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ON THE EIGENGRAPH FOR p-BIHARMONIC EQUATIONS WITH RELLICH POTENTIALS AND WEIGHT

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Abstract

Using a variational technique and inequality of Hardy-Rellich, we prove the existence of infinitely many eigencurve sequences of the p-biharmonic operator involving a Rellich potentials. A variational formulation of the first curve (eigengraph) is given.

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1 Introduction

Nonlinear eigenvalue problems involving Rellich potential have been studied by many authors; see e.g. [2, 4, 6]. The study of eigencurve problems is a subject of several works, see [1, 5] and the references therein.

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We investigate in the present paper the following nonlinear eigenvalue problem

$$\begin{gathered}\Delta(|\Delta u|^{p-2}\Delta u) = \lambda w(x)\frac{|u|^{p-2}u}{\delta(x)^{2p}} + \mu\frac{|u|^{p-2}u}{\delta(x)^{2p}} \ \text{ in } \ \Omega,\\ u \in W_0^{2,p}(\Omega),\end{gathered} \tag{1}$$

where $\Delta_p^2 u := \Delta(|\Delta u|^{p-2}\Delta u)$ denotes the fourth order differential operator p-biharmonic, Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary $\partial\Omega$, $\delta(x) := \min_{y\in\partial\Omega}|x-y|$ denotes the distance between a given $x \in \Omega$ and the boundary $\partial\Omega$, $1 < p < \frac{N}{2}$, w is an indefinite weight in $L^\infty(\Omega)$ with (the Lebesgue measure)

$$\text{mes}\,(\{x \in \Omega : w(x) \neq 0\}) \neq 0. \tag{2}$$

λ is a real parameter such that $0 \leq \lambda < \frac{H}{\|w\|_\infty}$, where

$$H := \Big[\frac{N(p-1)(N-2p)}{p^2}\Big]^p,$$

is the best constant in the following classical Hardy's inequality (see[8]):

$$\int_\Omega |\Delta u|^p dx \geq H\int_\Omega \frac{|u|^p}{\delta(x)^{2p}}dx, \quad \forall u \in W_0^{2,p}(\Omega). \tag{3}$$

While μ stands for a function depending on λ generating the corresponding eigengraphs. More precisely, we mean by eigengraphs whose sets in $\mathbb{R}^2$ defined by $\{(\lambda, \mu(\lambda)$ such that $\lambda \in \mathbb{R}\}$.

In [6], authors have considered the case: $\mu = 0$ *and* $w(x) = 1$. For which problem (1) has a sequence of positive eigenvalues. The smallest eigenvalue λ_1 of $(\Delta_p^2, W_0^{2,p}(\Omega))$ is positive and admits the following variational characterization:

$$\lambda_1 = \inf\left\{\|\Delta v\|_p^p, v \in W_0^{2,p}(\Omega)\,\middle|\, \int_\Omega \frac{|v|^p}{\delta(x)^{2p}}\,dx = 1\right\}, \tag{4}$$

where $\|\Delta v\|_p = \left(\int_\Omega |\Delta v|^p\,dx\right)^{\frac{1}{p}}$ denotes the norm of $W_0^{2,p}(\Omega)$.

In this paper our result is partly motivated by these nice papers. More precisely, the Ljusternik-Schnirelmann principle on C^1-manifolds [9] provides a whole sequence of eigencurves $(\mu_k(\lambda))_{k\geq 1}$, such that $\mu_k(\lambda) \nearrow +\infty$.

The paper is organized as follows: In Section 2, we recall and we prove some preliminary results which will be used later. In Section 3, we establish the existence of at least one non-decreasing sequence of nonnegative eigencurve to problem (1).

2 Preliminaries and useful results

Let X be a real reflexive Banach space and X^* its topological dual with respect to the pairing $\langle\cdot,\cdot\rangle$. The strong convergence in X (and in X^*) is denoted by $\to$ and the weak convergence by $\rightharpoonup$.

We solve problem (1) in the space $X := W_0^{2,p}(\Omega)$ equipped with the norm

$$\|\Delta v\|_p := \left(\int_\Omega \mid \Delta v \mid^p dx\right)^{\frac{1}{p}}.$$

Let us notice that $W_0^{2,p}(\Omega)$ endowed with this norm is a uniformly convex Banach space for $1 < p < +\infty$. The norm $\|\Delta(\cdot)\|_p$ is uniformly equivalent on $W_0^{2,p}(\Omega)$ to the usual norm of $W_0^{2,p}(\Omega)$ [7].

By the compact embedding $W_0^{2,p}(\Omega) \hookrightarrow L^p(\Omega)$, there exists a positive constant K such that

$$\|u\|_{L^p(\Omega)} \leq K\|\Delta u\|_p \ \ \forall u \in W_0^{2,p}(\Omega),$$

where K is the best constant of the embedding.

We will introduce the following formulation involving a mini-max argument over sets of genus greater than k. We set

$$\mu_1(\lambda) = \inf\left\{\frac{\|\Delta u\|_p^p - \lambda\int_\Omega w(x)\frac{|u|^p}{\delta(x)^{2p}}\,dx}{\int_\Omega \frac{|u|^p}{\delta(x)^{2p}}dx}\,\middle|\, u \in W_0^{2,p}(\Omega)\setminus\{0\}\right\}. \tag{5}$$

Definition 1. *$u \in W_0^{2,p}(\Omega)$ is a weak solution of (1), if for all $v \in W_0^{2,p}(\Omega)$,*

$$\int_\Omega |\Delta u|^{p-2}\Delta u\Delta v\,dx = \int_\Omega \left(\lambda w(x) + \mu\right)\frac{|u|^{p-2}u}{\delta(x)^{2p}}v\,dx \ . \tag{6}$$

If $u \in W_0^{2,p}(\Omega)\backslash\{0\}$, then u shall be called an eigenfunction of (1) associated with the eigenpair (λ, μ).

Set

$$\mathcal{V} = \left\{u \in W_0^{2,p}(\Omega)\,\middle|\, \int_\Omega \frac{|u|^p}{\delta(x)^{2p}}\,dx = 1\right\}. \tag{7}$$

We say that a principal eigenfunction of (1), any eigenfunction $u \in \mathcal{V}$, $u \geq 0$ a.e. on $\overline{\Omega}$ associated to pair $(\lambda, \mu_1(\lambda))$. The graph of the function $\lambda \to \mu_1(\lambda)$ from $[0, \frac{C_H}{\|w\|_\infty}[$ into $\mathbb{R}$, where $\mu_1(\lambda)$ defined by (5), is called the principal eigengraph of problem(1).

Definition 2. A Gâteaux differentiable functional I satisfies the Palais-Smale condition (in short $(P.S)$-condition) if any sequence $\{u_n\}$ in $W_0^{2,p}(\Omega)$ such that

$(PS)_1$ $\{I(u_n)\}$ is bounded;

$(PS)_2$ $\lim_{n\to+\infty}\|I'(u_n)\|_{X^*} = 0$;

has a convergent subsequence.

The energy functional corresponding to problem (1) is defined on $W_0^{2,p}(\Omega)$ as

$$H(.) = \Phi(.) + \varphi(.) - \mu\Psi(.),$$

where

$$\Phi(u) = \frac{1}{p}\int_\Omega |\Delta u|^p dx,$$

$$\varphi(u) = -\frac{\lambda}{p}\int_\Omega w(x)\frac{|u|^p}{\delta(x)^{2p}}\, dx,$$

$$\Psi(u) = \frac{1}{p}\int_\Omega \frac{|u|^p}{\delta(x)^{2p}}\, dx,$$

and

$$\Phi_\lambda(.) = \Phi(.) + \varphi(.).$$

Lemma 1. *We have the following hold true:*

(i) Φ_λ, Ψ *and* φ *are even, and of class* C^1 *on* $W_0^{2,p}(\Omega)$.

(ii) $\mathcal{V}$ *is a closed* C^1*-manifold.*

Proof. (i). It is clear that Φ_λ, Ψ and φ are even and of class C^1 on $W_0^{2,p}(\Omega)$.
(ii). $\mathcal{V} = \Psi^{-1}\{\frac{1}{p}\}$. Therefore $\mathcal{V}$ is closed. The derivative operator Ψ' satisfies $\Psi'(u) \neq 0\ \forall u \in \mathcal{V}$, because

$$\langle \Psi'(u), u\rangle = \int_\Omega \frac{|u|^p}{\delta(x)^{2p}}\, dx = 1 \neq 0,\ \ if\ \ u \in \mathcal{V}.$$

That mean $\Psi'(u)$ is onto for all $u \in \mathcal{V}$. Hence φ is a submersion. Then $\mathcal{V}$ is a C^1-manifold. □

Remark 1. *The functional* $J : W_0^{2,p}(\Omega) \to W^{-2,p'}(\Omega)$, *defined by*

$$J(u) = \begin{cases} \|\Delta u\|_p^{2-p}\Delta_p^2 u & if\ u \neq 0 \\ 0 \quad if\ u = 0 \end{cases}$$

is the duality mapping of $(W_0^{2,p}(\Omega), \|\Delta.\|_p)$ *associated with the Gauge function* $\eta(t) = |t|^{p-2}t$.

Lemma 2. *For any* $\lambda \in [0, \frac{H}{\|w\|_\infty}[$, *we have*

I. φ' *and* Ψ' *are completely continuous, namely,* $u_n \rightharpoonup u$ *in* $W_0^{2,p}(\Omega)$ *implies*

$$\varphi'(u_n) \to \varphi'(u)\ \ (\Psi'(u_n) \to \Psi'(u))\ \ in\ \ W^{-2,p'}(\Omega).$$

II. Φ_λ *is bounded from below on* $\mathcal{V}$.

Proof. Note that $\|.\|_*$ is the dual norm of $W^{-2,p'}(\Omega)$ associated with $\|\Delta(\cdot)\|_p$.

I. First let us prove that φ' is well defined. Let $u, v \in W_0^{2,p}(\Omega)$. We have

$$\langle \varphi'(u), v\rangle = -\lambda \int_\Omega w(x)\frac{|u|^{p-2}}{\delta(x)^{2p}}uv\,dx.$$

Then

$$\Big|\langle \varphi'(u), v\rangle\Big| \le \lambda\|w\|_\infty\bigg(\int_{\{x\in\Omega/\delta(x)>1\}}\frac{|u|^{p-1}}{\delta(x)^{2p}}|v|\,dx + \int_{\{x\in\Omega/\delta(x)\le 1\}}\frac{|u|^{p-1}}{\delta(x)^{2p}}|v|\,dx\bigg),$$

thus

$$\begin{aligned}&\Big|\langle \varphi'(u), v\rangle\Big|\\ \le\ & \lambda\|w\|_\infty\bigg(\int_{\{x\in\Omega/\delta(x)>1\}}|u|^{p-1}|v|\,dx + \int_{\{x\in\Omega/\delta(x)\le 1\}}\frac{1}{\delta(x)^2}\frac{|u|^{p-1}}{\delta(x)^{2(p-1)}}|v|\,dx\bigg).\end{aligned}$$

By Hölder's inequality, it follows that

$$\begin{aligned}&\Big|\langle \varphi'(u), v\rangle\Big|\\ \le\ & \lambda\|w\|_\infty\left(\int_{\{x\in\Omega/\delta(x)>1\}}|u|^{(p-1)p'}\,dx\right)^{\frac{1}{P'}}\left(\int_{\{x\in\Omega/\delta(x)>1\}}|v|^p\,dx\right)^{\frac{1}{p}}\\ & +\lambda\|w\|_\infty\left(\int_{\{x\in\Omega/\delta(x)\le 1\}}\frac{|u|^{(p-1)p'}}{\delta(x)^{2(p-1)p'}}\,dx\right)^{\frac{1}{p'}}\left(\int_{\{x\in\Omega/\delta(x)\le 1\}}\frac{|v|^p}{\delta(x)^{2p}})\,dx\right)^{\frac{1}{p}},\end{aligned}$$

thanks to Rellich inequality (3), we have

$$\begin{aligned}&\Big|\langle \varphi'(u), v\rangle\Big|\\ \le\ & \lambda\|w\|_\infty\|u\|^{p-1}_{L^p(\Omega)}\|v\|_{L^p(\Omega)} + \frac{\lambda\|w\|_\infty}{H}\left(\int_\Omega|\Delta u|^{(p-1)p'}\,dx\right)^{\frac{1}{p'}}\left(\int_\Omega|\Delta v|^p\,dx\right)^{\frac{1}{p}},\end{aligned}$$

then

$$\Big|\langle \varphi'(u), v\rangle\Big| \le \lambda\|w\|_\infty\|u\|^{p-1}_{L^p(\Omega)}\|v\|_{L^p(\Omega)} + \frac{\lambda\|w\|_\infty}{H}\|\Delta u\|_p^{p-1}\|\Delta v\|_p,$$

where p and p' are conjugate by the equality $pp' = p + p'$. Therefore

$$\Big|\langle \varphi'(u), v\rangle\Big| \le \lambda\|w\|_\infty K^2\|\Delta u\|_p^{p-1}\|\Delta v\|_p + \frac{\lambda\|w\|_\infty}{H}\|\Delta u\|_p^{p-1}\|\Delta v\|_p.$$

Hence

$$\|\varphi'(u)\|_* \le \lambda\|w\|_\infty\left(K^2 + \frac{1}{H}\right)\|\Delta u\|_p^{p-1}.$$

For the complete continuity of φ', we argue as follows. Let $(u_n)_n \subset W_0^{2,p}(\Omega)$ be a bounded sequence and $u_n \rightharpoonup u$ in $W_0^{2,p}(\Omega)$. Due to the fact that the embedding $W_0^{2,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact, u_n converges strongly to u in $L^p(\Omega)$. Consequently, there exists a positive function $g \in L^p(\Omega)$ such that

$$| u_n | \leq g \quad \text{a.e. in } \Omega.$$

Since $g^{p-1} \in L^{p'}(\Omega)$, it follows from the Dominated Convergence Theorem that

$$w(x) \mid u_n \mid^{p-2} u_n \to w(x) \mid u \mid^{p-2} u \quad \text{in} \quad L^{p'}(\Omega),$$

$$w(x) \frac{\mid u_n \mid^{p-2} u_n}{\delta(x)^2 p} \to w(x) \frac{\mid u \mid^{p-2} u}{\delta(x)^2 p} \quad \text{in} \quad L^{p'}(\Omega).$$

That is,

$$\varphi'(u_n) \to \varphi'(u) \quad \text{in} \quad L^{p'}(\Omega).$$

Recall that the embedding

$$L^{p'}(\Omega) \hookrightarrow W^{-2,p'}(\Omega)$$

is compact. Thus,

$$\varphi'(u_n) \to \varphi'(u) \quad \text{in} \quad W^{-2,p'}(\Omega).$$

This proves the complete continuity of φ'. We follow the same steps to prove that Ψ' is completely continuous.

II. We have

$$\Phi_\lambda(u) = \frac{1}{p} \int_\Omega |\Delta u|^p dx - \frac{\lambda}{p} \int_\Omega w(x) \frac{|u|^p}{\delta(x)^{2p}}\, dx,$$

then

$$\Phi_\lambda(u) \geq \frac{1}{p} \|\Delta u\|_p^p - \frac{\lambda \|w\|_\infty}{p} \int_\Omega \frac{|u|^p}{\delta(x)^{2p}}\, dx, \tag{8}$$

by Rellich inequality (3), it follows that

$$\Phi_\lambda(u) \geq \frac{1}{p} \Big(H - \lambda \|w\|_\infty \Big) \int_\Omega \frac{|u|^p}{\delta(x)^{2p}}\, dx,$$

since $0 \leq \lambda < \frac{H}{\|w\|_\infty}$ and $u \in \mathcal{V}$, we obtain

$$\Phi_\lambda(u) \geq \frac{1}{p} \Big(H - \lambda \|w\|_\infty \Big) > -\infty.$$

This completes the proof of the lemma. □

Proposition 1. *The functional Φ_λ satisfies the Palais-Smale condition (PS) on $\mathcal{V}$.*

Proof. Let $(u_n)_n$ be a sequence of Palais-Smale of Φ_λ in $W_0^{2,p}(\Omega)$. For $\{u_n\} \subset \mathcal{V}$, thus there exists $M > 0$ such that

$$|\Phi_\lambda(u_n)| \leq M, \tag{9}$$

and

$$(\Phi_\lambda|\mathcal{V})'(u_n) \to 0. \tag{10}$$

Thanks to (8), and (9), that means that $\|\Delta u_n\|_p$ is bounded in $\mathbb{R}$.
Thus, without loss of generality, we can assume that u_n converges weakly in $W_0^{2,p}(\Omega)$ to some function $u \in W_0^{2,p}(\Omega)$ and $\|\Delta u_n\|_p \to \ell$. For the rest we distinguish two cases:
◇ If $\ell = 0$, then $\{u_n\}_n$ converges strongly to 0 in $W_0^{2,p}(\Omega)$.
◇ If $\ell \neq 0$, equation (10) implies that

$$\alpha_n = \Phi'_\lambda(u_n) - \beta_n \Psi'(u_n) \to 0 \quad \text{as} \quad n \to +\infty, \tag{11}$$

where

$$\beta_n = \frac{\langle \Phi'_\lambda(u_n), u_n\rangle}{\langle \Psi'(u_n), u_n\rangle},$$

then let us prove that

$$\limsup_{n\to\infty} \langle \Delta_p^2 u_n, u_n - u\rangle \leq 0.$$

Indeed, notice that

$$\langle \Delta_p^2 u_n, u_n - u\rangle = \|\Delta u_n\|_p^p - \langle \Delta_p^2 u_n, u\rangle.$$

Applying α_n of (11) to u, we deduce that

$$\theta_n := \langle \Delta_p^2 u_n, u\rangle + \langle \varphi'(u_n), u\rangle - \beta_n \langle \Psi'(u_n), u\rangle \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore

$$\langle \Delta_p^2 u_n, u_n - u\rangle = \|\Delta u_n\|_p^p + \langle \varphi'(u_n), u\rangle - \theta_n - \Big(\frac{\langle \Phi'_\lambda(u_n), u_n\rangle}{\langle \Psi'(u_n), u_n\rangle}\Big)\langle \Psi'(u_n), u\rangle.$$

That is,

$$\begin{aligned}\langle \Delta_p^2 u_n, u_n - u\rangle = {} & \frac{\|\Delta u_n\|_p^p}{\langle \Psi'(u_n), u_n\rangle}\Big(\langle \Psi'(u_n), u_n\rangle - \langle \Psi'(u_n), u\rangle\Big) - \theta_n + \langle \varphi'(u_n), u\rangle \\ & - \Big(\frac{\langle \varphi'(u_n), u_n\rangle}{\langle \Psi'(u_n), u_n\rangle}\Big) \cdot \langle \Psi'(u_n), u\rangle.\end{aligned}$$

On the other hand, from Lemma 2, φ' is completely continuous. Thus

$$\varphi'(u_n) \to \varphi'(u), \quad \langle \varphi'(u_n), u_n\rangle \to \langle \varphi'(u), u\rangle \quad \text{and} \quad \langle \varphi'(u_n), u\rangle \to \langle \varphi'(u), u\rangle.$$

From Lemma 2, Ψ' is also completely continuous. So

$$\Psi'(u_n) \to \Psi'(u), \quad \text{and} \quad \langle \Psi'(u_n), u_n\rangle \to \langle \Psi'(u), u\rangle.$$

Then

$$\Big|\langle\Psi'(u_n),u_n\rangle-\langle\Psi'(u_n),u\rangle\Big|\leq\Big|\langle\Psi'(u_n),u_n\rangle-\langle\Psi'(u),u\rangle\Big|+\Big|\langle\Psi'(u_n),u\rangle-\langle\Psi'(u),u\rangle\Big|.$$

It follows that

$$\Big|\langle\Psi'(u_n),u_n\rangle-\langle\Psi'(u_n),u\rangle\Big|\leq\Big|\langle\Psi'(u_n),u_n\rangle-\langle\Psi'(u),u\rangle\Big|+\|\Psi'(u_n)-\Psi'(u)\|_*\|u\|.$$

This implies that

$$\langle\Psi'(u_n),u_n\rangle-\langle\Psi'(u_n),u\rangle\to 0\quad\text{as}\quad n\to\infty. \tag{12}$$

Combining with the above equalities, we obtain

$$\limsup_{n\to+\infty}\langle\Delta_p^2 u_n,u_n-u\rangle\leq\frac{\ell^p}{\langle\Psi'(u),u\rangle}\limsup_{n\to\infty}\Big(\langle\Psi'(u_n),u_n\rangle-\langle\Psi'(u_n),u\rangle\Big).$$

We deduce

$$\limsup_{n\to\infty}\langle\Delta_p^2 u_n,u_n-u\rangle\leq 0. \tag{13}$$

On the other hand, we can write $\Delta_p^2 u_n=\|\Delta u_n\|_p^{p-2}J(u_n)$, since $\|\Delta u_n\|_p\neq 0$ for n large enough. Therefore

$$\limsup_{n\to\infty}\langle\Delta_p^2 u_n,u_n-u\rangle=\ell^{p-2}\limsup_{n\to\infty}\langle Ju_n,u_n-u\rangle.$$

According to (13), we conclude that

$$\limsup_{n\to\infty}\langle Ju_n,u_n-u\rangle\leq 0,$$

in view of Remark 1, J is the duality mapping, Thus satisfies the condition S^+ given in [10]. Therefore, $u_n\to u$ strongly in $W_0^{2,p}(\Omega)$. This completes the proof of the proposition. □

3 Existence of a sequence of eigencurves

In this section, we show that problem (1) has at least one increasing sequence of positive eigencurves by using the results of Ljusternik-Schnirelman.

Let

$$\Sigma_j=\Big\{K\subset\mathcal{V}:K\text{ is symmetric, compact and }\ \gamma(K)\geq j\Big\},$$

where $\gamma(K)=j$ is the Krasnoselskii genus of set K, i.e., the smallest integer j, such that there exists an odd continuous map from K to $\mathbb{R}^j\setminus\{0\}$.

Our first main result is to prove the following result:

Theorem 1. *For any $\lambda \in [0, \frac{H}{\|w\|_\infty}[$ and for any integer $j \in \mathbb{N}^*$,*

$$\mu_j(\lambda) := \inf_{K \in \Sigma_j} \max_{u \in K} p\Phi_\lambda(u)$$

is a critical value of Φ_λ restricted on $\mathcal{V}$. More precisely, there exists $u_j \in \mathcal{V}$, $\mu_j(\lambda) \in \mathbb{R}$ such that

$$\mu_j(\lambda) = p\Phi_\lambda(u_j) = \sup_{u \in K} p\Phi_\lambda(u),$$

and u_j is an eigenfunctin of problem (1) associated to the eigenvalue (λ, μ_j). Moreover,

$$\mu_j(\lambda) \to \infty, \quad as \; j \to \infty.$$

We start with two auxiliary results.

Lemma 3. *for any $j \in \mathbb{N}^*$, $\Sigma_j \neq \emptyset$.*

Proof. Since $W_0^{2,p}(\Omega)$ is separable, there exists $(e_i)_{i \geq 1}$ linearly dense in $W_0^{2,p}(\Omega)$ such that $\operatorname{supp} e_i \cap \operatorname{supp} e_n = \emptyset$ if $i \neq n$. We may assume that $e_i \in \mathcal{M}$ (if not, we take $e'_i \equiv \frac{e_i}{[p\Psi(e_i)]^{\frac{1}{p}}}$).

Let now $j \in \mathbb{N}^*$ and denote

$$F_j = \operatorname{span}\{e_1, e_2, \ldots, e_j\}.$$

Clearly, F_j is a vector subspace with dim $F_j = j$. If $v \in F_j$, then there exist $\alpha_1, \ldots, \alpha_j$ in $\mathbb{R}$, such that $v = \sum_{i=1}^{j} \alpha_i e_i$. Thus

$$\Psi(v) = \sum_{i=1}^{j} |\alpha_i|^p \Psi(e_i) = \frac{1}{p} \sum_{i=1}^{j} |\alpha_i|^p.$$

It follows that the map

$$v \mapsto (p\Psi(v))^{\frac{1}{p}} = ||v||$$

defines a norm on F_j. Consequently, there is a constant $c > 0$ such that

$$c\,\|\Delta v\|_p \leq ||v|| \leq \frac{1}{c}\|\Delta v\|_p.$$

This implies that the set

$$\mathcal{V}_j = F_j \cap \left\{ v \in W_0^{2,p}(\Omega) : \Psi(v) \leq \frac{1}{p} \right\}$$

is bounded because $\mathcal{V}_j \subset B(0, \frac{1}{c})$, where

$$B\left(0, \frac{1}{c}\right) = \left\{ u \in W_0^{2,p}(\Omega) \text{ such that } \|\Delta u\|_p \leq \frac{1}{c} \right\}.$$

Thus, $\mathcal{V}_j$ is a symmetric bounded neighborhood of $0 \in F_j$. Moreover, $F_j \cap \mathcal{V}$ is a compact set. By the property of genus, we get $\gamma(F_j \cap \mathcal{V}) = j$ and then we obtain finally that $\Sigma_j \neq \emptyset$. □

Lemma 4.
$$\mu_j(\lambda) \to \infty \quad as \quad j \to \infty.$$
.

Proof. Let $(e_j, e_n^*)_{j,n}$ be a bi-orthogonal system such that $e_j \in W_0^{2,p}(\Omega)$ and $e_n^* \in W^{-2,p'}(\Omega)$, the $(e_j)_j$ are linearly dense in $W_0^{2,p}(\Omega)$ and the $(e_n^*)_n$ are total for the dual $W^{-2,p'}(\Omega)$. For $j \in \mathbb{N}^*$, set

$$F_j = \text{span}\{e_1, \dots, e_j\} \quad \text{and} \quad F_j^{\perp} = \text{span}\{e_{j+1}, e_{j+2}, \dots\}.$$

By the property of genus, we have for any $A \in \Sigma_j$, $A \cap F_{j-1}^{\perp} \neq \emptyset$. Thus

$$t_j = \inf_{A \in \Sigma_j} \sup_{u \in A \cap F_{j-1}^{\perp}} p\Phi_\lambda(u) \to \infty \quad \text{as} \quad j \to \infty.$$

Indeed, if not, for large j there exists $u_j \in F_{j-1}^{\perp}$ with $\int_\Omega \frac{|u_j|^p}{\delta(x)^{2p}}\,dx = 1$ such that $t_j \leq p\Phi_\lambda(u_j) \leq M$, for some $M > 0$ independent of j. Thus from (8)

$$\|\Delta u_j\|_p \leq \left(pM + \lambda \|m\|_\infty\right)^{\frac{1}{p}}.$$

This implies that $(u_j)_j$ is bounded in $W_0^{2,p}(\Omega)$. For a subsequence of $\{u_j\}$ if necessary, we can assume that $\{u_j\}$ converges weakly in $W_0^{2,p}(\Omega)$ and strongly in $L^p(\Omega)$.
By our choice of $F_{j-1}^{\perp}$, we have $u_j \rightharpoonup 0$ in $W_0^{2,p}(\Omega)$ because $\langle e_n^*, e_j \rangle = 0$, for any $j > n$. This contradicts the fact that $1 = \int_\Omega \frac{|u_j|^p}{\delta(x)^{2p}}\,dx \to 0$ for all j. Since $\mu_j(\lambda) \geq t_k$ the claim is proved. □

Proof of Theorem 1. Applying lemma 3, lemma 4 and Ljusternik-schnireleman theory to the problem (1), we have for each $j \in \mathbb{N}^*$, $\mu_j(\lambda)$ is a critical value of Φ_λ on C^1-manifold $\mathcal{V}$, such that

$$\mu_j(\lambda) \to \infty, \ \text{as} \ j \to \infty.$$

□

Corollary 1. *The following statements hold true*

(i) $\mu_1(\lambda) = \inf \left\{ \frac{\|\Delta u\|_p^p - \lambda \int_\Omega w(x) \frac{|u|^p}{\delta(x)^{2p}}\,dx}{\int_\Omega \frac{|u|^p}{\delta(x)^{2p}}\,dx} \middle| u \in W_0^{2,p}(\Omega) \setminus \{0\} \right\}.$

(ii) $0 < \mu_1(\lambda) \leq \mu_2(\lambda) \leq \cdots \leq \mu_n(\lambda) \to +\infty.$

Proof. (i) For $u \in \mathcal{V}$, set $K_1 = \{u, -u\}$. It is clear that $\gamma(K_1) = 1$, Φ_λ is even and

$$p\Phi_\lambda(u) = \max_{K_1} p\Phi_\lambda \geq \inf_{K \in \Gamma_1} \max_{u \in K} p\Phi_\lambda(u).$$

Thus

$$\inf_{u\in\mathcal{V}} p\Phi_\lambda(u) \geq \inf_{K\in\Sigma_1} \max_{u\in K} p\Phi_\lambda(u) = \mu_1(\lambda).$$

On the other hand, for all $K \in \Gamma_1$ and $u \in K$, we have

$$\sup_{u\in K} p\Phi_\lambda \geq p\Phi_\lambda(u) \geq \inf_{u\in\mathcal{V}} p\Phi_\lambda(u).$$

It follows that

$$\inf_{K\in\Sigma_1} \max_{K} p\Phi_\lambda = \mu_1(\lambda) \geq \inf_{u\in\mathcal{V}} p\Phi_\lambda(u).$$

Then

$$\mu_1(\lambda) = \inf\left\{ \frac{\|\Delta u\|_p^p - \lambda \int_\Omega w(x)\frac{|u|^p}{\delta(x)^{2p}}\,dx}{\int_\Omega \frac{|u|^p}{\delta(x)^{2p}}\,dx} \,\middle|\, u \in W_0^{2,p}(\Omega)\setminus\{0\} \right\}.$$

(ii) For all $i \geq j$, we have $\Sigma_i \subset \Sigma_j$ and in view of the definition of $\lambda_i, i \in \mathbb{N}^*$, we get $\mu_i(\lambda) \geq \mu_j(\lambda)$. As regards $\mu_n(\lambda) \to \infty$, it has been proved in Theorem 1. □

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