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SOME RESULTS ON LORENTZIAN PARA-KENMOTSU MANIFOLDS

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Abstract

In the present paper, we define Lorentzian para-Kenmotsu manifolds and study Ricci-pseudosymmetric, Ricci-generalized pseudosymmetric and symmetric conditions to characterize Lorentzian para-Kenmotsu manifolds. Next, we study Lorentzian para-Kenmotsu manifolds satisfying the curvature condition $S \cdot R = 0$. Moreover, we study Ricci solitons on Lorentzian para-Kenmotsu manifolds. Finally, we give an example of a 5-dimensional Lorentzian para-Kenmotsu manifold to verify some results of the paper.

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1 Introduction

A Riemannian manifold is called semisymmetric if $R(X,Y) \cdot R = 0$ [14]. R. Deszcz [8] generalized the concept of semisymmetry and introduced pseudosymmetric manifolds. Let (M,g) be an n-dimensional $(n \geq 3)$ differentiable manifold of class C^∞. We denote by ∇, R, S, Q and r the Levi-Civita connection, the curvature tensor, the Ricci tensor, the Ricci operator and the scalar curvature of (M,g), respectively. We define endomorphism $X \wedge_A Y$ for an arbitrary vector field Z and $(0,k)$ tensor T, $k \geq 1$ by

$$(X \wedge_A Y)Z = A(Y,Z)X - A(X,Z)Y, \tag{1}$$

and

$$((X \wedge_A Y) \cdot T)(X_1, X_2, ...X_k) = -T((X \wedge_A Y)X_1, X_2.....X_k) \tag{2}$$

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$$-T(X_1,(X\wedge_A Y)X_2.....X_k)-.......-T(X_1,X_2.....(X\wedge_A Y)X_k),$$

respectively, where $X,Y,Z\in\chi(M)$; $\chi(M)$ being the Lie algebra of vector fields on M and A is the symmetric $(0,2)$-tensor. For a $(0,k)$-tensor field T, the $(0,k+2)$ tensor fields $R\cdot T$ and $Q(A,T)$ are defined by [3, 8]

$$(R(X,Y)\cdot T)(X_1,X_2,......X_k)=-T(R(X,Y)X_1,X_2.....X_k) \tag{3}$$

$$-T(X_1,R(X,Y)X_2.....X_k)-.......-T(X_1,X_2.....R(X,Y)X_k),$$

and

$$Q(A,T)(X_1,X_2,......X_k;X,Y)=-T((X\wedge_A Y)X_1,X_2.....X_k) \tag{4}$$

$$-T(X_1,(X\wedge_A Y)X_2.....X_k)-.......-T(X_1,X_2.....(X\wedge_A Y)X_k),$$

respectively.

By setting $T=R$ or $T=S$, $A=g$ or $A=S$ in the above formulas, we get the tensors $R\cdot R$, $R\cdot S$, $Q(g,S)$ and $Q(S,R)$.

A Riemannian manifold M is said to be Ricci-generalized pseudosymmetric if the tensors $R\cdot R$ and $Q(S,R)$ are linearly dependent at every point of M, i.e.,

$$R\cdot R=L_R Q(S,R). \tag{5}$$

This is equivalent to

$$(R(X,Y)\cdot R)(U,V,W)=L_R[((X\wedge_S Y)\cdot R)(U,V,W)] \tag{6}$$

holding on the set $U_R=\{x\in M: Q(S,R)\neq 0 \;\; at \;\; x\}$, where L_R is some function on U_R [8]. Particularly, if $L_R=0$, then M is a semisymmetric manifold. The manifold is said to be locally symmetric if $\nabla R=0$. Clearly, locally symmetric spaces are semisymmetric.

If the tensors $R\cdot S$ and $Q(g,S)$ are linearly dependent at every point of M, i.e.,

$$R\cdot S=L_S Q(g,S), \tag{7}$$

then M is called Ricci-pseudosymmetric. This is equivalent to

$$(R(X,Y)\cdot S)(U,V)=L_S[((X\wedge_g Y)\cdot S)(U,V)] \tag{8}$$

holding on the set $U_S=\{x\in M: S-\frac{r}{n}g\neq 0 \;\; at \;\; x\}$, with some function L_S on U_S [12]. Particularly, if $L_S=0$, then M is a Ricci-semisymmetric manifold. We note that $U_S\subset U_R$ and on 3-dimensional Riemannian manifolds we have $U_S=U_R$. Every Ricci-generalized pseudosymmetric manifold is Ricci-pseudosymmetric but the converse is not true.

Furthermore, tensors $R\cdot R$ and $R\cdot S$ on (M,g) are defined by

$$(R(X,Y)\cdot R)(U,V)W=R(X,Y)R(U,V)W-R(R(X,Y)U,V)W \tag{9}$$

$$-R(U,R(X,Y)V)W-R(U,V)R(X,Y)W,$$

and

$$(R(X,Y)\cdot S)(U,V) = -S(R(X,Y)U,V) - S(U,R(X,Y)V), \tag{10}$$

respectively.

Recently, pseudosymmetric and Ricci-pseudosymmetric conditions have been studied by many authors in several ways to a different extent such as K. K. Baishya and P. R. Chowdhury [2], U. C. De and D. Tarafdar [7], N. Malekzadeh et al. [11] and many others.

A Ricci soliton (g, V, λ) on a Riemannian manifold (M, g) is a generalization of an Einstein metric such that [9]

$$(\pounds_V g + 2S + 2\lambda g)(X,Y) = 0, \tag{11}$$

where S is the Ricci tensor, $\pounds_V$ is the Lie derivative operator along the vector field V on M and λ is a real number. The Ricci soliton is said to be shrinking, steady and expanding according to λ being negative, zero and positive, respectively. For more details we refer to the readers $[4-6]$.

Motivated by the above studies, in this paper we characterize Lorentzian para-Kenmotsu manifolds satisfying certain curvature conditions: $R\cdot S = L_S Q(g,S)$, $R\cdot R = L_R Q(S,R)$, $S\cdot R = 0$, symmetric Lorentzian para-Kenmotsu manifolds and Lorentzian para-Kenmotsu manifolds admitting Ricci solitions. The paper is organized as follows: In Section 2, we give a brief introduction of Lorentzian para-Kenmotsu manifolds. Sections 3, 4 and 5 are devoted to the study of Ricci-pseudosymmetric, Ricci-generalized pseudosymmetric and symmetric Lorentzian para-Kenmotsu manifolds, respectively. In Section 6, we discuss Lorentzian para-Kenmotsu manifolds satisfying the curvature condition $S\cdot R = 0$. In Section 7, we show that if a Lorentzian para-Kenmotsu manifold admits a Ricci solition, then the manifold is an η-Einstein manifold and the Ricci solition is always shrinking.

2 Preliminaries

An n-dimensional differentiable manifold M with a structure (ϕ, ξ, η, g) is said to be a Lorentzian almost paracontact metric manifold, if it admits a $(1,1)$-tensor field ϕ, a contravariant vector field ξ, a 1-form η and a Lorentzian metric g satisfying [1]

$$\eta(\xi) = -1, \tag{12}$$

$$\phi^2 X = X + \eta(X)\xi, \tag{13}$$

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \tag{14}$$

$$g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y), \tag{15}$$

$$g(X,\xi) = \eta(X), \tag{16}$$

$$\Phi(X,Y) = \Phi(Y,X) = g(X,\phi Y) \tag{17}$$

for any vector fields X, Y on M.
If ξ is a killing vector field, the (para) contact structure is called a K-(para) contact. In such a case, we have

$$\nabla_X \xi = \phi X. \tag{18}$$

A Lorentzian almost paracontact manifold M is called a Lorentzian para-Sasakian manifold if

$$(\nabla_X \phi)Y = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi \tag{19}$$

for any vector fields X, Y on M.

Now, we define a new manifold called Lorentzian para-Kenmostu manifold:

Definition 1. *A Lorentzian almost paracontact manifold M is called Lorentzian para-Kenmostu manifold if* [10]

$$(\nabla_X \phi)Y = -g(\phi X, Y)\xi - \eta(Y)\phi X \tag{20}$$

for any vector fields X, Y on M.

In a Lorentzian para-Kenmostu manifold, we have

$$\nabla_X \xi = -X - \eta(X)\xi, \tag{21}$$

$$(\nabla_X \eta)Y = -g(X, Y) - \eta(X)\eta(Y), \tag{22}$$

where ∇ is the Levi-Civita connection with respect to the Lorentzian metric g. Furthermore, on a Lorentzian para-Kenmotsu manifold M, the following relations hold [10]:

$$g(R(X, Y)Z, \xi) = \eta(R(X, Y)Z) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y), \tag{23}$$

$$R(\xi, X)Y = -R(X, \xi)Y = g(X, Y)\xi - \eta(Y)X, \tag{24}$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{25}$$

$$R(\xi, X)\xi = X + \eta(X)\xi, \tag{26}$$

$$S(X, \xi) = (n-1)\eta(X), \quad S(\xi, \xi) = -(n-1), \tag{27}$$

$$Q\xi = (n-1)\xi, \tag{28}$$

$$S(\phi X, \phi Y) = S(X, Y) + (n-1)\eta(X)\eta(Y) \tag{29}$$

for any vector fields $X, Y, Z \in \chi(M)$.
Let $\{e_1, e_2, e_3......., e_n = \xi\}$ be a frame of orthonormal basis of the tangent space at any point of the manifold. Then the Ricci tensor S and the scalar curvature r of the manifold are defined by

$$S(X, Y) = \sum_{i=1}^{n} \epsilon_i g(R(e_i, X)Y, e_i),$$

$$r = \sum_{i=1}^{n} \epsilon_i S(e_i, e_i),$$

respectively. Also, we have

$$g(X, Y) = \sum_{i=1}^{n} \epsilon_i g(X, e_i) g(Y, e_i),$$

where $X, Y \in \chi(M)$ and $\epsilon_i = g(e_i, e_i) = +1 \quad \text{or} \quad -1$.

Definition 2. *A Lorentzian para-Kenmotsu manifold M is said to be an η-Einstein manifold if its Ricci tensor S is of the form*

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y), \tag{30}$$

where a and b are scalar functions on M. In particular, if $b = 0$, then the manifold is said to be an Einstein manifold.

Let $\{e_1, e_2, e_3......., e_{n-1}, e_n = \xi\}$ be a frame of orthonormal basis of the tangent space at any point of the manifold. If we put $X = Y = e_i$ in (30) and sum up with respect to $i(1 \leq i \leq n)$, then we have

$$r = an - b. \tag{31}$$

On the other hand, putting $X = Y = \xi$ in (30) and using (12), (13) and (27), we also have

$$-(n-1) = -a + b. \tag{32}$$

Hence it follows from (31) and (32) that

$$a = \frac{r}{n-1} - 1, \quad b = \frac{r}{n-1} - n.$$

So the Ricci tensor S of an η-Einstein Lorentzian para-Kenmotsu manifold is given by

$$S(X, Y) = (\frac{r}{n-1} - 1)g(X, Y) + (\frac{r}{n-1} - n)\eta(X)\eta(Y). \tag{33}$$

It is known that every 3-dimensional Kenmotsu manifold is an η-Einstein manifold and its Ricci tensor is given by [13]

$$S(X, Y) = (\frac{r}{2} + 1)g(X, Y) - (3 + \frac{r}{2})\eta(X)\eta(Y),$$

where r is the scalar curvature of the manifold.
Now we can easily prove the following:

Proposition 1. *Let M be a 3-dimensional Lorentzian para-Kenmotsu manifold. Then, we have*

$$R(X, Y)Z = (\frac{r}{2} - 2)[g(Y, Z)X - g(X, Z)Y] + (\frac{r}{2} - 3)[\eta(Y)X - \eta(X)Y]\eta(Z) \tag{34}$$
$$+ (\frac{r}{2} - 3)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)]\xi,$$

$$S(X, Y) = (\frac{r}{2} - 1)g(X, Y) + (\frac{r}{2} - 3)\eta(X)\eta(Y) \tag{35}$$

for any vector fields $X, Y, Z \in \chi(M)$.

3 Ricci pseudo-symmetric Lorentzian para-Kenmotsu manifolds

Let M be a Ricci-pseudosymmetric Lorentzian para-Kenmotsu manifold, that is, the manifold satisfying the condition $R \cdot S = L_S Q(g, S)$. Then from (7) we have

$$(R(X,Y) \cdot S)(U,V) = L_S Q(g,S)(X,Y;U,V) \tag{36}$$

for any vector fields $X, Y, U, V \in \chi(M)$. It is equivalent to

$$(R(X,Y) \cdot S)(U,V) = L_S[((X \wedge_g Y) \cdot S)(U,V)]. \tag{37}$$

By virtue of (2) and (10), (37) becomes

$$-S(R(X,Y)U,V) - S(U,R(X,Y)V)$$

$$= L_S[-S((X \wedge_g Y)U,V) - S(U,(X \wedge_g Y)V)]$$

which by using (1) takes the form

$$-S(R(X,Y)U,V) - S(U,R(X,Y)V) \tag{38}$$

$$= L_S[-g(Y,U)S(X,V) + g(X,U)S(Y,V)$$

$$-g(Y,V)S(U,X) + g(X,V)S(U,Y)].$$

Putting $X = U = \xi$ in (38) then using (16), (26) and (27), we get

$$(1 - L_S)[S(Y,V) - (n-1)g(Y,V)] = 0. \tag{39}$$

Thus, we have either (i) $L_S = 1$, or (ii) $S(Y,V) = (n-1)g(Y,V)$ from which we get $r = n(n-1)$. Hence we have the following:

Proposition 2. *Every n-dimensional Ricci-pseudosymmetric Lorentzian para-Kenmotsu manifold is of the form $R \cdot S = Q(g,S)$, provided the manifold is an Einstein manifold of the form $S(Y,V) = (n-1)g(Y,V)$ with the scalar curvature $n(n-1)$.*

Conversely, if the manifold is an Einstein manifold of the form $S(Y,V) = (n-1)g(Y,V)$, then it is clear that $R \cdot S = L_S Q(g,S)$. This leads to the following theorem:

Theorem 1. *An n-dimensional Lorentzian para-Kenmotsu manifold is Ricci-pseudosymmetric if and only if the manifold is an Einstein manifold of the form $S(Y,V) = (n-1)g(Y,V)$ with the scalar curvature $n(n-1)$, provided $L_S \neq 1$.*

4 Ricci-generalized pseudosymmetric Lorentzian para-Kenmotsu manifolds

Let M be an n-dimensional Ricci-generalized pseudosymmetric Lorentzian para-Kenmotsu manifold. Then from (5), we have

$$R \cdot R = L_R Q(S, R). \tag{40}$$

It is equivalent to

$$(R(X,Y) \cdot R)(U,V)W = L_R[((X \wedge_S Y) \cdot R)(U,V)W] \tag{41}$$

for any $X, Y, U, V, W \in \chi(M)$. By using (2) and (8) in (41), we have

$$\begin{aligned} &R(X,Y)R(U,V)W - R(R(X,Y)U,V)W \\ &-R(U,R(X,Y)V)W - R(U,V)R(X,Y)W \\ &= L_R[(X \wedge_S Y)R(U,V)W - R((X \wedge_S Y)U,V)W \\ &-R(U,(X \wedge_S Y)V)W - R(U,V)(X \wedge_S Y)W]. \end{aligned} \tag{42}$$

By virtue of (1), (42) takes the form

$$\begin{aligned} &R(X,Y)R(U,V)W - R(R(X,Y)U,V)W \\ &-R(U,R(X,Y)V)W - R(U,V)R(X,Y)W \\ &= L_R[S(Y,R(U,V)W)X - S(X,R(U,V)W)Y \\ &-S(Y,U)R(X,V)W + S(X,U)R(Y,V)W \\ &-S(Y,V)R(U,X)W + S(X,V)R(U,Y)W \\ &-S(Y,W)R(U,V)X + S(X,W)R(U,V)Y]. \end{aligned} \tag{43}$$

Putting $X = U = \xi$ in (43) and making use of (24), (25) and (27), we get

$$\begin{aligned} &g(V,W)Y - R(Y,V)W - g(Y,W)V \\ &= L_R[(n-1)g(V,W)Y - \eta(W)S(Y,V)\xi \\ &-(n-1)R(Y,V)W + (n-1)g(Y,W)\eta(V)\xi \\ &-S(Y,W)V - S(Y,W)\eta(V)\xi + (n-1)g(V,Y)\eta(W)\xi] \end{aligned}$$

which by taking the inner product with Z becomes

$$\begin{aligned} &g(V,W)g(Y,Z) - g(R(Y,V)W,Z) - g(Y,W)g(V,Z) \\ &= L_R[(n-1)g(V,W)g(Y,Z) - S(Y,V)\eta(W)\eta(Z) \\ &-(n-1)g(R(Y,V)W,Z) + (n-1)g(Y,W)\eta(V)\eta(Z) \\ &-S(Y,W)g(V,Z) - S(Y,W)\eta(V)\eta(Z) + (n-1)g(V,Y)\eta(W)\eta(Z)]. \end{aligned} \tag{44}$$

Let $\{e_1, e_2, e_3......., e_{n-1}, \xi\}$ be a frame of orthonormal basis of the tangent space at any point of the manifold. If we put $V = W = e_i$ in (44) and sum up with respect to $i(1 \leq i \leq n)$, then we have

$$\sum_{i=1}^{n} \epsilon_i[g(e_i, e_i)g(Y, Z) - g(R(Y, e_i)e_i, Z) - g(Y, e_i)g(e_i, Z)$$

$$= L_R \sum_{i=1}^{n} \epsilon_i[(n-1)g(e_i, e_i)g(Y, Z) - S(Y, e_i)\eta(e_i)\eta(Z)$$
$$-(n-1)g(R(Y, e_i)e_i, Z) + (n-1)g(Y, e_i)\eta(e_i)\eta(Z)$$
$$-S(Y, e_i)g(e_i, Z) - S(Y, e_i)\eta(e_i)\eta(Z) + (n-1)g(e_i, Y)\eta(e_i)\eta(Z)]$$

from which it follows that

$$S(Y, Z) - (n-1)g(Y, Z) = nL_R[S(Y, Z) - (n-1)g(Y, Z)]. \tag{45}$$

Thus, we have either (i) $L_R = \frac{1}{n}$ or (ii) $S(Y, Z) = (n-1)g(Y, Z)$ from which we get $r = n(n-1)$. Hence we have the following:

Proposition 3. *Every n-dimensional Ricci-generalized pseudosymmetric Lorentzian para-Kenmotsu manifold is of the form $R \cdot R = \frac{1}{n}Q(g, S)$, provided the manifold is an Einstein manifold of the form $S(Y, Z) = (n-1)g(Y, Z)$ with the scalar curvature $n(n-1)$.*

Theorem 2. *An n-dimensional Ricci-generalized pseudosymmetric Lorentzian para-Kenmotsu manifold is an Einstein manifold of the form $S(Y, Z) = (n-1)g(Y, Z)$ with the scalar curvature $n(n-1)$, provided that $L_R \neq \frac{1}{n}$.*

5 Symmetric Lorentzian para-Kenmotsu manifolds

Definition 3. *A Lorentzian para-Kenmotsu manifold M is said to be symmetric if*

$$(\nabla_X R)(Y, Z)W = 0 \tag{46}$$

for all vector fields X, Y, Z and W on M, where R is the curvature tensor with respect to connection ∇.

Let M be a symmetric Lorentzian para-Kenmotsu manifold, then $(\nabla_X R)(Y, Z)W = 0$. By a suitable contraction of equation (46), we have

$$(\nabla_X S)(Z, W) = \nabla_X S(Z, W) - S(\nabla_X Z, W) - S(Z, \nabla_X W) = 0.$$

Taking $W = \xi$ in the last equation, we have

$$\nabla_X S(Z, \xi) - S(\nabla_X Z, \xi) - S(Z, \nabla_X \xi) = 0. \tag{47}$$

By using (21) and (27), (47) takes the form

$$(n-1)(\nabla_X \eta)Z + S(X,Z) + (n-1)\eta(X)\eta(Z) = 0. \tag{48}$$

In view of (22), (48) gives

$$S(X,Z) = (n-1)g(X,Z). \tag{49}$$

By contracting (49) over X and Z, it follows that

$$r = n(n-1). \tag{50}$$

Thus we have the following:

Theorem 3. *Let M be an n-dimensional symmetric Lorentzian para-Kenmotsu manifold. Then the manifold is an Einstein manifold of the form $S(X,Z) = (n-1)g(X,Z)$ with the scalar curvatutre $n(n-1)$.*

6 Lorentzian para-Kenmotsu manifolds satisfying the curvature condition $S \cdot R = 0$

Let M be a Lorentzian para-Kenmotsu manifold satisfying the curvature condition $(S(X,Y) \cdot R)(U,V)W = 0$. This implies that

$$\begin{aligned}(X \wedge_S Y)R(U,V)W + R((X \wedge_S Y)U,V)W \\ +R(U,(X \wedge_S Y)V)W + R(U,V)(X \wedge_S Y)W = 0\end{aligned} \tag{51}$$

for any vector fields $X, Y, U, V, W \in \chi(M)$. By virtue of (1), (51) takes the form

$$\begin{aligned}S(Y,R(U,V)W)X - S(X,R(U,V)W)Y + S(Y,U)R(X,V)W \\ -S(X,U)R(Y,V)W + S(Y,V)R(U,X)W - S(X,V)R(U,Y)W \\ +S(Y,W)R(U,V)X - S(X,W)R(U,V)Y = 0.\end{aligned} \tag{52}$$

Taking $U = W = \xi$ in (52) then using (24) and (25), we have

$$\begin{aligned}2S(Y,V)X - 2S(X,V)Y + 2(n-1)\eta(Y)\eta(V)X \\ -2(n-1)\eta(X)\eta(V)Y + \eta(X)S(Y,V)\xi - \eta(Y)S(X,V)\xi \\ +(n-1)g(V,X)\eta(Y)\xi - (n-1)g(V,Y)\eta(X)\xi = 0\end{aligned}$$

which by taking the inner product with ξ and using (12) and (16) reduces to

$$S(Y,V)\eta(X) - S(X,V)\eta(Y) + (n-1)g(Y,V)\eta(X) - (n-1)g(X,V)\eta(Y). \tag{53}$$

Now putting $X = \xi$ in (53) and using (12) and (27) to get

$$S(Y,V) = -(n-1)g(V,Y) - 2(n-1)\eta(Y)\eta(V). \tag{54}$$

Thus we have the following:

Theorem 4. *If an n-dimensional Lorentzian para-Kenmotsu manifold satisfying the curvature condition $S \cdot R = 0$, then the manifold is an η-Einstein manifold of the form* (54).

Remark. If we take $r = -2$ in a 3-dimensional Lorentzian para-Kenmotsu manifold, then (35) verifies (54).

7 Ricci solitons

Suppose that a Lorentzian para-Kenmotsu manifold admits a Ricci soliton (g, ξ, λ). Then we have

$$(\pounds_\xi g + 2S + 2\lambda g)(X, Y) = 0 \tag{55}$$

which implies that

$$g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) + 2S(X, Y) + 2\lambda g(X, Y) = 0. \tag{56}$$

Using (21) in (56), we get

$$S(X, Y) + (\lambda - 1)g(X, Y) - \eta(X)\eta(Y) = 0 \tag{57}$$

which by taking $Y = \xi$ yields

$$S(X, \xi) = -\lambda\eta(X) \implies \quad \lambda = -(n - 1). \tag{58}$$

Putting this value of λ in (57), we get

$$S(X, Y) = ng(X, Y) + \eta(X)\eta(Y). \tag{59}$$

Thus in view of (58) and (59), we have the following:

Theorem 5. *If an n-dimensional Lorentzian para-Kenmotsu manifold admits a Ricci soliton, then the manifold is an η-Einstein manifold of the form* (59) *and the Ricci soliton is always shrinking.*

Now, let (g, V, λ) be a Ricci soliton on a Lorentzian para-Kenmotsu manifold such that V is pointwise collinear with ξ, i.e., $V = b\xi$, where b is a function. Then (11) holds and thus, we have

$$bg(\nabla_X \xi, Y) + (Xb)\eta(Y) + bg(X, \nabla_Y \xi)$$

$$+(Yb)\eta(X) + 2S(X, Y) + 2\lambda g(X, Y) = 0$$

which in view of (21) takes the form

$$-2bg(X, Y) - 2b\eta(X)\eta(Y) + (Xb)\eta(Y) \tag{60}$$

$$+(Yb)\eta(X) + 2S(X, Y) + 2\lambda g(X, Y) = 0.$$

Putting $Y = \xi$ in (60) then using (12), (16) and (27), we have

$$-(Xb) + (\xi b)\eta(X) + 2(n - 1)\eta(X) + 2\lambda\eta(X) = 0. \tag{61}$$

Again taking $X = \xi$ in (61) and using (12), we get

$$(\xi b) + (n - 1) + \lambda = 0. \tag{62}$$

Combining the equations (61) and (62) it follows that

$$db = [\lambda + (n-1)]\eta. \tag{63}$$

Now applying d on (63), we have

$$[\lambda + (n-1)]d\eta = 0 \quad \Longrightarrow \quad \lambda = -(n-1), \quad d\eta \neq 0. \tag{64}$$

Thus from (63) and (64), we find $db = 0$, i.e., b is constant. Hence (60) takes the form

$$S(X,Y) = (b-\lambda)g(X,Y) + b\eta(X)\eta(Y). \tag{65}$$

Thus in view of (64) and (65), we have the following theorem:

Theorem 6. *If (g, V, λ) is a Ricci soliton on a Lorentzian para-Kenmotsu manifold such that V is pointwise collinear with ξ, then V is a constant multiple of ξ and the manifold is an η-Einstein manifold of the form* (65) *and the Ricci soliton is always shrinking.*

Example. We consider the 5-dimensional manifold $M = \{(x_1, x_2, y_1, y_2, z) \in \in R^5 : z > 0\}$, where (x_1, x_2, y_1, y_2, z) are the standard coordinates in R^5. Let e_1, e_2, e_3, e_4 and e_5 be the vector fields on M defined by

$$e_1 = z\frac{\partial}{\partial x_1}, \quad e_2 = z\frac{\partial}{\partial x_2}, \quad e_3 = z\frac{\partial}{\partial y_1}, \quad e_4 = z\frac{\partial}{\partial y_2}, \quad e_5 = z\frac{\partial}{\partial z} = \xi,$$

which are linearly independent at each point p of M. Let g be the Lorentzian metric defined by

$$g(e_i, e_i) = 1, \quad \text{for} \quad 1 \leq i \leq 4 \quad \text{and} \quad g(e_5, e_5) = -1,$$

$$g(e_i, e_j) = 0, \quad \text{for} \quad i \neq j, \quad 1 \leq i, j \leq 5.$$

Let η be the 1-form defined by $\eta(X) = g(X, e_5) = g(X, \xi)$ for all $X \in \chi(M)$, and let ϕ be the $(1,1)$-tensor field defined by

$$\phi e_1 = -e_2, \quad \phi e_2 = -e_1, \quad \phi e_3 = -e_4, \quad \phi e_4 = -e_3, \quad \phi e_5 = 0.$$

By applying linearity of ϕ and g, we have

$$\eta(\xi) = g(\xi, \xi) = -1, \quad \phi^2 X = X + \eta(X)\xi \quad \text{and} \quad g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y)$$

for all $X, Y \in \chi(M)$. Thus for $e_5 = \xi$, the structure (ϕ, ξ, η, g) defines a Lorentzian almost paracontact metric structure on M. Then we have

$$[e_i, e_j] = 0, \quad \text{if} \quad i \neq j, \quad \text{and} \quad 1 \leq i, j \leq 4,$$

$$[e_i, e_5] = -e_i, \quad \text{for} \quad 1 \leq i \leq 4.$$

The Levi-Civita connection ∇ of the Lorentzian metric g is given by

$$\begin{aligned} &2g(\nabla_X Y, Z) \\ = \;& Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) - g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y]), \end{aligned}$$

which is known as Koszul's formula. Using Koszul's formula, we find

$$\nabla_{e_1}e_1 = -e_5, \quad \nabla_{e_1}e_2 = 0, \quad \nabla_{e_1}e_3 = 0, \quad \nabla_{e_1}e_4 = 0, \quad \nabla_{e_1}e_5 = -e_1,$$

$$\nabla_{e_2}e_1 = 0, \quad \nabla_{e_2}e_2 = -e_5, \quad \nabla_{e_2}e_3 = 0, \quad \nabla_{e_2}e_4 = 0, \quad \nabla_{e_2}e_5 = -e_2,$$

$$\nabla_{e_3}e_1 = 0, \quad \nabla_{e_3}e_2 = 0, \quad \nabla_{e_3}e_3 = -e_5, \quad \nabla_{e_3}e_4 = 0, \quad \nabla_{e_3}e_5 = -e_3,$$

$$\nabla_{e_4}e_1 = 0, \quad \nabla_{e_4}e_2 = 0, \quad \nabla_{e_4}e_3 = 0, \quad \nabla_{e_4}e_4 = -e_5, \quad \nabla_{e_4}e_5 = -e_4,$$

$$\nabla_{e_5}e_1 = 0, \quad \nabla_{e_5}e_2 = 0, \quad \nabla_{e_5}e_3 = 0, \quad \nabla_{e_5}e_4 = 0, \quad \nabla_{e_5}e_5 = 0.$$

Now let

$$X = \sum_{i=1}^{5} X^i e_i = X^1 e_1 + X^2 e_2 + X^3 e_3 + X^4 e_4 + X^5 e_5,$$

$$Y = \sum_{j=1}^{5} Y^j e_j = Y^1 e_1 + Y^2 e_2 + Y^3 e_3 + Y^4 e_4 + Y^5 e_5,$$

$$Z = \sum_{k=1}^{5} Z^k e_k = Z^1 e_1 + Z^2 e_2 + Z^3 e_3 + Z^4 e_4 + Z^5 e_5$$

for all $X, Y, Z \in \chi(M)$. Also, one can easily verify that

$$\nabla_X \xi = -X - \eta(X)\xi \quad \text{and} \quad (\nabla_X \phi)Y = -g(\phi X, Y)\xi - \eta(Y)\phi X.$$

Therefore, the manifold is a Lorentzian para-Kenmotsu manifold.
From the above results, we can easily obtain the non-vanishing components of the curvature tensor as follows:

$$R(e_1, e_2)e_1 = -e_2, \quad R(e_1, e_2)e_2 = e_1, \quad R(e_1, e_3)e_1 = -e_3, \quad R(e_1, e_3)e_3 = e_1,$$

$$R(e_1, e_4)e_1 = -e_4, \quad R(e_1, e_4)e_4 = e_1, \quad R(e_1, e_5)e_1 = -e_5, \quad R(e_1, e_5)e_5 = -e_1,$$

$$R(e_2, e_3)e_2 = -e_3, \quad R(e_2, e_3)e_3 = e_2, \quad R(e_2, e_4)e_2 = -e_4, \quad R(e_2, e_4)e_4 = e_2,$$

$$R(e_2, e_5)e_2 = -e_5, \quad R(e_2, e_5)e_5 = -e_2, \quad R(e_3, e_4)e_3 = -e_4, \quad R(e_3, e_4)e_4 = e_3,$$

$$R(e_3, e_5)e_3 = -e_5, \quad R(e_3, e_5)e_5 = -e_3, \quad R(e_4, e_5)e_4 = -e_5, \quad R(e_4, e_5)e_5 = -e_4$$

from which it is clear that

$$R(X, Y)Z = g(Y, Z)X - g(X, Z)Y. \tag{66}$$

Thus, the manifold is of constant curvature. Also, we calculate the Ricci tensors as follows:

$$S(e_1, e_1) = S(e_2, e_2) = S(e_3, e_3) = S(e_4, e_4) = 4, \quad S(e_5, e_5) = -4.$$

Hence we find

$$r = S(e_1, e_1) + S(e_2, e_2) + S(e_3, e_3) + S(e_4, e_4) - S(e_5, e_5) = 20.$$

By contracting (66), it follows that

$$S(Y,Z) = 4g(Y,Z), \quad r = 20, \tag{67}$$

which are same as the values of Ricci tensor and scalar curvature obtained in sections 3, 4 and 5. Now taking $Z = \xi$ in (67), we get

$$S(Y,\xi) = 4\eta(Y). \tag{68}$$

Thus from (58) and (68) we obtain $\lambda = -4$, i.e., the Ricci soliton is shrinking which verifies Section 7.

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