Bulletin of the *Transilvania* University of Braşov • Vol 13(62), No. 1 - 2020
Series III: Mathematics, Informatics, Physics, 165-184
https://doi.org/10.31926/but.mif.2020.13.62.1.13

NONLINEAR DIFFERENTIAL POLYNOMIALS SHARING A SET OF VALUES

Samar HALDER[1,2] **and Pulak SAHOO**[3,4]

Abstract

In this article, we investigate the uniqueness problem of meromorphic functions when certain nonlinear differential polynomials generated by them share a set of values with finite weight and obtain some results that generalize the recent results due to P. Sahoo and G. Biswas [Filomat 32 (2018), 457-472]. Our results also improve and generalize the results due to H.Y. Xu [J. Comput. Anal. Appl., 16 (2014), 942-954].

2000 *Mathematics Subject Classification:* Primary 30D35.
Key words: meromorphic function, shared set, uniqueness, differential polynomial.

1 Introduction, Definitions and Results

Throughout the paper, by a meromorphic function we shall always mean a function that is meromorphic in the complex plane $\mathbb{C}$. In what follows, we assume that the reader is familiar with the standard notations of Nevanlinna value distribution theory such as $T(r, f)$, $m(r, f)$, $N(r, f)$, $\overline{N}(r, f)$, $S(r, f)$ and so on, that can be found, for instance, in [5, 10, 20].

Let f and g be any two nonconstant meromorphic functions. For $a \in \mathbb{C} \cup \{\infty\} = \overline{\mathbb{C}}$ and $S \subset \overline{\mathbb{C}}$, we define

$$E(S, f) = \bigcup_{a\in S}\{z : f(z) - a = 0, \ \text{counting multiplicities}\},$$
$$\overline{E}(S, f) = \bigcup_{a\in S}\{z : f(z) - a = 0, \ \text{ignoring multiplicities}\}.$$

[1]Department of Mathematics, University of *Kalyani*, West Bengal-741235, India, e-mail: samarhalder.mtmh@gmail.com
[2]The first author is supported by UGC-NFSC scheme of research fellowship.
[3]Department of Mathematics, University of *Kalyani*, West Bengal-741235, India, e-mail: sahoopulak1@gmail.com
[4]Corresponding author, the author is thankful to DST-PURSE for financial assistance.

If $E(S,f)=E(S,g)$, we say that f and g share the set S CM; if $\overline{E}(S,f)=\overline{E}(S,g)$, we say that f and g share the set S IM. As a special case, let $S=\{a\}$, then we say that f and g share the value a CM (resp. IM), provided that $E(S,f)=E(S,g)$ (resp. $\overline{E}(S,f)=\overline{E}(S,g)$) (see [4]).

For a positive integer m, we denote by $E_{m)}(a;f)$ the set of all a-points of f with multiplicities not greater than m, where an a-point is counted according to its multiplicities. Also by $\overline{E}_{m)}(a;f)$ we denote the set of distinct a-points of f with multiplicities not exceeding m. If $E_{\infty)}(a;f)=E_{\infty)}(a;g)$ for $a\in\overline{\mathbb{C}}$, we say that f and g share the value a CM. We also define $E_{m)}(S,f)=\bigcup_{a\in S}E_{m)}(a;f)$ and $\overline{E}_{m)}(S,f)=\bigcup_{a\in S}\overline{E}_{m)}(a;f)$ for any positive integer m. The research on the uniqueness theory related to meromorphic functions has brought out a good number of interesting results due to sharing of values by different functions. In fact, uniqueness problems regarding differential polynomials and their shared values have been studied in a large extent (see [1, 6, 9, 12, 15, 17, 18]). Recently, there has been an increasing interest to consider the differential polynomials with the shared set of values.

In 1997, C.C. Yang and X.H. Hua [19] proved the following result.

Theorem A. *Let f and g be two nonconstant meromorphic functions, $n\geq 11$ an integer and $a\in\mathbb{C}\setminus\{0\}$. If f^nf' and g^ng' share the value a CM, then either $f=tg$ for some $(n+1)$th root of unity t or $g(z)=c_1e^{cz}$, $f(z)=c_2e^{-cz}$, where c, c_1, c_2 are constants satisfying $(c_1c_2)^{n+1}c^2=-a^2$.*

Regarding Theorem A, one may ask the following question:

Question 1.1. *Whether there exists a differential polynomial d such that for any pair of nonconstant meromorphic functions f and g we can get $f\equiv g$ whenever $d(f)$ and $d(g)$ share one value CM?*

Some of the earlier works in this direction can be found in [2, 3, 6, 8, 9, 11]. Among them, Fang-Fang [2] and Lin-Yi [11] gave an affirmative answer to the above question and proved the following results respectively.

Theorem B. *Let f and g be two nonconstant meromorphic functions and n be a positive integer. If $E_{k)}(1,f^n(f-1)^2f')=E_{k)}(1,g^n(g-1)^2g')$ and one of the following conditions is satisfied: (a) $k\geq 3$, $n\geq 13$, (b) $k=2$, $n\geq 15$, (c) $k=1$, $n\geq 23$, then $f\equiv g$.*

Theorem C. *Let f and g be two nonconstant meromorphic functions satisfying $\Theta(\infty,f)>2/(n+1)$, $n\geq 12$ be an integer. If $f^n(f-1)f'$ and $g^n(g-1)g'$ share 1 CM, then $f\equiv g$.*

In 2014, H.Y. Xu [16] investigated the uniqueness of meromorphic functions for the case of two nonlinear differential polynomials sharing a set S_m containing m roots of unity, namely $S_m=\{1,\omega,\omega^2,\ldots,\omega^{m-1}\}$, where $\omega=\exp(\frac{2\pi}{m}i)$ (m being an integer) and obtained the following two results:

Theorem D. *Let f and g be two nonconstant meromorphic functions such that one of f and g has only multiple poles, n and $m(\geq 2)$ be two positive integers. For any distinct $a, b \in \mathbb{C} \setminus \{0\}$, let $E_{k)}(S_m, f^n(f-a)(f-b)f') = E_{k)}(S_m, g^n(g-a)(g-b)g')$ and let the expressions $\frac{a+b}{n+2} g \sum_{j=0}^{n+1} \left(\frac{f}{g}\right)^j - \frac{ab}{n+1} \sum_{j=0}^{n} \left(\frac{f}{g}\right)^j$ and $\sum_{j=0}^{n+2} \left(\frac{f}{g}\right)^j$ have no common simple zeros. If one of the following conditions is satisfied:*

(a) $k \geq 3 : n > 4 + \frac{8}{m}$ when $2 \leq m \leq 3$ and $n > 4 + \frac{4}{m}$ when $m \geq 4$;
(b) $k = 2 : n > 4 + \frac{11}{m}$ when $2 \leq m \leq 3$ and $n > 4 + \frac{4}{m}$ when $m \geq 4$;
(c) $k = 1 : n > 4 + \frac{20}{m}$ when $2 \leq m \leq 3$ and $n > 4 + \frac{4}{m}$ when $m \geq 4$,

then $f \equiv g$.

Theorem E. *Let f and g be two nonconstant meromorphic functions, $n, m(\geq 2)$ be two positive integers. If $E_{k)}(S_m, f^n(f-a)^2 f') = E_{k)}(S_m, g^n(g-a)^2 g')$ and one of the following conditions is satisfied:*

(a) $k \geq 3 : n > 4 + \frac{8}{m}$;
(b) $k = 2 : n > \max\{4 + \frac{4}{m}, 2 + \frac{10}{m}\}$;
(c) $k = 1 : n > 4 + \frac{20}{m}$ when $2 \leq m \leq 3$ and $n > 4 + \frac{4}{m}$ when $m \geq 4$,

then $f \equiv g$.

Remark 1.1. It is to be noticed that there are some lacunas in Lemma 2.3 (see [16]) as well as in the lower bound of n in Theorem E (see [16]). In the proof of Theorem E (though not given in details), Case 1 of Lemma 2.4 is needed, where the lower bound of n is taken as $n \geq 8$.

Recently, P. Sahoo and G. Biswas [14], overcoming the lacunas, extended Theorems D and E and obtained the following results.

Theorem F. *Let f and g be two nonconstant meromorphic functions such that one of f and g has only multiple poles, n and $m(\geq 2)$ be two positive integers. For any distinct $a, b, c \in \mathbb{C} \setminus \{0\}$, let $E_{k)}(S_m, f^n(f-a)(f-b)(f-c)f') = E_{k)}(S_m, g^n(g-a)(g-b)(g-c)g')$ and let the expressions $\frac{a+b+c}{n+3} g^2 \sum_{j=0}^{n+2} \left(\frac{f}{g}\right)^j - \frac{ab+bc+ca}{n+2} g \sum_{j=0}^{n+1} \left(\frac{f}{g}\right)^j + \frac{abc}{n+1} \sum_{j=0}^{n} \left(\frac{f}{g}\right)^j$ and $\sum_{j=0}^{n+3} \left(\frac{f}{g}\right)^j$ have no common simple zeros. If one of the following conditions is satisfied:*

(a) $k \geq 3 : n > 5 + \frac{8}{m}$ when $2 \leq m \leq 3$ and $n > 5 + \frac{3}{m}$ when $m \geq 4$;
(b) $k = 2 : n > 5 + \frac{23}{2m}$ when $2 \leq m \leq 3$ and $n > 5 + \frac{3}{m}$ when $m \geq 4$;
(c) $k = 1 : n > 5 + \frac{22}{m}$ when $2 \leq m \leq 3$ and $n > 5 + \frac{3}{m}$ when $m \geq 4$;

then $f \equiv tg$, where $t^m = 1$.

Theorem G. *Let f and g be two nonconstant meromorphic functions such that one of f and g has only multiple poles, n and $m(\geq 2)$ be two positive integers. For any distinct $a, b \in \mathbb{C} \setminus \{0\}$, let $E_{k)}(S_m, f^n(f-a)^2(f-b)f') = E_{k)}(S_m, g^n(g-a)^2(g-b)g')$ and let the expressions $\frac{2a+b}{n+3} g^2 \sum_{j=0}^{n+2} \left(\frac{f}{g}\right)^j - \frac{a^2+2ab}{n+2} g \sum_{j=0}^{n+1} \left(\frac{f}{g}\right)^j + \frac{a^2 b}{n+1} \sum_{j=0}^{n} \left(\frac{f}{g}\right)^j$ and $\sum_{j=0}^{n+3} \left(\frac{f}{g}\right)^j$ have no common simple zeros. If one of the following conditions is satisfied:*

(a) $k \geq 3 : n > 3 + \frac{8}{m}$ when $2 \leq m \leq 3$ and $n > 5 + \frac{2}{m}$ when $m \geq 4$;

(b) $k=2: n>3+\frac{11}{m}$ *when* $2\leq m\leq 3$ *and* $n>5+\frac{2}{m}$ *when* $m\geq 4$;
(c) $k=1: n>3+\frac{20}{m}$ *when* $2\leq m\leq 3$ *and* $n>5+\frac{2}{m}$ *when* $m\geq 4$;
then $f\equiv tg$, *where* $t^m=1$.

Theorem H. *Let* f *and* g *be two nonconstant meromorphic functions such that one of* f *and* g *has only multiple poles,* n *and* $m(\geq 2)$ *be two positive integers. For* $a\in\mathbb{C}\setminus\{0\}$, *let* $E_{k)}(S_m, f^n(f-a)^3f')=E_{k)}(S_m, g^n(g-a)^3g')$ *and let the expressions* $\frac{3a}{n+3}g^2\sum_{j=0}^{n+2}\left(\frac{f}{g}\right)^j-\frac{3a^2}{n+2}g\sum_{j=0}^{n+1}\left(\frac{f}{g}\right)^j+\frac{a^3}{n+1}\sum_{j=0}^{n}\left(\frac{f}{g}\right)^j$ *and* $\sum_{j=0}^{n+3}\left(\frac{f}{g}\right)^j$ *have no common simple zeros. If one of the following conditions is satisfied:*
(a) $k\geq 3: n>max\{2+\frac{8}{m},10\}$ *when* $2\leq m\leq 3$ *and* $n>max\{\frac{8}{m},10\}$ *when* $m\geq 4$;
(b) $k=2: n>max\{2+\frac{10}{m},10\}$ *when* $2\leq m\leq 3$ *and* $n>max\{\frac{10}{m},10\}$ *when* $m\geq 4$;
(c) $k=1: n>max\{2+\frac{16}{m},10\}$ *when* $2\leq m\leq 3$ *and* $n>max\{\frac{16}{m},10\}$ *when* $m\geq 4$;
then $f\equiv tg$, *where* $t^m=1$.

In connection to Theorems D-H the following question is inevitable.

Question 1.2. *Is it possible to deduce a generalized result in which all the Theorems D-H will be included?*

In this paper, we will concentrate on the above question and give a positive answer by considering differential polynomials in a generalized form with respect to that of Theorems D-H. Henceforth, we assume that $a_1,a_2,\ldots,a_\mu\in\mathbb{C}\setminus\{0\}$ such that $a_i\neq a_j$ $(i,j=1,2,\ldots,\mu)$ and $\mu(\geq 2)$, $s(\geq\mu)$, η, $s_i(\geq 1)$ $(i=1,2,\ldots,\mu)$ are nonnegative integers satisfying $\sum_{i=1}^{\mu}s_i=s$ and $P(z)=(z-a_1)^{s_1}(z-a_2)^{s_2}\ldots(z-a_\mu)^{s_\mu}$ is a nonzero polynomial of degree s such that P has exactly η roots with multiplicity greater than 1.

We collect all the a_i's counting multiplicities and arrange them in a sequence according to their monotonically increasing subscripts. In what follows, by a'_i we shall mean the i-th term of the sequence $\{a'_i\}_{i=1}^{s}$, where

$$\begin{aligned} a'_i &= a_1,\ \ 1\leq i\leq s_1\\ &= a_2,\ \ s_1+1\leq i\leq s_1+s_2\\ &\ \ \ \ \ldots\\ &= a_\mu,\ \ 1+\sum_{j=1}^{\mu-1}s_j\leq i\leq s. \end{aligned}$$

Therefore $P(z)=(z-a_1)^{s_1}(z-a_2)^{s_2}\ldots(z-a_\mu)^{s_\mu}=(z-a'_1)(z-a'_2)\ldots(z-a'_s)$. However, f' has its usual meaning for any meromorphic function f.

We now state our main results of the paper.

Theorem 1.1. *Let* f *and* g *be two nonconstant meromorphic functions with one of them having only multiple poles,* $n,m(\geq 2)$ *be positive integers and* $E_{k)}(S_m, f^nP(f)f')$

$= E_{k)}(S_m, g^nP(g)g')$. *If* $\frac{\sum_{i=1}^{s} a'_i}{n+s} g^{s-1} \sum_{\zeta=0}^{n+s-1} \left(\frac{f}{g}\right)^{\zeta} - \frac{\sum_{1\le i<j\le s} a'_i a'_j}{n+s-1} g^{s-2} \sum_{\zeta=0}^{n+s-2} \left(\frac{f}{g}\right)^{\zeta} + \ldots + \frac{(-1)^{s-1}a'_1a'_2\ldots a'_s}{n+1} \sum_{\zeta=0}^{n} \left(\frac{f}{g}\right)^{\zeta}$ *and* $\sum_{\zeta=0}^{n+s} \left(\frac{f}{g}\right)^{\zeta}$ *have no common simple zero and one of the following conditions is satisfied:*
(i) $k \ge 3 : n > max\{\mu' + 2 + \frac{8}{m}, s + 2 + max\{0, \frac{3-s+\mu}{m}\}\}$ *when* $2 \le m \le 3$ *and* $n > s + 2 + max\{0, \frac{3-s+\mu}{m}\}$ *when* $m \ge 4$;
(ii) $k = 2 : n > max\{\mu' + 2 + \frac{20+\mu}{2m}, s + 2 + max\{0, \frac{3-s+\mu}{m}\}\}$ *when* $2 \le m \le 3$ *and* $n > s + 2 + max\{0, \frac{3-s+\mu}{m}\}$ *when* $m \ge 4$;
(iii) $k = 1 : n > max\{\mu' + 2 + \frac{16+2\mu}{m}, s + 2 + max\{0, \frac{3-s+\mu}{m}\}\}$ *when* $2 \le m \le 3$ *and* $n > s + 2 + max\{0, \frac{3-s+\mu}{m}\}$ *when* $s > 2,\ m \ge 4$,
then $f \equiv tg$, *where* $t^m = 1$ *and* $\mu' = \mu - \eta$.

Taking $s_1 = s_2 = \ldots = s_\mu = 1$ and $\mu = s$ in the above theorem we can obtain the following corollary.

Corollary 1.1. *Let f and g be two nonconstant meromorphic functions with one of them having only multiple poles, $s(\ge 2), n, m(\ge 2)$ be positive integers. For all distinct $a_i \in \mathbb{C}\setminus\{0\}$, let $E_{k)}(S_m, f^n(f-a_1)(f-a_2)\ldots(f-a_s)f') = E_{k)}(S_m, g^n(g-a_1)(g-a_2)\ldots(g-a_s)g')$. If* $\frac{\sum_{i=1}^{s} a_i}{n+s} g^{s-1} \sum_{\zeta=0}^{n+s-1} \left(\frac{f}{g}\right)^{\zeta} - \frac{\sum_{1\le i<j\le s} a_i a_j}{n+s-1} g^{s-2} \sum_{\zeta=0}^{n+s-2} \left(\frac{f}{g}\right)^{\zeta} + \ldots + \frac{(-1)^{s-1}a_1a_2\ldots a_s}{n+1} \sum_{\zeta=0}^{n} \left(\frac{f}{g}\right)^{\zeta}$ *and* $\sum_{\zeta=0}^{n+s} \left(\frac{f}{g}\right)^{\zeta}$ *have no common simple zero and one of the following conditions is satisfied:*
(i) $k \ge 3 : n > s + 2 + \frac{8}{m}$ *when* $2 \le m \le 3$ *and* $n > s + 2 + \frac{3}{m}$ *when* $m \ge 4$;
(ii) $k = 2 : n > s + 2 + \frac{20+s}{2m}$ *when* $2 \le m \le 3$ *and* $n > s + 2 + \frac{3}{m}$ *when* $m \ge 4$;
(iii) $k = 1 : n > s+2+\frac{16+2s}{m}$ *when* $2 \le m \le 3$ *and* $n > s+2+\frac{3}{m}$ *when* $s > 2, m \ge 4$,
then $f \equiv tg$, *where* $t^m = 1$.

Theorem 1.2. *Let f and g be two nonconstant meromorphic functions with one of f and g having only multiple poles and $n, m(\ge 2), s(> 2)$ be positive integers. Let $E_{k)}(S_m, f^n(f-a)^s f') = E_{k)}(S_m, g^n(g-a)^s g')$ where $a \in \mathbb{C}\setminus\{0\}$. If the expressions* ${}^sC_1 \frac{a}{n+s} g^{s-1} \sum_{\zeta=0}^{n+s-1} \left(\frac{f}{g}\right)^{\zeta} - {}^sC_2 \frac{a^2}{n+s-1} g^{s-2} \sum_{\zeta=0}^{n+s-2} \left(\frac{f}{g}\right)^{\zeta} + \ldots + \frac{(-1)^{s-1}a^s}{n+1} \sum_{\zeta=0}^{n} \left(\frac{f}{g}\right)^{\zeta}$ *and* $\sum_{\zeta=0}^{n+s} \left(\frac{f}{g}\right)^{\zeta}$ *have no common simple zero and one of the following conditions is satisfied:*
(i) $k \ge 3 : n > max\{2 + \frac{8}{m}, 3s + 1\}$ *when* $2 \le m \le 3$ *and* $n > max\{\frac{8}{m}, 3s + 1\}$ *when* $m \ge 4$;
(ii) $k = 2 : n > max\{2 + \frac{10}{m}, 3s + 1\}$ *when* $2 \le m \le 3$ *and* $n > max\{\frac{10}{m}, 3s + 1\}$ *when* $m \ge 4$;
(iii) $k = 1 : n > max\{2 + \frac{16}{m}, 3s + 1\}$ *when* $2 \le m \le 3$ *and* $n > max\{\frac{16}{m}, 3s + 1\}$

when $m \geq 4$,
then $f \equiv tg$, *where* $t^m = 1$.

Remark 1.2. If we take $s = 2$ in Corollary 1.1, then it reduces to Theorem D with a little improvement of lower bound of n for $m \geq 4, k \geq 2$. Thus Corollary 1.1 generalizes and improves Theorem D except for the case $k = 1$ when $m \geq 4$. Also Corollary 1.1 generalizes Theorem F.

Remark 1.3. If we take $s = 3, \mu = 2, \eta = 1$ in Theorem 1.1, then it reduces to Theorem G. Thus Theorem 1.1 generalizes Theorem G.

Remark 1.4. Theorem 1.2 generalizes Theorem H, since taking $s = 3$ in Theorem 1.2 it reduces to Theorem H. Again Theorem 1.2 is an extension of Theorem E.

Note 1.1. By following our technique it follows that for $s = 2$ in Theorem 1.2, the bounds of n in case of $k = 1$ are obtained as: $n > max\{2 + \frac{18}{m}, 3s + 1\}$ when $2 \leq m \leq 3$ and $n > \max\{\frac{18}{m}, 3s + 1\}$ when $m \geq 4$.

Although the standard definitions and notations of the value distribution theory are available in [5, 20], we explain the following definitions which are used in the paper.

Definition 1. *[7] Let* $a \in \mathbb{C} \cup \{\infty\}$. *We denote by* $N(r, a; f \mid= 1)$ *the counting function of simple a-points of* f. *For a positive integer* p *we denote by* $N(r, a; f \mid\leq p)$ *the counting function of those a-points of* f *(counted with proper multiplicities) whose multiplicities are not greater than* p. *By* $\overline{N}(r, a; f \mid\leq p)$ *we denote the corresponding reduced counting function.*

Analogously we can define $N(r, a; f \mid\geq p)$ *and* $\overline{N}(r, a; f \mid\geq p)$.

Definition 2. *[7] Let* k *be positive integer or infinity. We denote by* $N_k(r, a; f)$ *the counting function of a-points of* f, *where an a-point of multiplicity* m *is counted* m *times if* $m \leq k$ *and* k *times if* $m > k$. *Then*

$$N_k(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f \mid\geq 2) + ... + \overline{N}(r, a; f \mid\geq k).$$

Clearly $N_1(r, a; f) = \overline{N}(r, a; f)$.

Definition 3. *[20, p. 222] Let* f *and* g *be nonconstant meromorphic functions such that* f *and* g *share* 1 *IM. We denote by* $N_E^{1)}(r, 1; f)$ *the counting function of common simple* 1*-points of* f *and* g.

2 Lemmas

In order to prove our results, we need the following lemmas.

Lemma 2.1. *[20, p. 36, p. 39] Suppose that* f *is a nonconstant meromorphic function in the complex plane and* k *is a positive integer. Then*

$$T(r, f^{(k)}) \leq T(r, f) + k\overline{N}(r, \infty; f) + S(r, f).$$

and

$$N(r,0;f^{(k)}) \le N(r,0;f) + k\overline{N}(r,\infty;f) + S(r,f).$$

Lemma 2.2. *[13] Let f be a nonconstant meromorphic function, $a_i (i = 0,1,2,\ldots,p)$, $b_j (j = 0,1,2,\ldots,q)$ be constants such that $a_p \ne 0$ and $b_q \ne 0$ and let $\wp(f) = \sum_{i=0}^{p} a_i f^i / \sum_{j=0}^{q} b_j f^j$ be an irreducible rational function in f. Then $T(r,\wp(f)) = dT(r,f) + S(r,f)$, where $d = max\{p,q\}$.*

Lemma 2.3. *Let f and g be two nonconstant meromorphic functions and $s(\ge 2)$, n,m be positive integers such that $n > \frac{2}{m} + \frac{4}{m^2} - 1$. If f or g is meromorphic function having only multiple poles and the two expressions $\frac{\sum_{i=1}^{s} a_i}{n+s} g^{s-1} \sum_{\zeta=0}^{n+s-1} \left(\frac{f}{g}\right)^{\zeta} - \frac{\sum_{1\le i<j\le s} a_i a_j}{n+s-1} g^{s-2} \sum_{\zeta=0}^{n+s-2} \left(\frac{f}{g}\right)^{\zeta} + \ldots + \frac{(-1)^{s-1} a_1 a_2 \ldots a_s}{n+1} \sum_{\zeta=0}^{n} \left(\frac{f}{g}\right)^{\zeta}$ and $\sum_{\zeta=0}^{n+s} \left(\frac{f}{g}\right)^{\zeta}$ have no common simple zero, and*

$$\left(\frac{f^{n+s+1}}{n+s+1} - \frac{\sum_{i=1}^{s} a_i}{n+s} f^{n+s} + \frac{\sum_{1\le i<j\le s} a_i a_j}{n+s-1} f^{n+s-1} - \ldots + \frac{(-1)^s a_1 a_2 \ldots a_s}{n+1} f^{n+1}\right)^m$$
$$\equiv \left(\frac{g^{n+s+1}}{n+s+1} - \frac{\sum_{i=1}^{s} a_i}{n+s} g^{n+s} + \frac{\sum_{1\le i<j\le s} a_i a_j}{n+s-1} g^{n+s-1} - \ldots + \frac{(-1)^s a_1 a_2 \ldots a_s}{n+1} g^{n+1}\right)^m,$$

where $a_i \in \mathbb{C} \setminus \{0\}$, $i = 1,2,\ldots,s$, then $f \equiv tg$, where $t^m = 1$.

Proof. From the assumption of the lemma it follows that

$$\frac{f^{n+s+1}}{n+s+1} - \frac{\sum_{i=1}^{s} a_i}{n+s} f^{n+s} + \frac{\sum_{1\le i<j\le s} a_i a_j}{n+s-1} f^{n+s-1} - \ldots + \frac{(-1)^s a_1 a_2 \ldots a_s}{n+1} f^{n+1}$$
$$\equiv t\left(\frac{g^{n+s+1}}{n+s+1} - \frac{\sum_{i=1}^{s} a_i}{n+s} g^{n+s} + \frac{\sum_{1\le i<j\le s} a_i a_j}{n+s-1} g^{n+s-1} - \ldots + \frac{(-1)^s a_1 a_2 \ldots a_s}{n+1} g^{n+1}\right), \tag{2.1}$$

where $t^m = 1$. From (2.1) we get that f and g share ∞ CM. Without loss of generality, from the assumption of the lemma we may suppose that g has some multiple poles. Let $h = \frac{f}{g}$. Suppose that h is not constant. Then from (2.1), we have

$$A_0 g^s (h^{n+s+1} - t) + A_1 g^{s-1} (h^{n+s} - t) + \ldots + A_{s-1} g (h^{n+2} - t) + A_s (h^{n+1} - t) \equiv 0,$$

$$\text{i.e.,} \quad A_0 g^s = -A_1 g^{s-1} \frac{(h^{n+s} - t)}{(h^{n+s+1} - t)} - A_2 g^{s-2} \frac{(h^{n+s-1} - t)}{(h^{n+s+1} - t)} - \ldots$$

$$-A_s\frac{(h^{n+1}-t)}{(h^{n+s+1}-t)}, \tag{2.2}$$

where $A_0=\frac{1}{n+s+1},\ A_1=-\frac{\sum\limits_{i=1}^{s}a_i}{n+s},\ A_2=\frac{\sum\limits_{1\le i<j\le s}a_ia_j}{n+s-1},\ldots,\ A_s=\frac{(-1)^sa_1a_2\ldots a_s}{n+1}$.

Let z_0 be a pole of g with multiplicity $p_0(\ge 2)$, which is not a zero of $h-v_{k,r}$, where $(v_{k,r})^{n+s+1}=t=\omega^r$ $(k=0,1,2,...,n+s; r=0,1,2,...,m-1)$ such that $\omega=\cos\frac{2\pi}{m}+i\sin\frac{2\pi}{m}$. Then from (2.2), we have $sp_0=(s-1)p_0$ i.e., $p_0=0$. Thus we arrive at a contradiction. Therefore, we see that the poles of g are precisely the zeros of $h-v_{k,r}$.

Let z_1 be a zero of $h-v_{k,r}$ with multiplicity q_1, which is a pole of g with multiplicity p_1. Then from (2.2), we get $sp_1=q_1+(s-1)p_1$ i.e., $p_1=q_1$. Since g has no simple pole, it follows that such points are multiple zeros of $h-v_{k,r}$. Now, for $r=0,1,2,...,m-1$, we obtain from (2.2) that

$$A_0g^s=-\frac{A_1g^{s-1}(h^{n+s}-\omega^r)+A_2g^{s-2}(h^{n+s-1}-\omega^r)+\ldots+A_s(h^{n+1}-\omega^r)}{h^{n+s+1}-\omega^r}. \tag{2.3}$$

Suppose that z_2 is a simple zero of $h-v_{k,r}$ $(k=0,1,2,\ldots,n+s;\ r=0,1,2,...,m-1)$ which is also a zero of multiplicity $q_2(\ge 2)$ of numerator of right hand side of (2.3). Then from (2.3), we see that z_2 would be a zero of g^s of order (q_2-1). Therefore z_2 would be a zero of $(h^{n+1}-\omega^r)$. We observe that the number of common factors of $(h^{n+1}-\omega^r)$ and $(h^{n+s+1}-\omega^r)$ are less than or equal to the number of common factors of $(h^{m(n+1)}-1)$ and $(h^{m(n+s+1)}-1)$ for $r=0,1,2,...,m-1$. Since the greatest common divisor of $m(n+1)$ and $m(n+s+1)$ cannot exceed sm, it follows that $h^{n+1}-\omega^r$ and $h^{n+s+1}-\omega^r$ may have at most sm common factors for $r=0,1,2,...,m-1$. Moreover, a nonconstant meromorphic function cannot have more than two Picard exceptional values. Therefore we see that $h-v_{k,r}$ has multiple zeros for at least $m(n+s+1)-sm-2$ values of $k\in\{0,1,2,...,m(n+s+1)-1\}$ when $r=0,1,2,...,m-1$. Thus, $\Theta(v_{k,r};h)\ge\frac{1}{2}$ for at least $m(n+s+1)-sm-2$ values of $k\in\{0,1,2,...,m(n+s+1)-1\}$ where $r=0,1,2,...,m-1$, which is a contradiction as $n>\frac{2}{m}+\frac{4}{m^2}-1$. Hence h is a constant. If $h\ne t$, then from (2.2), it follows that the function g becomes a constant, which is impossible. Hence we get $f\equiv tg$, where $t^m=1$. This completes the proof of the lemma. □

Lemma 2.4. *(see [2]) Let f and g be two nonconstant meromorphic functions, and let k be a positive integer. If $E_{k)}(1,f)=E_{k)}(1,g)$, then one of the following cases holds:*

$$\begin{aligned}(i)\quad T(r,f)+T(r,g) \le\ & N_2(r,\infty;f)+N_2(r,0;f)+N_2(r,\infty;g)+N_2(r,0;g)\\ &+\overline{N}(r,1;f)+\overline{N}(r,1;g)-\overline{N}^{1)}_E(r,1;f)\\ &+\overline{N}(r,1;f|\ge k+1)+\overline{N}(r,1;g|\ge k+1)\\ &+S(r,f)+S(r,g),\end{aligned}$$

(ii) $f=\frac{(B+1)g+(A-B-1)}{Bg+(A-B)}$, *where $A(\ne 0), B$ are constants.*

Lemma 2.5. *Let f and g be two nonconstant meromorphic functions and n, m, μ, $s(\geq \mu)$, $s_i(i=1,2,\ldots,\mu)$ be any positive integers such that $n+s > 1+\frac{4s}{\mu}$. Then $(f^n(f-a_1)^{s_1}(f-a_2)^{s_2}\ldots(f-a_\mu)^{s_\mu}f')^m\,(g^n(g-a_1)^{s_1}(g-a_2)^{s_2}\ldots(g-a_\mu)^{s_\mu}g')^m \not\equiv 1$, where $a_i \in \mathbb{C}\setminus\{0\}$, $a_i \neq a_j$ $(i,j=1,2,\ldots,\mu)$ and $\sum_{i=1}^{\mu} s_i = s$.*

Proof. Let, if possible

$$\left(f^n(f-a_1)^{s_1}(f-a_2)^{s_2}\ldots(f-a_\mu)^{s_\mu}f'\right)^m\left(g^n(g-a_1)^{s_1}(g-a_2)^{s_2}...(g-a_\mu)^{s_\mu}g'\right)^m \equiv 1.$$

Therefore we must have

$$f^n(f-a_1)^{s_1}(f-a_2)^{s_2}\ldots(f-a_\mu)^{s_\mu}f'g^n(g-a_1)^{s_1}(g-a_2)^{s_2}\ldots(g-a_\mu)^{s_\mu}g' \equiv t, \tag{2.4}$$

where $t^m = 1$.

Since each $a_i \neq a_j$ (i,j=1,2,...,μ), we may count the zeros and poles of f and g as follows: Let z_0 be a zero of f with multiplicity $p_0(\geq 1)$. Then, from (2.4) it follows that z_0 is a pole of g with multiplicity $q_0(\geq 1)$, say. Then $(n+1)p_0 - 1 = (n+s+1)q_0 + 1$ i.e., $sq_0 = (n+1)(p_0-q_0) - 2 \geq n-1$, i.e., $q_0 \geq \frac{n-1}{s}$. Hence

$$(n+1)p_0 \geq \frac{(n+s+1)(n-1)}{s} + 2 \;\text{ i.e., }\; p_0 \geq \frac{n+s-1}{s}. \tag{2.5}$$

Let z_i be a zero of $f-a_i$ $(i=1,2,\ldots,\mu)$ with multiplicity $p_i(\geq 1)$. Then it is a pole of g with multiplicity $q_i(\geq 1)$, say. Therefore we have $s_ip_i + p_i - 1 = (n+s+1)q_i + 1 \geq n+s+2$

$$\text{i.e., } \quad p_i \geq \frac{n+s+3}{s_i+1} \;\text{ for }\; i=1,2,\ldots,\mu. \tag{2.6}$$

Since a pole of f is either a zero of $g(g-a_1)(g-a_2)\ldots(g-a_\mu)$ or a zero of g', using (2.5), (2.6) and $\sum_{i=1}^{\mu} s_i = s$, we get

$$\begin{aligned}
\overline{N}(r,\infty;f) &\leq \overline{N}(r,0;g) + \sum_{i=1}^{\mu}\overline{N}(r,a_i;g) + \overline{N}_0(r,0;g') + S(r,f) + S(r,g)\\
&\leq \frac{s}{n+s-1}N(r,0;g) + \sum_{i=1}^{\mu}\frac{s_i+1}{n+s+3}N(r,a_i;g) + \overline{N}_0(r,0;g')\\
&\quad +S(r,f) + S(r,g)\\
&\leq \left(\frac{s}{n+s-1} + \frac{\mu+s}{n+s+3}\right)T(r,g) + \overline{N}_0(r,0;g') + S(r,f)\\
&\quad +S(r,g),
\end{aligned} \tag{2.7}$$

where $\overline{N}_0(r,0;g')$ denotes the reduced counting function of those zeros of g' which are not the zeros of $g(g-a_1)(g-a_2)\ldots(g-a_\mu)$. By the second fundamental theorem of Nevanlinna and from (2.5)-(2.7) we get

$$\mu T(r,f) \leq \overline{N}(r,\infty;f) + \overline{N}(r,0;f) + \sum_{i=1}^{\mu}\overline{N}(r,a_i;f) - \overline{N}_0(r,0;f') + S(r,f)$$

$$\begin{aligned} &\le \left(\frac{s}{n+s-1}+\frac{\mu+s}{n+s+3}\right)\{T(r,f)+T(r,g)\}+\overline{N}_0(r,0;g') \\ &\quad -\overline{N}_0(r,0;f')+S(r,f)+S(r,g). \end{aligned} \tag{2.8}$$

Similarly, we get

$$\begin{aligned} \mu T(r,g) &\le \left(\frac{s}{n+s-1}+\frac{\mu+s}{n+s+3}\right)\{T(r,f)+T(r,g)\}+\overline{N}_0(r,0;f') \\ &\quad -\overline{N}_0(r,0;g')+S(r,f)+S(r,g). \end{aligned} \tag{2.9}$$

Adding (2.8) and (2.9) we obtain

$$\left(\mu-\frac{2s}{n+s-1}-\frac{2(\mu+s)}{n+s+3}\right)\{T(r,f)+T(r,g)\} \le S(r,f)+S(r,g).$$

Since $n+s>1+\frac{4s}{\mu}$, we arrive at a contradiction. This proves the lemma. $\square$

Lemma 2.6. *Let f and g be two nonconstant meromorphic functions with one of f,g has only multiple poles and $n,m(\ge 2),\mu(\ge 2),s(\ge \mu)$ be positive integers such that $n>s+2+max\{0,\frac{3-s+\mu}{m}\}$. Let $F=f^nP(f)f'$ and $G=g^nP(g)g'$ and let $\frac{\sum_{i=1}^{s}a_i'}{n+s}g^{s-1}\sum_{\zeta=0}^{n+s-1}\left(\frac{f}{g}\right)^{\zeta}-\frac{\sum_{1\le i<j\le s}a_i'a_j'}{n+s-1}g^{s-2}\sum_{\zeta=0}^{n+s-2}\left(\frac{f}{g}\right)^{\zeta}+\ldots+\frac{(-1)^{s-1}a_1'a_2'\ldots a_s'}{n+1}\sum_{\zeta=0}^{n}\left(\frac{f}{g}\right)^{\zeta}$ and $\sum_{\zeta=0}^{n+s}\left(\frac{f}{g}\right)^{\zeta}$ have no common simple zero, and*

$$F^m=\frac{(B+1)G^m+A-B-1}{BG^m+A-B}, \tag{2.10}$$

where $A(\ne 0),B$ are constants, then $f\equiv tg$, where $t^m=1$.

Proof. Let

$$\begin{aligned} R(z) &= \frac{z^{n+s+1}}{n+s+1}-\frac{\sum_{i=1}^{s}a_i'}{n+s}z^{n+s}+\frac{\sum_{1\le i<j\le s}a_i'a_j'}{n+s-1}z^{n+s-1} \\ &\quad -\ldots+\frac{(-1)^s a_1'a_2'\ldots a_s'}{n+1}z^{n+1}. \end{aligned} \tag{2.11}$$

Then we have

$$\begin{aligned} F&=(R(f))'=f^n(f-a_1)^{s_1}(f-a_2)^{s_2}\ldots(f-a_\mu)^{s_\mu}f', \\ G&=(R(g))'=g^n(g-a_1)^{s_1}(g-a_2)^{s_2}\ldots(g-a_\mu)^{s_\mu}g'. \end{aligned} \tag{2.12}$$

Using Lemmas 2.1, 2.2 and $\sum_{i=1}^{\mu}s_i=s$ we obtain

$$\begin{aligned} T(r,F) &\le T(r,f^n(f-a_1)^{s_1}(f-a_2)^{s_2}...(f-a_\mu)^{s_\mu})+T(r,f') \\ &\le (n+s+2)T(r,f)+S(r,f). \end{aligned}$$

Since $\sum_{i=1}^{\mu} s_i = s$, we have

$$\begin{aligned}(n+s)T(r,f) &= T(r, f^n(f-a_1)^{s_1}(f-a_2)^{s_2}...(f-a_\mu)^{s_\mu}) \\ &\leq T(r,F) + T(r,f') + O(1) \\ &\leq T(r,F) + 2T(r,f) + S(r,f).\end{aligned}$$

So, we have

$$(n+s-2)T(r,f) + S(r,f) \leq T(r,F) \leq (n+s+2)T(r,f) + S(r,f).$$

Hence we get $S(r,F) = S(r,f)$. Similarly, we can get $S(r,G) = S(r,g)$.
Now employing Lemma 2.2 we get

$$\begin{aligned}(n+s+1)T(r,f) &= T(r, R(f)) \\ &\leq T(r,(R(f))') + N(r,0;R(f)) - N(r,0;(R(f))') + S(r,f) \\ &= T(r,F) + N(r,0;f) + \sum_{j=1}^{s} N(r,\alpha_j;f) - \sum_{i=1}^{\mu} s_i N(r,a_i;f) \\ &\quad -N(r,0;f') + S(r,f), \qquad (2.13)\end{aligned}$$

where $\alpha_1, \alpha_2, \dots, \alpha_s$ are the roots of the equation

$$\frac{1}{n+s+1}z^s = \frac{\sum\limits_{i=1}^{s} a_i'}{n+s}z^{s-1} - \frac{\sum\limits_{1\leq i<j\leq s} a_i'a_j'}{n+s-1}z^{s-2} + \ldots + (-1)^{s-1}\frac{a_1'a_2'\ldots a_s'}{n+1}.$$

Similarly, we get

$$\begin{aligned}(n+s+1)T(r,g) &\leq T(r,G) + N(r,0;g) + \sum_{j=1}^{s} N(r,\alpha_j;g) - \sum_{i=1}^{\mu} s_i N(r,a_i;g) \\ &\quad -N(r,0;g') + S(r,g). \qquad (2.14)\end{aligned}$$

Now, without any loss of generality we suppose that there exists a set M with infinite measure such that $T(r,g) \leq T(r,f), r \in M$. We now consider three cases as follows:

Case I. Suppose $B \neq 0, -1$. Then from (2.10), we see that $\overline{N}(r, \frac{B+1}{B}; F^m) = \overline{N}(r,\infty;G^m)$. By the second fundamental theorem of Nevanlinna and $S(r,F^m) = S(r,f)$, we get

$$\begin{aligned}mT(r,F) &= T(r,F^m) \\ &\leq \overline{N}(r,\infty;F^m) + \overline{N}(r,0;F^m) + \overline{N}\left(r, \frac{B+1}{B}; F^m\right) + S(r,f) \\ &= \overline{N}(r,\infty;F^m) + \overline{N}(r,0;F^m) + \overline{N}(r,\infty;G^m) + S(r,f) \\ &\leq \overline{N}(r,\infty;f) + \overline{N}(r,0;f) + \sum_{i=1}^{\mu} \overline{N}(r,a_i;f) + \overline{N}(r,0;f')\end{aligned}$$

$$+\overline{N}(r,\infty;g)+S(r,f). \tag{2.15}$$

Using (2.13), (2.15) and noting that $\sum_{i=1}^{\mu} s_i = s$, we get

$$\begin{aligned}(n+s+1)T(r,f) &\leq \frac{1}{m}\overline{N}(r,\infty;f)+\left(1+\frac{1}{m}\right)N(r,0;f)+\sum_{j=1}^{s}N(r,\alpha_j;f)\\ &\quad+\sum_{i=1}^{\mu}\left(\frac{1}{m}-s_i\right)N(r,a_i;f)+\frac{1}{m}\overline{N}(r,\infty;g)+S(r,f)\\ &\leq \left(s+1+\frac{2}{m}\right)T(r,f)+\frac{1}{m}T(r,g)+S(r,f)\end{aligned}$$

i.e., $\left(n-\frac{3}{m}\right)T(r,f)\leq S(r,f)$, a contradiction as $n > s+2+\max\{0,\frac{3-s+\mu}{m}\} > \frac{3}{m}$.

Case II. Suppose that $B=0$. From (2.10), we get that $\overline{N}(r,\frac{A-1}{A};F^m) = \overline{N}(r,0;G^m)$. We consider the following two subcases:

Subase (i). $A \neq 1$. Then similarly as in (2.15), using Nevanlinna's second fundamental theorem and $S(r,F^m)=S(r,f)$, we get

$$\begin{aligned}mT(r,F) &\leq \overline{N}(r,\infty;F^m)+\overline{N}(r,0;F^m)+\overline{N}\left(r,\frac{A-1}{A};F^m\right)+S(r,f)\\ &\leq \overline{N}(r,\infty;f)+\overline{N}(r,0;f)+\sum_{i=1}^{\mu}\overline{N}(r,a_i;f)+\overline{N}(r,0;f')+\overline{N}(r,0;g)\\ &\quad+\sum_{i=1}^{\mu}\overline{N}(r,a_i;g)+\overline{N}(r,0;g')+S(r,f).\end{aligned} \tag{2.16}$$

Using $\sum_{i=1}^{\mu} s_i = s$ we get from (2.13) and (2.16),

$$\begin{aligned}(n+s+1)T(r,f) &\leq \left(s+1+\frac{2}{m}\right)T(r,f)+\left(\frac{\mu+3}{m}\right)T(r,g)+S(r,f)\\ &\leq \left(s+1+\frac{\mu+5}{m}\right)T(r,f)+S(r,f)\end{aligned}$$

i.e., $\left(n-\frac{5+\mu}{m}\right)T(r,f)\leq S(r,f)$, a contradiction as $n > s+2+max\{0,\frac{3-s+\mu}{m}\} > \frac{5+\mu}{m}$.

Subase (ii). $A=1$. Then from (2.10), we obtain $F^m = G^m$, i.e., $F=tG$, where $t^m=1$. Integrating we get, $R(f)=tR(g)+t_0$, where t_0 is a constant.

If $t_0 \neq 0$, by Nevanlinna's second fundamental theorem and Lemma 2.2 we get

$$\begin{aligned}(n+s+1)T(r,f) &= T(r,R(f))\\ &\leq \overline{N}(r,\infty;R(f))+\overline{N}(r,0;R(f))+\overline{N}(r,t_0;R(f))+S(r,f)\\ &\leq \overline{N}(r,\infty;R(f))+\overline{N}(r,0;R(f))+\overline{N}(r,0;R(g))+S(r,f)\\ &\leq \overline{N}(r,\infty;f)+N(r,0;f)+\sum_{j=1}^{s}N(r,\alpha_j;f)+N(r,0;g)\end{aligned}$$

$$
\begin{aligned}
& +\sum_{j=1}^{s} N(r,\alpha_j;g)+S(r,f)\\
\leq\ & (s+2)T(r,f)+(s+1)T(r,g)+S(r,f)\\
\leq\ & (2s+3)T(r,f)+S(r,f)
\end{aligned}
$$

i.e., $(n-s-2)T(r,f)\leq S(r,f)$, a contradiction as $n>s+2+max\{0,\frac{3-s+\mu}{m}\}$. Thus $t_0=0$ and hence $R(f)=tR(g)$. Then by Lemma 2.3, we get $f\equiv tg$, where $t^m=1$.

Case III. Let $B=-1$. Proceeding in a similar manner as in the proof of Case II, we can get $F^mG^m\equiv 1$, a contradiction by Lemma 2.5. This completes the proof of the lemma. □

Lemma 2.7. *Let f and g be two nonconstant meromorphic functions and $n,m,s(>1)$ be positive integers such that $n>3s+1$. Let $F=f^n(f-a)^sf'$ and $G=g^n(g-a)^sg'$, where $a\in\mathbb{C}\setminus\{0\}$, and let one of f and g is meromorphic function having and only having multiple poles. If the two expressions* ${}^sC_1\dfrac{a}{n+s}g^{s-1}\displaystyle\sum_{\zeta=0}^{n+s-1}\left(\frac{f}{g}\right)^{\zeta}-{}^sC_2\dfrac{a^2}{n+s-1}g^{s-2}\sum_{\zeta=0}^{n+s-2}\left(\frac{f}{g}\right)^{\zeta}+\ldots+\frac{(-1)^{s-1}a^s}{n+1}\sum_{\zeta=0}^{n}\left(\frac{f}{g}\right)^{\zeta}$ *and* $\displaystyle\sum_{\zeta=0}^{n+s}\left(\frac{f}{g}\right)^{\zeta}$ *have no common simple zero, and (2.10) holds, then $f\equiv tg$, where $t^m=1$.*

Proof. Let

$$
\begin{aligned}
R_1(z) \;=\;& \frac{1}{n+s+1}z^{n+s+1}-{}^sC_1\frac{a}{n+s}z^{n+s}+{}^sC_2\frac{a^2}{n+s-1}z^{n+s-1}\\
& -\ldots+\frac{(-1)^s a^s}{n+1}z^{n+1}.
\end{aligned}
$$

Then we have $F=(R_1(f))'=f^n(f-a)^sf'$ and $G=(R_1(g))'=g^n(g-a)^sg'$. Now proceeding similarly as in the proof of Lemma 2.6, and using Lemmas 2.3 and 2.5 we can deduce the conclusion of the lemma. □

3 Proof of the Theorems

Proof of Theorem 1.1. Let F and G be given as in equation (2.12) and $R(z)$ as in (2.11). From the assumptions of Theorem 1.1 we have, $E_{k)}(S_m,F)=E_{k)}(S_m,G)$ i.e., $E_{k)}(1,F^m)=E_{k)}(1,G^m)$. It is obvious that

$$
\begin{aligned}
N_2(r,0;F^m)+N_2(r,\infty;F^m) \;\leq\;& 2N(r,0;f)+2\sum_{i=1}^{\mu}\overline{N}(r,a_i;f)+2\overline{N}(r,0;f')\\
& +2\overline{N}(r,\infty;f)+S(r,f), \qquad (3.1)
\end{aligned}
$$

and

$$
N_2(r,0;G^m)+N_2(r,\infty;G^m) \;\leq\; 2N(r,0;g)+2\sum_{i=1}^{\mu}\overline{N}(r,a_i;g)+2\overline{N}(r,0;g')
$$

$$+2\overline{N}(r,\infty;g)+S(r,g). \tag{3.2}$$

Without loss of generality we may assume that the first η elements in the sequence $\{s_1, s_2, \ldots, s_\mu\}$ are greater than 1. We now discuss the following three cases.

Case 1. Let $k \geq 3$. We have

$$\begin{aligned}
& \overline{N}(r,1;F^m)+\overline{N}(r,1;G^m)+\overline{N}(r,1;F^m|\geq k+1) \\
& +\overline{N}(r,1;G^m|\geq k+1)-\overline{N}_E^{1)}(r,1;F^m) \\
\leq\ & \frac{1}{2}N(r,1;F^m)+\frac{1}{2}N(r,1;G^m)+S(r,F^m)+S(r,G^m) \\
\leq\ & \frac{m}{2}T(r,F)+\frac{m}{2}T(r,G)+S(r,F)+S(r,G). \qquad (3.3)
\end{aligned}$$

Let us suppose that F^m and G^m satisfy (i) of Lemma 2.4. Then using Lemma 2.2 and (3.3) we obtain

$$\begin{aligned}
mT(r,F)+mT(r,G) &= T(r,F^m)+T(r,G^m) \\
&\leq \frac{m}{2}T(r,F)+\frac{m}{2}T(r,G)+N_2(r,0;F^m)+N_2(r,\infty;F^m) \\
&\quad +N_2(r,0;G^m)+N_2(r,\infty;G^m)+S(r,F)+S(r,G)
\end{aligned}$$

$$\begin{aligned}
\text{i.e.,}\quad T(r,F)+T(r,G) &\leq \frac{2}{m}N_2(r,\infty;F^m)+\frac{2}{m}N_2(r,0;F^m)+\frac{2}{m}N_2(r,\infty;G^m) \\
&\quad +\frac{2}{m}N_2(r,0;G^m)+S(r,F)+S(r,G). \qquad (3.4)
\end{aligned}$$

We now consider the following two subcases.

Subcase 1.1 Assume that $2 \leq m \leq 3$. Then $\left(\frac{4}{m}-s_i\right) \leq 0$, when $s_i \geq 2$ $(i = 1, 2, \ldots, \eta)$ and $\left(\frac{4}{m}-s_i\right) \leq 1$, when $s_i = 1$ $(i = \eta+1, \eta+2, \ldots, \mu)$. From (2.13), (2.14), (3.1), (3.2), (3.4) and noting that $\mu' = \mu - \eta$ we obtain

$$\begin{aligned}
& (n+s+1)T(r,f)+(n+s+1)T(r,g) \\
\leq\ & \left(1+\frac{4}{m}\right)N(r,0;f)+N(r,0;f')+\sum_{j=1}^{s}N(r,\alpha_j;f)+\sum_{i=\eta+1}^{\mu}N(r,a_i;f) \\
& +\frac{4}{m}\overline{N}(r,\infty;f)+\left(1+\frac{4}{m}\right)N(r,0;g)+N(r,0;g')+\sum_{j=1}^{s}N(r,\alpha_j;g) \\
& +\sum_{i=\eta+1}^{\mu}N(r,a_i;g)+\frac{4}{m}\overline{N}(r,\infty;g)+S(r,f)+S(r,g) \\
\leq\ & \left(s+\mu'+3+\frac{8}{m}\right)T(r,f)+\left(s+\mu'+3+\frac{8}{m}\right)T(r,g)+S(r,f)+S(r,g),
\end{aligned}$$

which implies

$$\left(n-\mu'-2-\frac{8}{m}\right)\{T(r,f)+T(r,g)\} \leq S(r,f)+S(r,g). \tag{3.5}$$

Since $n > max\{\mu' + 2 + \frac{8}{m},\ s + 2 + max\{0, \frac{3-s+\mu}{m}\}\}$, we get a contradiction from (3.5). Then from Lemma 2.4, we have

$$F^m = \frac{(B+1)G^m + A - B - 1}{BG^m + A - B},$$

where $A(\neq 0), B$ are constants. Thus, by Lemma 2.6 and $n > max\{\mu' + 2 + \frac{8}{m},\ s + 2 + max\{0, \frac{3-s+\mu}{m}\}\}$, we get $f \equiv tg$, where $t^m = 1$.

Subcase 1.2 Next we assume that $m \geq 4$. Then $\left(\frac{4}{m} - s_i\right) \leq 0$ for $s_i \geq 1$ $(i = 1, 2, \ldots, \mu)$. From (2.13), (2.14), (3.1), (3.2) and (3.4) we obtain

$$\begin{aligned}
& (n+s+1)T(r,f) + (n+s+1)T(r,g) \\
\leq\ & \left(1 + \frac{4}{m}\right) N(r,0;f) + \sum_{j=1}^{s} N(r,\alpha_j;f) + \frac{4}{m}\overline{N}(r,\infty;f) + \left(1 + \frac{4}{m}\right) N(r,0;g) \\
& + \sum_{j=1}^{s} N(r,\alpha_j;g) + \frac{4}{m}\overline{N}(r,\infty;g) + S(r,f) + S(r,g) \\
\leq\ & \left(s + 1 + \frac{8}{m}\right) T(r,f) + \left(s + 1 + \frac{8}{m}\right) T(r,g) + S(r,f) + S(r,g)
\end{aligned}$$

From this we obtain

$$\left(n - \frac{8}{m}\right)\{T(r,f) + T(r,g)\} \leq S(r,f) + S(r,g), \tag{3.6}$$

a contradiction since $n > s + 2 + max\{0, \frac{3-s+\mu}{m}\} > \frac{8}{m}$. So from Lemma 2.4, we obtain that (2.10) holds. Thus, by Lemma 2.6, we have $f \equiv tg$, where $t^m = 1$.

Case 2. Let $k = 2$. We can easily see that

$$\begin{aligned}
& \overline{N}(r,1;F^m) + \overline{N}(r,1;G^m) + \frac{1}{2}\overline{N}(r,1;F^m| \geq 3) + \frac{1}{2}\overline{N}(r,1;G^m| \geq 3) \\
& -\overline{N}_E^{1)}(r,1;F^m) \\
\leq\ & \frac{1}{2}N(r,1;F^m) + \frac{1}{2}N(r,1;G^m) + S(r,F) + S(r,G) \\
\leq\ & \frac{m}{2}T(r,F) + \frac{m}{2}T(r,G) + S(r,F) + S(r,G). \qquad (3.7)
\end{aligned}$$

Assume that F^m and G^m satisfy (i) of Lemma 2.4. Then using Lemma 2.2 and (3.7) we obtain

$$\begin{aligned}
mT(r,F) + mT(r,G) &= T(r,F^m) + T(r,G^m) \\
&\leq \frac{m}{2}T(r,F) + \frac{m}{2}T(r,G) + N_2(r,0;F^m) + N_2(r,\infty;F^m) \\
&\quad + N_2(r,0;G^m) + N_2(r,\infty;G^m) + \frac{1}{2}\overline{N}(r,1;F^m| \geq 3) \\
&\quad + \frac{1}{2}\overline{N}(r,1;G^m| \geq 3) + S(r,F) + S(r,G),
\end{aligned}$$

$$\text{i.e.,}\quad T(r,F)+T(r,G) \;\leq\; \frac{2}{m}N_2(r,\infty;F^m)+\frac{2}{m}N_2(r,0;F^m)+\frac{2}{m}N_2(r,\infty;G^m)$$
$$+\frac{2}{m}N_2(r,0;G^m)+\frac{1}{m}\overline{N}(r,1;F^m|\geq 3)$$
$$+\frac{1}{m}\overline{N}(r,1;G^m|\geq 3)+S(r,F)+S(r,G). \tag{3.8}$$

Now

$$\begin{aligned}\overline{N}(r,1;F^m|\geq 3) &\leq \frac{1}{2}N\left(r,\infty;\frac{F^m}{(F^m)'}\right)\leq \frac{1}{2}N\left(r,\infty;\frac{(F^m)'}{F^m}\right)+S(r,F)\\ &\leq \frac{1}{2}\overline{N}(r,\infty;F^m)+\frac{1}{2}\overline{N}(r,0;F^m)+S(r,F)\\ &\leq \frac{1}{2}\overline{N}(r,\infty;f)+\frac{1}{2}\overline{N}(r,0;f)+\frac{1}{2}\sum_{i=1}^{\mu}\overline{N}(r,a_i;f)\\ &\quad+\frac{1}{2}\overline{N}(r,0;f')+S(r,f). \end{aligned}\tag{3.9}$$

Similarly,

$$\begin{aligned}\overline{N}(r,1;G^m|\geq 3) &\leq \frac{1}{2}\overline{N}(r,\infty;g)+\frac{1}{2}\overline{N}(r,0;g)+\frac{1}{2}\sum_{i=1}^{\mu}\overline{N}(r,a_i;g)\\ &\quad+\frac{1}{2}\overline{N}(r,0;g')+S(r,g). \end{aligned}\tag{3.10}$$

We now consider the following two subcases.

Subcase 2.1 We assume that $2\leq m\leq 3$. Then $\left(\frac{4}{m}-s_i\right)\leq 0$, when $s_i\geq 2$ $(i=1,2,\ldots,\eta)$ and $\left(\frac{4}{m}-s_i\right)\leq 1$, when $s_i=1$ $(i=\eta+1,\eta+2,\ldots,\mu)$. From (2.13), (2.14), (3.1), (3.2) and (3.8)-(3.10) we obtain

$$\begin{aligned}&(n+s+1)T(r,f)+(n+s+1)T(r,g)\\ \leq\;& (1+\frac{9}{2m})N(r,0;f)+\sum_{j=1}^{s}N(r,\alpha_j;f)+\sum_{i=\eta+1}^{\mu}N(r,a_i;f)+\frac{1}{2m}\sum_{i=1}^{\mu}N(r,a_i;f)\\ &+\frac{9}{2m}\overline{N}(r,\infty;f)+\left(1+\frac{1}{2m}\right)N(r,0;f')+\sum_{j=1}^{s}N(r,\alpha_j;g)+\sum_{i=\eta+1}^{\mu}N(r,a_i;g)\\ &+\frac{1}{2m}\sum_{i=1}^{\mu}N(r,a_i;g)+\frac{9}{2m}\overline{N}(r,\infty;g)+\left(1+\frac{9}{2m}\right)N(r,0;g)\\ &+\left(1+\frac{1}{2m}\right)N(r,0;g')+S(r,f)+S(r,g)\\ \leq\;& \left(s+\mu'+3+\frac{20+\mu}{2m}\right)T(r,f)+\left(s+\mu'+3+\frac{20+\mu}{2m}\right)T(r,g)\\ &+S(r,f)+S(r,g).\end{aligned}$$

Therefore we have

$$\left(n-\mu'-2-\frac{20+\mu}{2m}\right)\{T(r,f)+T(r,g)\}\leq S(r,f)+S(r,g),$$

contradicts with $n > max\{\mu'+2+\frac{20+\mu}{2m},\ s+2+max\{0,\frac{3-s+\mu}{m}\}\}$. Thus by Lemma 2.4, we obtain (2.10). Hence $f \equiv tg$, by Lemma 2.6, where $t^m = 1$.

Subcase 2.2 Next we assume that $m \geq 4$. Then $\left(\frac{4}{m} - s_i\right) \leq 0$ for $s_i \geq 1$ $(i = 1, 2, \ldots, \mu)$. Proceeding similarly as in Subcase 1.2, we obtain

$$\begin{aligned}
& (n+s+1)T(r,f) + (n+s+1)T(r,g) \\
\leq\ & \left(1+\frac{9}{2m}\right)N(r,0;f) + \sum_{j=1}^{s} N(r,\alpha_j;f) + \frac{1}{2m}\sum_{i=1}^{\mu} N(r,a_i;f) + \frac{9}{2m}\overline{N}(r,\infty;f) \\
& + \frac{1}{2m}N(r,0;f') + \left(1+\frac{9}{2m}\right)N(r,0;g) + \sum_{j=1}^{s} N(r,\alpha_j;g) + \frac{1}{2m}\sum_{i=1}^{\mu} N(r,a_i;g) \\
& + \frac{9}{2m}\overline{N}(r,\infty;g) + \frac{1}{2m}N(r,0;g') + S(r,f) + S(r,g)
\end{aligned}$$

$$\text{i.e.,}\quad \left(n - \frac{20+\mu}{2m}\right)\{T(r,f)+T(r,g)\} \leq S(r,f) + S(r,g).$$

Since $n > s+2+max\{0,\frac{3-s+\mu}{m}\} > \frac{20+\mu}{2m}$, we get a contradiction. Therefore, by (ii) of Lemma 2.4 and Lemma 2.6, we obtain $f \equiv tg$, where $t^m = 1$.

Case 3. Let $k = 1$. We have

$$\begin{aligned}
& \overline{N}(r,1;F^m) + \overline{N}(r,1;G^m) - \overline{N}_E^{1)}(r,1;F^m) \\
\leq\ & \frac{1}{2}N(r,1;F^m) + \frac{1}{2}N(r,1;G^m) + S(r,F) + S(r,G) \\
\leq\ & \frac{m}{2}T(r,F) + \frac{m}{2}T(r,G) + S(r,F) + S(r,G). \qquad (3.11)
\end{aligned}$$

Suppose that F^m and G^m satisfy (i) of Lemma 2.4. Then by Lemma 2.2 and (3.11) we obtain

$$\begin{aligned}
mT(r,F) + mT(r,G) &= T(r,F^m) + T(r,G^m) \\
&\leq \frac{m}{2}T(r,F) + \frac{m}{2}T(r,G) + N_2(r,0;F^m) + N_2(r,\infty;F^m) \\
&\quad + N_2(r,0;G^m) + N_2(r,\infty;G^m) + \overline{N}(r,1;F^m| \geq 2) \\
&\quad + \overline{N}(r,1;G^m| \geq 2) + S(r,F) + S(r,G).
\end{aligned}$$

$$\begin{aligned}
\text{i.e.,}\quad T(r,F) + T(r,G) &\leq \frac{2}{m}N_2(r,\infty;F^m) + \frac{2}{m}N_2(r,0;F^m) + \frac{2}{m}N_2(r,\infty;G^m) \\
&\quad + \frac{2}{m}N_2(r,0;G^m) + \frac{2}{m}\overline{N}(r,1;F^m| \geq 2) \\
&\quad + \frac{2}{m}\overline{N}(r,1;G^m| \geq 2) + S(r,F) + S(r,G). \qquad (3.12)
\end{aligned}$$

Now

$$\overline{N}(r,1;F^m| \geq 2) \leq N\left(r,\infty;\frac{F^m}{(F^m)'}\right) \leq N\left(r,\infty;\frac{(F^m)'}{F^m}\right) + S(r,F)$$

$$\begin{aligned}
&\leq \overline{N}(r,\infty;F^m)+\overline{N}(r,0;F^m)+S(r,F)\\
&\leq \overline{N}(r,\infty;f)+\overline{N}(r,0;f)+\sum_{i=1}^{\mu}\overline{N}(r,a_i;f)\\
&\quad +\overline{N}(r,0;f')+S(r,f). \qquad (3.13)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\overline{N}(r,1;G^m|\geq 2) &\leq \overline{N}(r,\infty;g)+\overline{N}(r,0;g)++\sum_{i=1}^{\mu}\overline{N}(r,a_i;g)\\
&\quad +\overline{N}(r,0;g')+S(r,g). \qquad (3.14)
\end{aligned}$$

Subcase 3.1 We assume that $2\leq m\leq 3$. Then $\left(\frac{4}{m}-s_i\right)\leq 0$, when $s_i\geq 2$ $(i=1,2,\ldots,\eta)$ and $\left(\frac{4}{m}-s_i\right)\leq 1$, when $s_i=1$ $(i=\eta+1,\eta+2,\ldots,\mu)$. From (2.13), (2.14), (3.1), (3.2) and (3.12)-(3.14) we obtain

$$\begin{aligned}
&(n+s+1)T(r,f)+(n+s+1)T(r,g)\\
\leq\ & \left(1+\frac{6}{m}\right)N(r,0;f)+\sum_{j=1}^{s}N(r,\alpha_j;f)+\sum_{i=\eta+1}^{\mu}N(r,a_i;f)+\frac{2}{m}\sum_{i=1}^{\mu}N(r,a_i;f)\\
&+\frac{6}{m}\overline{N}(r,\infty;f)+\left(1+\frac{2}{m}\right)N(r,0;f')+\left(1+\frac{6}{m}\right)N(r,0;g)\\
&+\sum_{j=1}^{s}N(r,\alpha_j;g)+\sum_{i=\eta+1}^{\mu}N(r,a_i;g)+\frac{2}{m}\sum_{i=1}^{\mu}N(r,a_i;g)+\frac{6}{m}\overline{N}(r,\infty;g)\\
&+\left(1+\frac{2}{m}\right)N(r,0;g')+S(r,f)+S(r,g)\\
\leq\ & \left(s+\mu'+3+\frac{16+2\mu}{m}\right)T(r,f)+\left(s+\mu'+3+\frac{16+2\mu}{m}\right)T(r,g)\\
&+S(r,f)+S(r,g),
\end{aligned}$$

which implies

$$\left(n-\mu'-2-\frac{16+2\mu}{m}\right)\{T(r,f)+T(r,g)\}\leq S(r,f)+S(r,g).$$

Since $n>max\{\mu'+2+\frac{16+2\mu}{m},\ s+2+max\{0,\frac{3-s+\mu}{m}\}\}$, we get a contradiction. Then from Lemma 2.4, we conclude that (2.10) holds. Thus, applying Lemma 2.6 we obtain $f\equiv tg$, where $t^m=1$.

Subcase 3.2 Next we suppose that $m\geq 4$. Then proceeding similarly as in Subcase 1.2 and Subcase 3.1 we obtain

$$\begin{aligned}
&(n+s+1)T(r,f)+(n+s+1)T(r,g)\\
\leq\ & \left(1+\frac{6}{m}\right)N(r,0;f)+\sum_{j=1}^{s}N(r,\alpha_j;f)+\frac{2}{m}\sum_{i=1}^{\mu}N(r,a_i;f)+\frac{6}{m}\overline{N}(r,\infty;f)
\end{aligned}$$

$$+\frac{2}{m}N(r,0;f')+\left(1+\frac{6}{m}\right)N(r,0;g)+\sum_{j=1}^{s}N(r,\alpha_j;g)+\frac{2}{m}\sum_{i=1}^{\mu}N(r,a_i;g)$$
$$+\frac{6}{m}\overline{N}(r,\infty;g)+\frac{2}{m}N(r,0;g')+S(r,f)+S(r,g).$$

From this we have

$$\left(n-\frac{16+2\mu}{m}\right)\{T(r,f)+T(r,g)\}\leq S(r,f)+S(r,g),$$

a contradiction as $n > s+2+max\{0,\frac{3-s+\mu}{m}\}\geq\frac{16+2\mu}{m}$ for $m\geq 4$ and for all s satisfying $s>2, s\geq\mu$. Then using (ii) of Lemma 2.4 and Lemma 2.6 we obtain $f\equiv tg$, where $t^m=1$. This proves Theorem 1.1. □

Proof of Theorem 1.2. Arguing similarly as in the proof of Theorem 1.1 and applying Lemma 2.7, we can obtain the conclusion of the theorem. Here we omit the details. □

Acknowledgement. The authors are thankful to the referee and the editor for reading the manuscript carefully and making valuable comments and suggestions towards the improvement of the paper.

References

[1] Banerjee, A., *On uniqueness of meromorphic functions when two differential monomials share one value*, Bull. Korean Math. Soc. **44** (2007), 607-622.

[2] Fang, C.Y., and Fang, M.L., *Uniqueness of meromorphic functions and differential polynomials*, Comput. Math. Appl. **44** (2002), 607-617.

[3] Fang, M.L., and Hong, W., *A unicity theorem for entire functions concerning differential polynomials*, Indian J. Pure Appl. Math. **32** (2001), 1343-1348.

[4] Gundersen, G.G., *Meromorphic functions that share three or four values*, J. London Math. Soc. **20** (1979), 457-466.

[5] Hayman, W.K., *Meromorphic Functions*, The Clarendon Press, Oxford, 1964.

[6] Lahiri, I., *Uniqueness of meromorphic functions when two linear differential polynomials share the same* 1*-points*, Ann. Polon. Math. **71** (1999), 113-128.

[7] Lahiri, I., *Weighted value sharing and uniqueness of meromorphic functions*, Complex Var. Theory Appl. **46** (2001), 241-253.

[8] Lahiri, I., and Pal, R., *Nonlinear differential polynomials sharing* 1*-points*, Bull. Korean Math. Soc. **43** (2006), 161-168.

[9] Lahiri, I., and Sahoo, P., *Uniqueness of nonlinear differential polynomials sharing* 1*-points*, Georgian Math J. **12** (2005), 131-138.

[10] Laine, I., *Nevanlinna Theory and Complex Differential Equations*, Walter de Gruyter, Berlin, New York, 1993.

[11] Lin, W.C., and Yi, H.X., *Uniqueness theorems for meromorphic functions*, Indian J. Pure Appl. Math. **35** (2004), 121-132.

[12] Meng, C., *Uniqueness of meromorphic functions sharing one value*, Applied Math. E-Notes **7** (2007), 199-205.

[13] Mokhon'ko, A.Z., *On the Nevanlinna characteristics of some meromorphic functions*, Teor. Functs. Functs. Anal. Prilozh. **14** (1971), 83-87.

[14] Sahoo, P., and Biswas, G., *Differential polynomials of meromorphic functions sharing a set of values*, Filomat **32** (2018), 457-472.

[15] Sahoo, P., and Saha, B., *Uniqueness of meromorphic functions whose certain differential polynomials share a small function*, Acta Math Vietnamica **41** (2016), 25-36.

[16] Xu, H.Y., *The shared set of meromorphic functions and differential polynomials*, J. Comput. Anal. Appl. **16** (2014), 942-954.

[17] Xu, H.Y., Cao, T.B., and Liu, S., *Uniqueness results of meromorphic functions whose nonlinear differential polynomials have one nonzero pseudo value*, Mat. Vesnik **64** (2012), 1-16.

[18] Xu, H.Y., Yi, C.F., and Cao, T.B., *Uniqueness of meromorphic functions and differential polynomials sharing one value with finite weight*, Ann. Polon. Math. **95** (2009), 55-66.

[19] Yang, C.C., and Hua, X.H., *Uniqueness and value sharing of meromorphic functions*, Ann. Acad. Sci. Fenn. Math. **22** (1997), 395-406.

[20] Yang, C.C., and Yi, H.X., *Uniqueness theory of meromorphic functions*, Kluwer Academic Publishers, Dordrecht, Boston, London, 2003.