag{7}$$

$$\xi k = 0, \tag{8}$$

$$Q\xi = 2k\xi, \tag{9}$$

$$QX = (\frac{r}{2} - k)X + (-\frac{r}{2} + 3k)\eta(X)\xi + \mu hX, k \neq -1, \tag{10}$$

where X is any vector field on M, Q is the Ricci operator of M and r denotes the scalar curvature of M.

$$h\ grad\mu = grad\ k. \tag{11}$$

We recall the following lemmas:

Lemma 1. *[8] Let $M(\phi,\xi,\eta,g)$ be a generalized (k,μ)-paracontact metric manifold with $k>-1$ and $\xi\mu=0$. Then*

1. *at any point of M, precisely one of the following relations is valid: $\mu = 2(1+\sqrt{1+k})$, or $\mu = 2(1-\sqrt{1+k})$*
2. *at any point $P\in M$ there exists a chart $(U,(x,y,z))$ with $P\in U\subseteq M$, such that the functions k,μ depend only on the variable z.*

Lemma 2. *[8] Let $M(\phi,\xi,\eta,g)$ be a generalized (k,μ)-paracontact metric manifold. Then for any point $P\in M$, with $k>-1$ there exists a neighborhood U of P such that $hX=\lambda_1 X$, $\lambda_1=\sqrt{1+k}$ and for $k<-1$ there exists also a neighborhood U of P such that $hX=\lambda_2\phi X$, $\lambda_2=\sqrt{-1-k}$.*

3 Projective semisymmetric generalized (k,μ)-paracontact metric manifolds

In this section we study projective semisymmetric generalized (k,μ)-paracontact metric manifolds, that is the generalized (k,μ)-paracontact metric manifolds satisfying the curvature condition $R(X,Y)\cdot P=0$.
Thus, we have

$$\begin{aligned} &R(X,Y)P(U,V)W - P(R(X,Y)U,V)W \\ &-P(U,R(X,Y)V)W - P(U,V)R(X,Y)W = 0. \end{aligned} \tag{12}$$

Putting $X=U=W=\xi$ in (12), we obtain

$$\begin{aligned} R(\xi,Y)P(\xi,V)\xi - P(R(\xi,Y)\xi,V)\xi - P(\xi,R(\xi,Y)V)\xi \\ -P(\xi,V)R(\xi,Y)\xi = 0. \end{aligned} \tag{13}$$

Using (2) and (6) in (13), we have

$$\begin{aligned} k\mu\,[g(hY,V)\xi - \eta(V)hY] + \mu^2(k+1)[\eta(V)Y - \eta(V)\eta(Y)\xi] \\ +k^2[\eta(V)\eta(Y)\xi - \eta(V)Y + g(V,Y)\xi] - \frac{k}{2}S(V,Y)\xi - \frac{\mu}{2}S(V,hY)\xi = 0. \end{aligned} \tag{14}$$

Taking inner product of (14) with ξ, we get

$$\mu S(V,hY) + kS(Y,V) - 2k\mu g(hY,V) - 2k^2 g(Y,V) = 0. \tag{15}$$

Now replacing Y by hY in (15) yields

$$[k^2 - \mu^2(k+1)][S(Y,V) - 2kg(Y,V)] = 0. \tag{16}$$

Then $S(Y,V)-2kg(Y,V)=0$ or, $k^2-\mu^2(k+1)=0$.

Case 1: Let $S(Y,V)-2kg(Y,V)=0$, which is equivalent to $S(Y,V)=2kg(Y,V)$. Thus, the manifold is an Einstein manifold.

Case 2: Let $k^2-\mu^2(k+1)=0$. Then we get $(2k-\mu^2)(\xi k)-2\mu(k+1)(\xi\mu)=0$. Using (8) we have $\mu(k+1)(\xi\mu)=0$. Taking account of $\mu\neq 0$ and $k>-1$, we have $\xi\mu=0$. Using the above discussions and Lemma 1 we have the following:

Theorem 1. *If a generalized (k, μ)-paracontact metric manifold with k, μ non-zero functions and $k > -1$ is projective semisymmetric then one of the following conditions hold:*

1. *the manifold is an Einstein manifold.*
2. *at any point $P \in M$ there exists a chart $(U, (x, y, z))$ with $P \in U \subseteq M$, such that functions k, μ depend only on variable z and either $\mu = 2(1 + \sqrt{1+k})$, or $\mu = 2(1 - \sqrt{1+k})$ is valid.*

Also, it is clear that ([8], Corollary 4.1) any generalized (k, μ)-paracontact metric manifold with $k < -1$ and $\xi\mu = 0$ does not exist. Thus, we can state the following:

Theorem 2. *If a generalized (k, μ)-paracontact metric manifold with k, μ non-zero functions and $k < -1$ is projective semisymmetric, then the manifold is an Einstein manifold.*

4 Generalized (k, μ)-paracontact metric manifolds satisfying $P \cdot S = 0$

This section is devoted to studying generalized (k, μ)-paracontact metric manifolds satisfying condition $P \cdot S = 0$. From condition $P \cdot S = 0$, we have

$$S(P(X, Y)U, V) + S(U, P(X, Y)V) = 0. \tag{17}$$

Substituting $X = U = \xi$ in (17), we obtain

$$S(P(\xi, Y)\xi, V) + S(\xi, P(\xi, Y)V) = 0. \tag{18}$$

Now using (9) and (2) in (18), we get

$$S(-\mu hY, V) + 2k\eta(P(\xi, Y)V) = 0. \tag{19}$$

Using (2) in (19), yields

$$-\mu S(hY, V) + 2k^2 g(Y, V) + 2k\mu g(hY, V) - kS(Y, V) = 0. \tag{20}$$

Substituting hY for Y in (20), we have

$$-\mu(k + 1)S(Y, V) + 2k^2 g(hY, V) + 2k\mu(k + 1)g(Y, V) - kS(hY, V) \tag{21}$$

Now multiplying (20) by k and (21) by μ and subtracting the results, we obtain

$$[k^2 - \mu^2(k + 1)][S(Y, V) - 2kg(Y, V)] = 0. \tag{22}$$

Then $S(Y, V) - 2kg(Y, V) = 0$ or, $k^2 - \mu^2(k + 1) = 0$.

Case 1: Let $S(Y, V) - 2kg(Y, V) = 0$, which is equivalent to $S(Y, V) = 2kg(Y, V)$. Thus, the manifold is an Einstein manifold.

Case 2: Let $k^2 - \mu^2(k+1) = 0$. Then we get $(2k - \mu^2)(\xi k) - 2\mu(k+1)(\xi\mu) = 0$. Using (8) we have $\mu(k + 1)(\xi\mu) = 0$. Taking account of $\mu \neq 0$ and $k < -1$, we have $\xi\mu = 0$. Hence using Lemma 1 we have the following:

Theorem 3. *If a generalized (k,μ)-paracontact metric manifold with $k > -1$ satisfies the curvature condition $P \cdot S = 0$, then one of the following conditions hold:*

1. *the manifold is an Einstein manifold.*
2. *at any point $P \in M$ there exists a chart $(U,(x,y,z))$ with $P \in U \subseteq M$, such that functions k,μ depend only on variable z and either $\mu = 2(1+\sqrt{1+k})$, or $\mu = 2(1-\sqrt{1+k})$ is valid.*

Also, it is clear that ([8], Corollary 4.1) any generalized (k,μ)-paracontact metric manifold with $k < -1$ and $\xi\mu = 0$ does not exist. Then in this case we only have $S(Y,V) = 2kg(Y,V)$, that is, the manifold is an Einstein manifold.

Conversely, if the manifold is an Einstein manifold, then necessarily the curvature condition $P \cdot S = 0$ holds. Therefore, we have the following:

Theorem 4. *A generalized (k,μ)-paracontact metric manifold with k,μ non-zero functions and $k < -1$ is projective semisymmetric if and only if the manifold is an Einstein manifold.*

5 ξ-projectively flat generalized (k,μ)-paracontact metric manifolds

ξ-conformally flat K-contact manifolds have been studied by Zhen et al [17]. Since at each point p of the manifold M the tangent space $T_p(M)$ can be decomposed into the direct sum $T_p(M) = \phi(T_p(M)) \oplus \{\xi_p\}$, where $\{\xi_p\}$ is the one-dimensional linear subspace of $T_p(M)$ generated by ξ_p, the conformal curvature tensor C is a map

$$C : T_p(M) \times T_p(M) \times T_p(M) \to \phi(T_p(M)) \oplus \{\xi_p\}.$$

An almost contact metric manifold $M^n(n = 2m+1)$ is called ξ-conformally flat if the projection of the image of C onto $\{\xi_p\}$ is zero [17]. Analogously, we define ξ-projectively flat generalized (k,μ)-paracontact metric manifolds.

Definition 1. *A generalized (k,μ)-paracontact metric manifold is said to be ξ-projectively flat if $P(X,Y)\xi = 0$.*

In this section we consider ξ-projectively flat generalized (k,μ)-paracontact metric manifold. The projective curvature tensor P on a generalized $(k \neq -1,\mu)$-paracontact manifold defined by (11). Assuming $P(X,Y)\xi = 0$, we obtain

$$R(X,Y)\xi - \frac{1}{2}[S(Y,\xi)X - S(X,\xi)Y] = 0. \tag{23}$$

Making use of (6) and (9) in (23) gives

$$\mu[\eta(Y)hX - \eta(X)hY] = 0. \tag{24}$$

Putting hX for X in (24), yields

$$\mu(k+1)[X-\eta(X)\xi]=0. \tag{25}$$

Therefore, $\mu=0$, since $k\neq -1$. Using $\mu=0$ in (11) we have $gradk=0$. Hence, k is constant. Therefore, we have the following:

Theorem 5. *If a generalized (k,μ)-paracontact metric manifold is ξ-projectively flat, then the manifold reduces to a 3-dimensional $N(k)$-paracontact metric manifold.*

6 Generalized (k,μ)-paracontact metric manifolds satisfying $P(X,Y)\cdot\phi=0$

In this section we investigate generalized (k,μ)-paracontact metric manifolds satisfying the curvature condition $P(X,Y)\cdot\phi=0$. The above condition is equivalent to

$$P(X,Y)\phi Z-\phi(P(X,Y)Z)=0. \tag{26}$$

Substituting $X=\xi$ in (26), we get

$$P(\xi,Y)\phi Z-\phi(P(\xi,Y)Z)=0. \tag{27}$$

Using (11) in (27), yields

$$kg(Y,\phi Z)\xi+\mu g(hY,\phi Z)\xi-\frac{1}{2}S(Y,\phi Z)\xi+\mu\eta(Z)\phi hY=0. \tag{28}$$

Substituting ϕZ for Z in (28), we have

$$kg(Y,Z)\xi+\mu g(hY,Z)\xi-\frac{1}{2}S(Y,Z)\xi=0. \tag{29}$$

Taking inner product with ξ of the above equation, we obtain

$$S(Y,Z)=2kg(Y,Z)+2\mu g(hY,Z). \tag{30}$$

Now we consider the following cases.

Case 1: $k<-1$. Then from Lemma 2 and (30), we have

$$S(Y,Z)=2kg(Y,Z)+\mu\lambda_2 g(\phi Y,Z). \tag{31}$$

Interchanging Y and Z in (31) gives an equation, then add the resultant with (31), we obtain

$$S(Y,Z)=2kg(Y,Z). \tag{32}$$

Case 2: $k>-1$. Then from Lemma 2 and (30), we get

$$S(Y,Z)=(2k+\mu\lambda_1)g(Y,Z),$$

which shows that the manifold is an Einstein manifold. Combining the cases we can state the following:

Theorem 6. *If a generalized (k,μ)-paracontact metric manifold satisfies the curvature condition $P(X,Y)\cdot\phi=0$, then the manifold is an Einstein manifold.*

7 Example of a generalized (k,μ)-paracontact metric manifold

Example 1. *[4] Let $k : I \subset \mathbb{R} \to \mathbb{R}$ be a smooth function defined on an open interval I, such that $k(z) \geq -1$ for any $z \in I$. Then we construct a generalized (k,μ)-paracontact metric manifold $M(\phi,\xi,\eta,g)$ on the set $M = \mathbb{R}^2 \times I \subset \mathbb{R}^3$ as follows:*

We set $\sigma(z) = \sqrt{1+k(z)} \geq 0$, $\sigma'(z) = \frac{\partial\sigma}{\partial z}$ and the three linearly independent vector fields e_1, e_2 and e_3 are given as

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = \frac{\partial}{\partial y},$$
$$e_3 = 2y\frac{\partial}{\partial x} + [2\sigma(z)x - \frac{\sigma'(z)}{1+2\sigma(z)}y]\frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

Let g be the pseudo-Riemannian metric defined by

$$g(e_i,e_j) = \begin{cases} 1 & \text{if} \quad i=j\neq 3 \\ -1 & \text{if} \quad i=j=3 \\ 0 & \text{if} \quad i\neq j;\ i,j=1,2,3. \end{cases}$$

Let η be the 1-form defined by $\eta(Z) = g(Z,e_1)$ for any $Z \in T(M)$. Further, we consider ϕ be the $(1,1)$ tensor field defined by

$$\phi(e_1) = 0,\ \phi(e_2) = -e_3,\ \phi(e_3) = -e_2.$$

Let ∇ be the Levi-Civita connection with respect to the metric tensor g. Then we get

$$[e_1,e_2] = 0, \quad [e_1,e_3] = 2\sigma(z)e_2, \quad [e_2,e_3] = -\frac{\sigma'(z)}{1+2\sigma(z)}e_2 + 2e_1.$$

Then we have

$$\eta(e_1) = g(e_1,e_1) = 1, \quad \phi^2 Z = Z - \eta(Z)e_1, \quad g(\phi Z,\phi W) = -g(Z,W) + \eta(Z)\eta(W),$$

for any $W, Z \in T(M)$. Hence, (ϕ,ξ,η,g) defines a paracontact metric structure on M for $e_1 = \xi$.
The Levi-Civita connection ∇ of the metric g is given by the Koszul's formula

$$\begin{aligned} 2g(\nabla_X Y,Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ &\quad -g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]). \end{aligned}$$

Using the above formula we have

$$\nabla_{e_1}e_1 = 0, \quad \nabla_{e_1}e_2 = e_3 + \sigma(z)e_3, \quad \nabla_{e_1}e_3 = e_2 + \sigma(z)e_2, \nabla_{e_2}e_1 = e_3 + \sigma(z)e_3,$$
$$\nabla_{e_2}e_2 = -\frac{\sigma'(z)}{1+2\sigma(z)}e_3, \quad \nabla_{e_2}e_3 = e_1 + \sigma(z)e_1 - \frac{\sigma'(z)}{1+2\sigma(z)}e_2,$$
$$\nabla_{e_3}e_1 = e_2 - \sigma(z)e_2, \quad \nabla_{e_3}e_2 = -e_1 + \sigma(z)e_1, \quad \nabla_{e_3}e_3 = 0.$$

Comparing the above relations with $\nabla_X e_1 = -\phi X + \phi h X$ we get

$$he_1 = 0, \ \ he_2 = -\sigma(z)e_2, \ \ he_3 = \sigma(z)e_3.$$

Using the formula $R(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$, we calculate the following:

$$\begin{aligned} & R(e_2,e_1)e_1 = -\{1+\sigma(z)\}^2 e_2 \\ = \ & \{(\sigma(z))^2-1\}\{\eta(e_1)e_2-\eta(e_2)e_1\}+2\{1+\sigma(z)\}\{\eta(e_1)he_2-\eta(e_2)he_1\}, \end{aligned}$$

$$\begin{aligned} & R(e_3,e_1)e_1 = \{3(\sigma(z))^2+2\sigma(z)-1\}e_3 \\ = \ & \{(\sigma(z))^2-1\}\{\eta(e_1)e_3-\eta(e_3)e_1\}+2\{1+\sigma(z)\}\{\eta(e_1)he_3-\eta(e_3)he_1\}, \end{aligned}$$

$$\begin{aligned} & R(e_2,e_3)e_1 = 0 \\ = \ & \{(\sigma(z))^2-1\}\{\eta(e_3)e_2-\eta(e_2)e_3\}+2\{1+\sigma(z)\}\{\eta(e_3)he_2-\eta(e_2)he_3\}. \end{aligned}$$

By the above expressions of curvature tensor we conclude that M is a generalized (k,μ)-paracontact metric manifold with $k = \{(\sigma(z))^2-1\}$ and $\mu = 2\{1+\sigma(z)\}$.

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