

FINSLERIAN GEODESICS ON FRÉCHET MANIFOLDS

Kaveh EFTEKHARINASAB ^{*,1} and Valentyna PETRUSENKO²

Abstract

We establish a framework, namely, nuclear bounded Fréchet manifolds endowed with Riemann-Finsler structures to study geodesic curves on certain infinite dimensional manifolds such as the manifold of Riemannian metrics on a closed manifold. We prove on these manifolds geodesics exist locally and they are length minimizing in a sense. Moreover, we show that a curve on these manifolds is geodesic if and only if it satisfies a collection of Euler-Lagrange equations. As an application, without much difficulty, we prove that the solution to the Ricci flow on an Einstein manifold is not geodesic.

2000 *Mathematics Subject Classification*: 58E10, 58B20, 53C44 .

Key words: Fréchet nuclear manifold, Finsler structure, Geodesic .

1 Introduction

The Riemannian geometry, including geodesics, of the manifold of all Riemannian metrics on a closed manifold which is a Fréchet manifold was studied in [8, 11]. In these papers the geodesic equation is described explicitly, however, in practice it would be difficult to check if a curve is geodesic by the obtained formulas. On the other hand, geodesics of other spaces such as groups of diffeomorphisms that have the structure of Fréchet manifolds were investigated by viewing Fréchet manifolds as inverse limits of Hilbert (ILH) manifolds, cf. [2, 16, 7]. Another recent approach to study geodesics on Fréchet manifolds is by considering these manifolds as projective limits of Banach manifolds, cf. [9, 10].

The reasons for these difficulties and indirect approaches are because Fréchet analysis and geometry are rather restrictive. As for Fréchet spaces, there is no general solvability theory of differential equations and the inverse mapping theorem does not hold in general. Hence, for a Riemannian Fréchet manifold the exponential map may not exist, and even if it exists it is not necessarily a local

^{1*} *Corresponding author*, Topology lab., Institute of Mathematics of National Academy of Sciences of Ukraine, Ukraine, e-mail: kaveh@imath.kiev.ua

²The Higher Mathematics Department, National Aviation University, Ukraine, e-mail: petrusenko76@ukr.net

diffeomorphism at the identity. Another concern is that there exist only weak Riemannian metrics on these manifolds and as shown in [17, 18] a curve connecting two distinct points may have the zero length. Also, a torsion-free covariant derivative compatible with a weak Riemannian metric does not exist in general. These deficiencies inhibit the study of geodesics on these manifolds.

The purpose of this paper is to develop a new natural systematic way to study geodesics on certain Fréchet (bounded or MC^k) manifolds including the space of smooth sections of a fiber bundle on a closed manifold. Our approach is based on a strengthened notion of differentiability (bounded or MC^k -differentiability) introduced in [19]. The basics of Fréchet geometry is redeveloped under the assumption that transition functions between the coordinate charts possess this type of differentiability in [4]. Such generalized manifolds seem to extend the geometry of Fréchet manifolds: for example, an inverse function theorem is obtained for this class of differentiability [19, Theorem 4.10]. Also, an MC^k -vector field on an MC^k -Fréchet manifold M has a unique MC^k -integral curve ([4, Theorem 5.1]) and in this paper we prove that it has a local flow too, see Theorem 4. Also, we prove that this flow is MC^k -differentiable and its domain is open in $M \times \mathbb{R}$ (Lemma 1). This result is crucial for studying geodesics on manifolds.

To define geodesics we will apply the notion of spray as in the book of Lang [15] (cf. [22, 13] for other approaches to geodesics on infinite dimensional manifolds). A reason for this approach is that once we have the existence of integral curves, we can carry over important results such as the existence of exponential maps and parallel translation from the Banach case without much difficulty, indeed we shall face many similarities with the results in Banach geometry. We also prove that, for these generalized manifolds, exponential maps are local diffeomorphisms at the identity (Proposition 1).

As mentioned, since Fréchet manifolds are weakly Riemannian, the length of a curve with distinct endpoints can be zero. On an abstract infinite dimension Fréchet manifold M there are two ways to deal with this problem: use a graded weak Riemannian structure or a Finsler structure, see [24]. We use a collection of weak Riemannian metrics (for a graded weak Riemannian structure) and a collection of continuous functions on the tangent bundle TM (for a Finsler structure) so that together they are strong enough to induce a topology on the tangent spaces equivalent to the one induced from the manifold topology. Consequently, in both cases, a curve possesses a sequence of geodesic lengths.

Herein we will use a Finsler structure (in the sense of Palais [23] which is a Finsler structure in the sense of Upmeyer-Neeb [20]) as it is slightly less technical than a graded weak Riemannian structure. Roughly speaking a Finsler structure on an infinite dimensional Fréchet manifold M is a collection of continuous functions on the tangent bundle TM such that their restrictions to every tangent space is a collection of seminorms that generates the same topology as the Fréchet model space. In addition, they satisfy a certain local compatibility condition. We should mention that our definition of a Finsler structure differs and it is far more general than the one in the finite dimensional theory. As pointed out by Neeb [20] for infinite dimensional manifolds some crucial Finsler geometric results (such as

the Gauss's lemma) are not available in general and we cannot expect to have the usual machinery of Finsler geometry. However, in the case of nuclear bounded Fréchet manifolds since the topology of a model space is generated by a fundamental system of MC^∞ -Hilbertian seminorms $\|\cdot\|^n = \sqrt{\langle \cdot, \cdot \rangle_n}$, in fact they give rise to a Riemann-Finsler structure, we can define appropriately the concept of orthogonality. Moreover, another crucial advantage of nuclear Fréchet manifolds (even over Banach manifolds) is that for these manifolds smooth vector fields can be identified with continuous derivations in the space of smooth real-valued functions on manifolds. Using these properties for an MC^∞ - nuclear Fréchet manifold equipped with a Riemann-Finsler structure we prove the existence of covariant derivatives compatible with the Riemann-Finsler structure (Proposition 3) and the Gauss Lemma (Theorem 8).

In view of the arguments above we believe that the category of MC^∞ -nuclear Fréchet manifolds provide a suitable setting for studying geodesics. On these manifolds, we prove that geodesics exist locally (Theorem 7) and they are length minimizing in a sense (Theorem 9). Also, we prove that a curve is geodesic if and only if it satisfies a collection of Euler-Lagrange equations (Theorem 11). Finally, we show easily that the solution of the Ricci flow equation on an Einstein manifold is not geodesic.

It is worth noting that this category of infinite dimensional manifolds would provide an appropriate framework for studying configuration spaces of physical field theories. As pointed out in [16], these spaces lead to Fréchet manifolds and to discuss motions we need paths of minimal lengths.

2 Bounded Fréchet manifolds

In this section, we shall briefly recall the basics of bounded Fréchet manifolds but in a self-contained way for the convenience of readers, which also allows us to establish our notations for the rest of the paper. For more studies, we refer to [3, 4, 6, 19].

As mentioned, we use the notion of bounded or MC^k -differentiability. It is based on Keller's differentiability but much stronger. Originally, in [19] it is called bounded differentiability but later on the term MC^k -differentiability has been used equivalently.

Let E, F be Fréchet spaces, U an open subset of E and $\varphi : U \rightarrow F$ a continuous map. Let $CL(E, F)$ be the space of all continuous linear maps from E to F topologized by the compact-open topology. If the directional (Gâteaux) derivatives

$$d\varphi(x)h = \lim_{t \rightarrow 0} \frac{\varphi(x + th) - \varphi(x)}{t}$$

exist for all $x \in U$ and all $h \in E$, and the induced map $d\varphi(x) : U \rightarrow CL(E, F)$ is continuous for all $x \in U$, then we say that φ is a Keller's differentiable map of class C^1 . The higher directional derivatives and C^k -maps, $k \geq 2$, are defined in the obvious inductive fashion.

To define bounded differentiability, we endow a Fréchet space F with a translation invariant metric ϱ defining its topology, and then introduce the metric concepts which strongly depend on the choice of ϱ . We consider only metrics of the following form

$$\varrho(x, y) = \sup_{n \in \mathbb{N}} \frac{1}{2^n} \frac{\|x - y\|_F^n}{1 + \|x - y\|_F^n},$$

where $\{\|\cdot\|_F^n\}_{n \in \mathbb{N}}$ is a collection of seminorms generating the topology of F .

Let (E, σ) be another Fréchet space and let $\mathbb{L}_{\sigma, \varrho}(E, F)$ be the set of all linear maps $L : E \rightarrow F$ which are (globally) Lipschitz continuous as mappings between metric spaces E and F , that is

$$\mathcal{Lip}(L)_{\sigma, \varrho} := \sup_{x \in E \setminus \{0\}} \frac{\varrho(L(x), 0)}{\sigma(x, 0)} < \infty,$$

where $\mathcal{Lip}(L)$ is the (minimal) Lipschitz constant of L .

The translation invariant metric

$$\mathfrak{d}_{\sigma, \varrho} : \mathbb{L}_{\sigma, \varrho}(E, F) \times \mathbb{L}_{\sigma, \varrho}(E, F) \longrightarrow [0, \infty), \quad (L, H) \mapsto \mathcal{Lip}(L - H)_{\sigma, \varrho}, \quad (1)$$

on $\mathbb{L}_{\sigma, \varrho}(E, F)$ turns it into an Abelian topological group. We always topologize the space $\mathbb{L}_{\sigma, \varrho}(E, F)$ by the metric (1).

Let U be an open subset of E and let $\varphi : U \rightarrow F$ be a continuous map. If φ is Keller's differentiable, $d\varphi(x) \in \mathbb{L}_{\sigma, \varrho}(E, F)$ for all $x \in U$ and the induced map $d\varphi(x) : U \rightarrow \mathbb{L}_{\sigma, \varrho}(E, F)$ is continuous, then φ is called bounded differentiable or MC^1 and we write $\varphi^{(1)} = \varphi'$. We define for $(k > 1)$ maps of class MC^k , recursively. If $\lambda(t)$ is a curve in a Fréchet space, we denote its derivative by λ' or $d\lambda(t)/dt$. For product spaces, we denote by d_i (in the case of curves by ∂_i) the partial derivative with respect to the i -th variable.

An MC^k -Fréchet manifold is a Hausdorff second countable topological space modeled on a Fréchet space with an atlas of coordinate charts such that the coordinate transition functions are all MC^k -maps. We define MC^k -maps between Fréchet manifolds as usual.

We recall the definition of nuclear manifolds as we mainly work with these manifolds. Let $(B_1, |\cdot|_1)$ and $(B_2, |\cdot|_2)$ be Banach spaces. A linear operator $T : B_1 \rightarrow B_2$ is called nuclear or trace class if it can be written in the form

$$T(x) = \sum_{j=1}^{\infty} \lambda_j \langle x, x_j \rangle y_j,$$

where $\langle \cdot, \cdot \rangle$ is the duality pairing between B_1 and its dual $(B'_1, |\cdot|'_1)$, $x_j \in B'_1$ with $\|x_j\|'_1 \leq 1$, $y_j \in B_2$ with $\|y_j\|_2 \leq 1$, and λ_j are complex numbers such that $\sum_j |\lambda_j| < \infty$.

If $\|\cdot\|_F^i$ is a seminorm on a Fréchet space F , we denote by F_i the Banach space given by completing F using the seminorm $\|\cdot\|_F^i$, there is a natural map from F to F_i whose kernel is $\ker \|\cdot\|_F^i$. A Fréchet space F is called nuclear if for any seminorm $\|\cdot\|_F^i$ we can find a larger seminorm $\|\cdot\|_F^j$ so that the natural induced

map from F_j to F_i is nuclear. A nuclear Fréchet manifold is a manifold modeled on a nuclear Fréchet space. Each nuclear Fréchet space admits a fundamental system of Hilbertian seminorms, see [14]. There are no infinite dimensional Banach spaces that are nuclear. A simple example of Fréchet nuclear space is the space of smooth functions $C^\infty(U, \mathbb{R})$, $U \subset \mathbb{R}^n$ is open, with the fundamental system of seminorms

$$\|f\|^i = \sup_{\substack{x \in S_i, \\ |\alpha| \leq i}} |f^{(\alpha)}(x)|,$$

where $S_1 \subset S_2 \subset S_3 \dots$ is an exhaustion by open sets.

A very important example of a Fréchet nuclear (bounded) manifold is the manifold of all smooth sections of a fiber bundle (such as the manifold of Riemannian metrics) on a closed manifold. For more details on nuclear spaces we refer to [14].

Let M be an MC^k -Fréchet manifold modeled on a Fréchet space F . Let $p \in M$, tangent vectors $v \in T_pM$ are defined as equivalence classes of smooth curves passing through p , where the equivalency means that curves have the same derivative at p . We write $TM := \bigcup_{p \in M} T_pM$ for the tangent bundle of M . The bundle projection $\pi : TM \rightarrow M$ maps elements of T_pM to p , the tangent bundle TM carries a natural vector bundle structure, see [4, Theorem 3.1].

An important feature of an MC^k -Fréchet manifold M (which is not true for Fréchet manifolds in general) is that an MC^k -vector field $X : M \rightarrow TM$ has a unique integral curve. More precisely,

Theorem 1. [4, Theorem 5.1] *Let $X : M \rightarrow TM$ be a vector field of class MC^k , $k \geq 1$. Then there exists an integral curve for X at $x \in M$. Furthermore, any two such curves are equal on the intersection of their domains.*

Another important feature of MC^k -differentiability (which is not true for Keller's differentiability) is that an MC^k -vector field on a Fréchet space has an MC^k -local flow.

Theorem 2. [3, Theorem 2.2] *Let X be an MC^k -vector field on $U \subset F$, $k \geq 1$. There exists a real number $a > 0$ such that for each $x \in U$ there exists a unique integral curve $\ell_x(t)$ satisfying $\ell_x(0) = x$ for all $t \in I_a = (-a, a)$. Furthermore, the mapping $\mathbb{F} : I_a \times U \rightarrow F$ given by $\mathbb{F}_t(x) = \mathbb{F}(t, x) = \ell_x(t)$ is of class MC^k .*

In this paper, we define the local flow of an MC^k -vector field $X : M \rightarrow TM$ and prove that it has the unique MC^k -flow and its domain is open in $M \times \mathbb{R}$. This is indeed a critical result that allows defining exponential maps.

A motivation for defining this class of differentiability was to obtain the following inverse function theorem:

Theorem 3. [19, Theorem 4.10] *Let $x_0 \in U \subset M$ be open and $\varphi : U \rightarrow N$ a MC^k -map, $k \geq 2$. If $\varphi'(x_0)$ is an isomorphism. Then there exists $r > 0$ such that $V = \varphi(B(x_0, r))$ is open in N and $\varphi : B(x_0, r) \rightarrow V$ is a diffeomorphism.*

In this theorem a ball is defined with respect to a metric that induces the same manifold topology, we shall use a Finsler metric. As a consequence of this

theorem, we shall prove that exponential maps are local diffeomorphisms at the identity.

We stress again none of the above results and the ones that we shall prove are true for Fréchet manifolds in general. Most concepts and results from finite dimensional differential geometry cannot be generalized trivially and without restrictive approaches to Fréchet manifolds. Apart from the concepts that depend on the finite-dimensionality, there are obstructions of intrinsic character which are mainly related to dual spaces. The dual of a Fréchet space (non-Banachable) is never a Fréchet space and cotangent bundles do not admit differentiable (in any sense) manifold structures, see [21]. Therefore, some concepts such as the musical isomorphism and strong Riemannian metrics are not at hand. Other obstacles are of analytic nature which are caused by the lack of general solvability of differential equations and the absence of an inverse function theorem in general, therefore geometrical objects such as geodesics, exponential maps and parallel translation may not exist. In this paper we overcome the latter drawbacks by working out in the category of MC^k -manifolds.

3 Geodesics of sprays

Let M be an MC^k -Fréchet manifold modeled on F and let $\pi : TM \rightarrow M$ be its tangent bundle. Suppose X is an MC^k -vector field $X : M \rightarrow TM$, $k \geq 1$.

Let U be open, $x \in U \subset M$ and $I_a = (-a, a)$, $a \in (0, \infty]$. A local flow of X at x is an MC^k -function

$$\mathbb{F} : U \times I_a \rightarrow M$$

such that

1. for each $x \in U$, $\ell_x : I_a \rightarrow M$ defined by $\ell_x(t) = \mathbb{F}(x, t)$ is an integral curve of X at x ,
2. if $\mathbb{F}_t : U \rightarrow M$ is $\mathbb{F}_t(x) = \mathbb{F}(x, t)$ then for $t \in I_a$, $\mathbb{F}_t(U)$ is open and \mathbb{F}_t is an MC^k -diffeomorphism onto its image.

For $t + s \in I_a$ we have $\mathbb{F}_{t+s}(x) = \ell_x(t + s)$. But $\mathbb{F}_t(\mathbb{F}_s(x)) = \mathbb{F}_t(\ell_x(s))$ is the integral curve through $\ell_x(s)$, and $\ell_x(t + s)$ is also an integral curve at $\ell_x(s)$ so by Theorem 1 they coincide, and on U

$$\mathbb{F}_t(\mathbb{F}_s(x)) = \ell_x(t + s) = \mathbb{F}_{t+s}(x),$$

therefore, $\mathbb{F}_s \circ \mathbb{F}_t = \mathbb{F}_{s+t} = \mathbb{F}_{t+s} = \mathbb{F}_t \circ \mathbb{F}_s$. Since $\ell_x(t)$ is a curve at x , $\ell_x(0) = x$, so \mathbb{F}_0 is the identity. Moreover, $\mathbb{F}_t \circ \mathbb{F}_{-t} = \mathbb{F}_{-t} \circ \mathbb{F}_t$ is the identity therefore, if

$$V_t = \mathbb{F}_t(U) \cap U \neq \emptyset,$$

then $\mathbb{F}_t|_{V_{-t}} : V_{-t} \rightarrow V_t$ is a diffeomorphism and its inverse is $\mathbb{F}_{-t}|_{V_t}$.

Now we prove that an MC^k -vector field $X : M \rightarrow TM$ has a unique local flow.

Theorem 4. *Let X be an MC^k -vector field on M . For each $x \in M$ there exists an MC^k -local flow of X at x . Let $\mathbb{F}_1 : U_1 \times I_1 \rightarrow M$ and $\mathbb{F}_2 : U_2 \times I_2 \rightarrow M$ be two local flows then they are equal on $(U_1 \cap U_2) \times (I_1 \cap I_2)$.*

Proof. (Uniqueness). For each $u \in U_1 \cap U_2$ we have $\mathbb{F}_1|_{\{u\} \times I} = \mathbb{F}_2|_{\{u\} \times I}$, where $I = I_1 \cap I_2$. This follows from Theorem 1 and the definition of local flows. Thus, $\mathbb{F}_1 = \mathbb{F}_2$ on the set $(U_1 \cap U_2) \times I$.

(Existence). In order to prove the existence we use the local representation. Let $(x \in U, \psi)$ be a chart and let $\mathbb{F} : V \times I_a \rightarrow F$ be the local flow of the local representative of X at $\psi(x)$ given by Theorem 2 with

$$I_a = (-a, a), \quad V \subset \psi(U), \quad \mathbb{F}(V \times I_a) \subset \psi(U).$$

Define

$$\begin{aligned} \bar{\mathbb{F}} : \psi^{-1}(V) \times I_a &\rightarrow M \\ (u, t) &\rightarrow \psi^{-1}(\mathbb{F}(\psi(u), t)). \end{aligned}$$

Since $\bar{\mathbb{F}}$ is continuous, there exist an open neighborhood $W \subset \psi^{-1}(V)$ of x and $0 < b < a$ such that

$$\bar{\mathbb{F}}(W \times I_b) \subset \psi^{-1}(V).$$

The restriction of $\bar{\mathbb{F}}$ to $W \times I_b$ is the local flow of X at x . By the construction, $\bar{\mathbb{F}}$ is MC^k . The first condition of the definition of local flows holds because it is true for the local representative. To prove the second condition of the definition, note that for each $t \in I_b$, $\bar{\mathbb{F}}_t$ has an MC^k inverse $\bar{\mathbb{F}}_{-t}$ on $\psi^{-1}(V) \cap \bar{\mathbb{F}}_t(W) = \bar{\mathbb{F}}_t(W)$. It follows that $\bar{\mathbb{F}}_t(W)$ is open. And, since $\bar{\mathbb{F}}_t$ and $\bar{\mathbb{F}}_{-t}$ are both of class MC^k , $\bar{\mathbb{F}}_t$ is a MC^k -diffeomorphism. \square

It follows from Theorem 1 that the union of the domains of all integral curves of an MC^k -vector field $X : M \rightarrow TM$ ($k \geq 1$) through $x \in M$ is an open interval which we denote by $I_x = (T_x^-, T_x^+)$, where T_x^- (resp. T_x^+) are the sup (resp., inf) of the times of existence of the integral curves.

Let $\mathcal{D}_X := \bigcup_{x \in M} (\{x\} \times I_x)$, then we have a map $\mathbb{F} : \mathcal{D}_X \rightarrow M$ defined on the entire \mathcal{D}_X such that $\mathbb{F}(x, t)$ is the local flow of X at x . We call this the flow determined by X , and we call \mathcal{D}_X the domain of the flow. We prove that the sets

$$M_t = \{x \in M \mid (x, t) \in \mathcal{D}_X\}$$

are open subsets of M .

Lemma 1. *The domain \mathcal{D}_X is open in $M \times \mathbb{R}$. Moreover, the set M_t is open in M for each $t \in \mathbb{R}$.*

Proof. We follow the idea of [15, Theorem 2.6]. Let $x \in M$ and let $J_x \subseteq I_x$ be the set of points for which $U \times (t - a, t + a) \subseteq \mathcal{D}(X)$ for some positive number a and an open neighborhood $x \in U$, and such that the restriction of the flow \mathbb{F} of X to this product is an MC^k -map. Then, the interval J_x is open in I_x and it contains zero by Theorem 4.

We show that J_x is closed in I_x too. Let s belong to its closure $\overline{J_x}$. By Theorem 4 we can find a neighborhood V for $\mathbb{F}(x, s)$ such that there is a unique MC^k -local flow

$$\mathbb{E} : V \times I_b \rightarrow M,$$

for some positive number b and $\mathbb{E}(v, 0) = v$ for all $v \in V$.

Let a neighborhood $\mathbb{F}(x, s) \in V_1 \subseteq V$ be small enough. By the definition of J_x , there exist $t_1 \in J_x$ close enough to s and a small number \bar{a} and a small enough neighborhood $x \in W$ such that on this product \mathbb{F} is MC^k and

$$\mathbb{F}(W \times (t_1 - \bar{a}, t_1 + \bar{a})) \subseteq V_1.$$

Define

$$\mathcal{F}(w, t) = \mathbb{E}(\mathbb{F}(w, t_1), t - t_1)$$

for $w \in W$ and t belongs to the translation of I_b by t_1 , $I_b + t_1$. Then

$$\mathcal{F}(w, t_1) = \mathbb{E}(\mathbb{F}(w, t_1), 0) = \mathbb{F}(w, t_1),$$

and by the chain rule ([12, Lemma B.1 (f)])

$$\begin{aligned} \frac{d}{dt} \mathcal{F}(w, t) &= d_2 \mathbb{F}(w, t) \circ d_2 \mathbb{E}(\mathbb{F}(w, t_1), t - t_1,) \\ &= X(\mathbb{E}(\mathbb{F}(w, t_1), t - t_1)) = X(\mathcal{F}(w, t)). \end{aligned}$$

Therefore, both $\mathbb{F}(x, t)$ and $\mathcal{F}(x, t)$ are integral curves of X with

$$\mathbb{F}(x, t_1) = \mathcal{F}(x, t_1).$$

Thus, they coincide on the intersection of their domains and $\mathcal{F}(t, x)$ is an extension of $\mathbb{F}(x, t)$ to a bigger interval containing s , therefore, J_x is closed in I_x and consequently $J_x = I_x$. Since \mathbb{F} is MC^k on $W \times (t_1 - \bar{a}, t_1 + \bar{a})$ it follows that \mathcal{F} is MC^k on $W \times (I + t_1)$. Whence, $\mathcal{D}(X)$ is open in $M \times \mathbb{R}$ and consequently M_t is open in M , and \mathbb{F} is of class MC^k on the whole domain $\mathcal{D}(X)$. \square

The double tangent bundle $T(TM)$ over TM has two vector bundle structure, one determined by the natural projection $\pi_{TM} : T(TM) \rightarrow TM$ (see [4, Theorem 3.1]) and the other by the tangent map $\pi_* = T\pi : T(TM) \rightarrow TM$. Indeed, the tangent map is a vector bundle morphism (the arguments for Banach manifolds are valid for M , see [15, Page 52]).

Suppose M is of class MC^k , $k \geq 3$. Let $\alpha : I \rightarrow M$ be an MC^l ($l \geq 2$)-curve, a lift of α into TM is a curve $\hat{\alpha} : I \rightarrow TM$ such that $\pi\hat{\alpha} = \alpha$. The derivative $\alpha' : I \rightarrow TM$ is called the canonical lift. A second order vector field over M is a vector field $\mathcal{F} : TM \rightarrow T(TM)$ such that

$$\pi_* \circ \mathcal{F} = Id_{TM}.$$

An integral curve $\iota : I \rightarrow TM$ of \mathcal{F} is equal to the canonical lift of $\pi\iota$, that is

$$(\pi\iota)' = \iota.$$

A geodesic with respect to \mathcal{F} is a curve $\mathbf{g} : I \rightarrow M$ such that its derivative $\mathbf{g}' : I \rightarrow TM$ is an integral curve of \mathcal{F} , that is $\mathbf{g}'' = \mathcal{F}(\mathbf{g}')$.

Let $s \neq 0 \in \mathbb{R}$ be fixed, define the mapping

$$\begin{aligned} L_s : TM &\rightarrow TM \\ v &\mapsto sv. \end{aligned}$$

A second order vector field $\mathbf{S} : TM \rightarrow T(TM)$ is said to be spray if

1. $\pi_*\mathbf{S}(v) = v$,
2. $\mathbf{S}(sv) = (L_s)_*(s\mathbf{S}(v))$ for all $s \in \mathbb{R}$ and $v \in TM$.

If a manifold admits a partition of unity, then there exists a spray over M , cf. [15, Theorem 3.1]. Let $U \times F$ be a chart for TM and let $\phi : U \times F \rightarrow F \times F$ with $\phi = (\phi_1, \phi_2)$ be a map. By repeating the arguments of [15, Proposition 3.2] and the remarks after it we obtain that ϕ represents a spray \mathbf{S} if and only if $\phi_1(x, v) = v$ and

$$\phi_2(x, v) = \frac{1}{2} d_2^2 \phi_2(x, 0)(v, v).$$

Thus, at $x \in U$ in the chart the spray is determined by a symmetric bilinear map

$$\mathbf{S}(x) = \frac{1}{2} d_2^2 \phi_2(x, 0). \quad (2)$$

Let \mathbf{S} be a spray over M . If $\iota : I \rightarrow TM$ is an integral curve of \mathbf{S} , then ι is the canonical lift of the curve $\ell := \pi \circ \iota : I \rightarrow M$, that is, $\iota = \ell'$. Thus, ℓ is a geodesic of \mathbf{S} because $\ell'' = \iota' = \mathbf{S} \circ \iota = \mathbf{S} \circ \ell'$. If $\ell : I \rightarrow M$ is a geodesic of \mathbf{S} , then its canonical lift $\iota = \ell'$ is an integral curve of \mathbf{S} . Therefore, a curve $\ell : I \rightarrow M$ is a geodesic of \mathbf{S} if, and only if, ℓ' is an integral curve of \mathbf{S} .

Lemma 2. *Let \mathbf{S} be a spray of class MC^k , $k \geq 2$, over M . If $x \in M$ and v is a tangent vector in $T_x M$, then there exists the unique integral curve $\iota : I \rightarrow TM$ of \mathbf{S} such that $\iota(0) = v$.*

Proof. The spray \mathbf{S} is a vector field on TM so by Theorem 1 it has a unique integral curve $\iota : I \rightarrow TM$ such that $\iota(0) = v$. The integral curve ι is the canonical lift of the geodesic $\ell = \pi \circ \iota$ and $\ell'(0) = \iota(0) = v$.

If $\ell_1 : J \rightarrow M$ is another geodesic with $\ell_1'(0) = v$, then $\iota_1 = \ell_1'$ is also an integral curve of \mathbf{S} such that $\iota_1(0) = v$ and so $\iota_1 = \iota$. \square

Let $v \in TM$. By the previous lemma there exists a unique integral curve $\iota_v : I_v \rightarrow TM$ of \mathbf{S} such that $\iota_v(0) = v$. For $v \in TM$ we have the following result:

Lemma 3. *Let $s, t \in \mathbb{R}$, then for a fixed s and all t such $st \in I_v$ we have*

$$\ell_{sv}(t) = s\ell_v(st).$$

Proof. Let a fixed s be given and $t \in \mathbb{R}$ be such that $st \in I_v$, then the curve $\ell_v(st)$ is defined and

$$\frac{d}{dt}(s\ell_v(st)) = (L_s)_*s\ell'_v(st) = (L_s)_*s\mathbf{S}(\ell_v(st)) = \mathbf{S}(s\ell_v(st)). \quad (3)$$

Therefore, the curve $s\ell_v(st)$ is a unique integral curve of \mathbf{S} such that $s\ell_v(0) = sv$ and the uniqueness of the integral curve implies that $\ell_{sv}(t) = s\ell_v(st)$. \square

Let \mathbf{S} be a spray on M of class MC^k , $k \geq 2$. Let ℓ_v be the integral curve of \mathbf{S} with the initial condition $v \in TM$. Let

$$\mathcal{D} := \{v \in TM \mid \ell_v \text{ is defined at least on } [0, 1]\}.$$

By Lemma 1, \mathcal{D} is an open set in TM and $v \mapsto \ell_v(1)$ is an MC^k -map.

We define the exponential map by

$$\begin{aligned} \exp : \mathcal{D} &\rightarrow M \\ \exp(v) &= \pi\ell_v(1). \end{aligned} \quad (4)$$

We denote by $\exp_x : T_xM \rightarrow M$ the restriction to the tangent space T_xM for $x \in M$. By the definition of spray for $s = 0$ at the zero vector 0_x in T_xM we have $\mathbf{S}(0_x) = 0$ so $\exp(0_x) = x$.

Proposition 1. *Let M be an MC^k -Fréchet manifold, $k \geq 3$, and let $\exp : \mathcal{D} \rightarrow M$ be the exponential map. Then for each $x \in M$, $\exp_x : T_xM \rightarrow M$ is a local diffeomorphism at 0_x .*

Proof. Let $v \in T_xM$ and let I_v be an interval containing zero. Consider the parameterized straight line

$$\begin{aligned} \iota_v : I_v &\rightarrow TM \\ t &\mapsto tv. \end{aligned}$$

In view of Lemma 3 for $s = 1$ we obtain $\exp(tv) = \pi\ell_{tv}(1) = \pi\ell_v(t)$. Thereby,

$$(\exp(tv))' = (\pi\ell_v(t))' = \ell_v(t),$$

but

$$(\exp(tv))' = \exp_* i'_v(t).$$

Then, by evaluating at $t = 0$ we get $(\exp_*)(0_x) = Id$. Thus, the map $(\exp_*)(0_x)$ is a linear isomorphism and hence the inverse mapping theorem, Theorem 3, implies that \exp_x is a local diffeomorphism at 0_x . \square

Given a point $x \in M$, by the preceding proposition and the inverse mapping theorem there exists a star-shaped open neighborhood \mathcal{W} of $0_x \in T_xM$ and an open neighborhood \mathcal{U} of x such that $\exp_x : \mathcal{W} \rightarrow \mathcal{U}$ is a diffeomorphism. The pair $(\mathcal{U}, \mathcal{W})$ is called a normal neighborhood of x in M .

We should note that our notion of a normal neighborhood differs from the normal coordinates in the classical sense. We shall give normal neighborhoods in terms of the so-called injectivity radius later on.

Proposition 2. *Let $x \in M$, $v \in T_x M$ and $\alpha_v(t) = \exp_x(tv)$. Then $\alpha_v(t)$ is a geodesic. Conversely, if $\alpha : I \rightarrow M$ is an MC^2 geodesic with $\alpha(0) = x$ and $\alpha'(0) = v$. Then $\alpha(t) = \exp_x(tv)$.*

Proof. The proof is standard so we omit it. □

4 Covariant derivatives

In this section, we work in the category of MC^∞ -Fréchet manifolds.

Let M be an MC^∞ -Fréchet manifold modeled on a Fréchet space F and $\mathcal{E}(M)$ the set of smooth real-valued maps on M . Let $\mathcal{V}(M) = MC^\infty(M \rightarrow TM)$ be the set of all MC^∞ -vector fields and $X, Y \in \mathcal{V}(M)$.

The Lie derivative of $\varphi \in \mathcal{E}(M)$ with respect to a vector field X with the flow \mathbb{F} is defined as usual by

$$\mathcal{L}_X \varphi(x) = \lim_{t \rightarrow 0} \frac{\varphi(\mathbb{F}(x, t)) - \varphi(x)}{t}.$$

It is easily seen that $\mathcal{L}_X \varphi = X(\varphi)$ belongs to $\mathcal{E}(M)$.

Let (U_i, ψ_i) be an atlas of M . We endow $\mathcal{E}(\psi_i(U_i)) := \mathcal{E}(\psi_i(U_i), \mathbb{R})$ with the topology of uniform convergence on compact sets, for the function and all its derivatives, that is, the weakest topology for which the maps

$$\varphi \rightarrow d^n \varphi \in C(\psi_i(U_i) \times F^n, \mathbb{R})$$

are continuous, where $C(\psi_i(U_i) \times F^n, \mathbb{R})$ is the space of continuous linear functions endowed with the compact-open topology.

Then, we equip $\mathcal{E}(M)$ with the weakest topology for which the maps

$$\varphi \mapsto \varphi \circ \psi_i^{-1}$$

from $\mathcal{E}(M)$ to $\mathcal{E}(\psi_i(U_i))$ are continuous. The topology of $\mathcal{E}(M)$ can also be viewed as the weakest topology for which the restrictions $\mathcal{E}(M) \rightarrow \mathcal{E}(U_i)$ are continuous. This topology is independent of the choice of atlas, see [25, Lemma 2].

We identify $\mathcal{V}(U_i)$ with $\mathcal{E}(U_i, U_i \times F)$, then we similarly define the topology of $\mathcal{V}(M)$ to be the weakest topology for which the restrictions $\mathcal{V}(M) \rightarrow \mathcal{V}(U_i)$ are continuous, see [25, Page 280].

The following theorem is proved for Fréchet manifolds in [25] for smoothness in the sense of Keller. Careful analysis of the proof of the theorem shows that it has a topological nature and since MC^k -differentiable maps are Keller's differentiable so the theorem is also valid for the subcategory of MC^k -Fréchet manifolds.

Theorem 5. [25, Theorem] *Let M be a regular smooth nuclear Fréchet manifold. Then the map $X \mapsto \mathcal{L}_X$ is a linear topological isomorphism of the space $\mathcal{V}(M)$ onto the space of continuous derivations in $\mathcal{E}(M)$.*

In [6] a covariant derivative for MC^k -Fréchet manifolds is defined by means of a connection map and Christoffel symbols. However, that definition is not consistent with our context here as we need that a covariant derivative comes from a spray. Herein, we adapt the definition of a covariant derivative in the sense of Lang [15].

If $\varphi \in \mathcal{E}(M)$ and $X \in \mathcal{V}(M)$, then we obtain an MC^∞ -function on M via

$$X.\varphi := d\varphi \circ X : M \rightarrow \mathbb{R}.$$

For $X, Y \in \mathcal{V}(M)$, there exists a unique a vector field $[X, Y] \in \mathcal{V}(M)$ determined by the property that on each open subset $U \subset M$ we have

$$[X, Y].\varphi = X.(Y.\varphi) - Y.(X.\varphi)$$

for all $\varphi \in MC^\infty(U, \mathbb{R})$, see [3]. If we again denote the local representatives of X, Y in an open set $U \subset F$ by themselves, then the local representation of $[X, Y]$ is given by

$$[X, Y](x) = X'(x)Y(x) - Y'(x)X(x).$$

By the definition we see that $[X, Y]$ is bilinear in both arguments and

$$[X, Y] = -[Y, X],$$

and

$$[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]].$$

Definition 1. Let $\pi : TM \rightarrow M$ be the tangent bundle. A covariant derivative ∇ is an \mathbb{R} -bilinear map

$$\begin{aligned} \nabla : \mathcal{V}(M) \times \mathcal{V}(M) &\rightarrow \mathcal{V}(M) \\ (X, Y) &\rightarrow \nabla_X Y \end{aligned}$$

such that for all $\varphi \in \mathcal{E}(M)$ and $X, Y \in \mathcal{V}(M)$ the following hold

1. $\nabla_{\varphi X} Y = \varphi \nabla_X Y$,
2. $\nabla_X(\varphi Y) = (\mathcal{L}_X \varphi)Y + \varphi \nabla_X Y$,
3. $\nabla_X Y - \nabla_Y X = [X, Y]$.

In a chart U we index objects by U to show their representatives. Let \mathbf{S} be a spray on M and let $\mathbf{S}_U(x)$ as in (2) be the symmetric function associated with \mathbf{S} in U . In a chart U , define

$$(\nabla_X Y)_U(x) = Y'_U(x)X_U(x) - \mathbf{S}_U(x)(X_U(x), Y_U(x)). \quad (5)$$

It is a covariant derivative over U and it does not depend on the choice of a local chart, the proof is straightforward and similar to [15, Theorem 2.1].

Now, we define a covariant derivative along a curve. Let I be an open interval in \mathbb{R} , $\lambda : I \rightarrow M$ a curve and $\hat{\lambda} : I \rightarrow TM$ its lift. Let $\text{Lift}(\lambda)$ be the vector space of lifts of λ . In a chart U , define the operator

$$\begin{aligned} \nabla_{\lambda'} : \text{Lift}(\lambda) &\rightarrow \text{Lift}(\lambda) \\ (\nabla_{\lambda'} \gamma)_U(t) &= \gamma'_U(t) - \mathbf{S}_U(\lambda(t))(\lambda'_U(t), \gamma_U(t)). \end{aligned} \quad (6)$$

This defines a covariant derivative and it does not depend on the choice of a local chart and for a mapping φ it satisfies the derivation property

$$(\nabla_{\lambda'}(\varphi\gamma))(t) = \varphi'(t)(\nabla_{\lambda'}(\gamma))(t) + \varphi(t)(\nabla_{\lambda'}\gamma)(t),$$

the proof is standard so we omit it, cf. [15, Theorem 3.1]. Let X be a vector field such that $\gamma(t) = X(\lambda(t))$ for $t \in I$ and let Y be a vector field such that $Y(\lambda(t_0)) = \lambda'(t_0)$ for some $t_0 \in I$. Then by the chain rule and (6) we have

$$(\nabla_{\lambda'}\gamma)(t_0) = (\nabla_Y X)(\lambda(t_0)).$$

Let J be an open interval in \mathbb{R} , $\mu : J \rightarrow M$ a MC^k -curve ($k \geq 2$), and $\gamma : J \rightarrow TM$ a lift of μ . We say that γ is μ -parallel if $\nabla_{\mu'}\gamma = 0$. By (6) in a local chart we have

$$\gamma'_U(t) = S_U(\mu(t))(\mu'_U(t), \gamma_U(t)),$$

and hence μ is a geodesic for the spray S if and only if $\nabla_{\mu'}\mu' = 0$.

5 Finsler structures and geodesics

As mentioned on a Fréchet manifold there exist only weak Riemannian metrics with unsatisfactory properties. Thus, we use a graded weak Riemannian structure or a Finsler structure instead. The idea behind a graded weak Riemannian metric structure is considering not one weak metric but a collection of weak metrics such that the family of induced seminorms generates the same topology as the Fréchet model space. Nevertheless, this is not enough to produce a strong enough topology on the tangent spaces, in addition, the induced seminorms need to satisfy an estimation of a tame type.

In the finite dimensional theory of Finsler manifolds, a Finsler structure is a function $F : TM \rightarrow \mathbb{R}^+$ which is smooth on the complement of the zero section and positively homogeneous and strongly convex on each tangent space. This definition is too restrictive and insufficient for infinite dimensional Fréchet manifolds. By contrast, in the infinite dimensional theory there are two definitions of Finsler structures: one in the sense of Palais and another in the sense of Upmeyer-Neeb which are different by their local compatibility conditions. Roughly speaking a Finsler structure is a collection of continuous functions on the tangent bundle such that their restrictions to every tangent space is a collection of seminorms that generates the same topology as the Fréchet model space. In addition, this family of seminorms needs to satisfy a certain local compatibility condition. The infinite dimensional theory of Finsler manifolds is much less general than the finite dimensional theory and analogue notions and results may not be available.

In this paper we use the definition of a Finsler structure in the sense of Palais [23].

Definition 2. [6, Definition 4.2] Let F be a Fréchet space T a topological space, and $V = T \times F$ the trivial bundle with fiber F over T . A Finsler structure for V is a collection of continuous functions $\|\cdot\|^n : V \rightarrow \mathbb{R}^+$, $n \in \mathbb{N}$, such that

1. For $b \in T$ fixed, $\| (b, f) \|^n = \| f \|_b^n$ is a collection of seminorms on F which gives the topology of F .
2. Given $k > 1$ and $t_0 \in T$, there exists a neighborhood \mathcal{U} of t_0 such that

$$\frac{1}{k} \| f \|_{t_0}^n \leq \| f \|_u^n \leq k \| f \|_{t_0}^n \quad (7)$$

for all $u \in \mathcal{U}$, $n \in \mathbb{N}$, $f \in F$.

Suppose M is a bounded Fréchet manifold modeled on F . Let $\pi_M : TM \rightarrow M$ be the tangent bundle and let $\| \cdot \|_n : TM \rightarrow \mathbb{R}^+$ be a collection of functions, $n \in \mathbb{N}$. We say $\{\| \cdot \|_n\}_{n \in \mathbb{N}}$ is a Finsler structure for TM if for a given $x \in M$, there exists a bundle chart $\psi : U \times F \simeq TM|_U$ with $x \in U$ such that

$$\{\| \cdot \|_n \circ \psi^{-1}\}_{n \in \mathbb{N}}$$

is a Finsler structure for $U \times F$.

A bounded Fréchet Finsler manifold is a bounded Fréchet manifold together with a Finsler structure on its tangent bundle. If $\{\| \cdot \|_n\}_{n \in \mathbb{N}}$ is a Finsler structure for M , then eventually we can obtain a graded Finsler structure, $(\| \cdot \|_n)_{n \in \mathbb{N}}$, for M , that is $\| \cdot \|^{i+1} \leq \| \cdot \|^{i+1}$ for all i .

We define the length of an MC^1 -curve $\gamma : [a, b] \rightarrow M$ with respect to the n -th component by

$$L_n(\gamma) = \int_a^b \| \gamma'(t) \|_{\gamma(t)}^n dt.$$

The length of a piecewise path with respect to the n -th component is the sum over the curves constituting the path. So, a curve γ possesses a sequence of geodesic lengths $L_n(\gamma)$. By abuse of language, we say that the length of a curve γ is minimal if for all other such curves λ , we have $L_n(\gamma) \leq L_n(\lambda)$ for all n . On each connected component of M , the distance is defined by

$$\rho_n(x, y) = \inf_{\gamma} L_n(\gamma),$$

where infimum is taken over all continuous piecewise MC^1 -curve connecting x to y . Thus, we obtain an increasing sequence of metrics $\rho_n(x, y)$ and define the distance ρ by

$$\rho(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{\rho_n(x, y)}{1 + \rho_n(x, y)}. \quad (8)$$

Theorem 6. [6, Theorem 4.6] *Suppose M is connected and endowed with a Finsler structure $(\| \cdot \|_n)_{n \in \mathbb{N}}$. Then the distance ρ defined by (8) is a metric for M , called the Finsler metric. Furthermore, the topology induced by this metric coincides with the original topology of M .*

If a manifold admits a partition of unity, then it possesses a Finsler structure, in particular, nuclear Fréchet manifolds can be equipped with Finsler structures, cf. [6, Proposition 4.4].

Definition 3. Let F be a Fréchet space. A continuous function $|\cdot|: F \rightarrow \mathbb{R}^+$ is said to be the pre-Finsler norm on F if

1. it is positive homogeneous of order 1,
2. it is sub-additive.

Definition 4. Let $(F, |\cdot|)$ be a pre-Finsler space, a function $\ll \cdot, \cdot \gg: F \times F \rightarrow \mathbb{R}$ is said to be the Finslerian product if

1. it is positive homogeneous of order 1 in its first argument,
2. it is linear in its second variable.

We say that a vector $v \in F$ is F-orthogonal to $u \in F$ if $\ll u, v \gg_n = 0, \forall n \in \mathbb{N}$.

Let M be a nuclear Fréchet manifold of class MC^∞ with a Finsler structure $(\|\cdot\|^n)_{n \in \mathbb{N}}$. Let $x \in M$ and $u, v \in T_x M$. The tangent space $T_x M$ admits semi-inner products by Hilbertian seminorms $\|v\|_x^n = \sqrt{\langle v, v \rangle_{n,x}}$. We define the Finslerin products on $T_x M$ simply by

$$\ll u, v \gg_{n,x} = \langle u, v \rangle_{n,x}, \forall n \in \mathbb{N}. \tag{9}$$

For the sake of brevity we write $\ll u, v \gg_n$ instead of $\ll u, v \gg_{n,x}$ where the confusion may not occur.

In local charts, mappings $\ll \cdot, \cdot \gg_n$ are linear so smooth in the sense of Keller. Also, in local charts, the Cauchy-Schwartz inequality yields that they are globally Lipschitz and so of class MC^∞ by Lemma B.1(a) [12].

Remark 1. For nuclear Féchet manifolds a Finsler structure $(\|\cdot\|^n)_{n \in \mathbb{N}}$ in fact is given by semi-inner products and the products (9) are Riemannian. Therefore, on each tangent space the topology is induced by a family of weak Riemannian metrics that satisfy the Finsler condition. In such a case, we call $(\|\cdot\|^n)_{n \in \mathbb{N}}$ a Riemann-Finsler structure. It is to be observed that we cannot use an arbitrary collection of weak metrics they need to satisfy the Finsler condition (Definition (2)); this justifies the terminology “Riemann-Finsler structure”.

If X, Y are vector fields, then $\ll X, Y \gg_n$ is a function on M with the value $\ll X(x), Y(x) \gg_n$ at a point $x \in M$.

Proposition 3. Let M be an MC^∞ -nuclear Fréchet manifold with a Riemann-Finsler structure $(\|\cdot\|^n)_{n \in \mathbb{N}}$. Then for each $n \in \mathbb{N}$ there exists a unique covariant derivative ∇^n such that

$$\nabla_Z^n \ll X, Y \gg_n = \ll \nabla_Z^n X, Y \gg_n + \ll X, \nabla_Z^n Y \gg_n; X, Y, Z \in \mathcal{V}(M). \tag{10}$$

Proof. (Uniqueness). Suppose there exists such a covariant derivative. If for all X, Y and Z we compute $\nabla_Z^n \ll X, Y \gg_n, \nabla_X^n \ll Y, Z \gg_n$ and $\nabla_Y^n \ll Z, X \gg_n$

by (10), then by subtracting the sum of the first two from the last one and applying the torsion-free property of a covariant derivative we obtain

$$\begin{aligned} K_n(X, Y, Z) &= \mathcal{L}_Z \ll X, Y \gg_n + \mathcal{L}_X \ll Y, Z \gg_n - \mathcal{L}_Y \ll Z, X \gg_n \\ &\quad - \ll X, [Y, Z] \gg_n + \ll Y, [Z, X] \gg_n + \ll Z, [X, Y] \gg_n \\ &= 2 \ll \nabla_X^n Y, Z \gg_n. \end{aligned} \quad (11)$$

Let $\widehat{\nabla}^n$ be the other covariant derivatives satisfying (10). The right-hand side of (11) does not depend on the covariant derivatives, therefore, for all $n \in \mathbb{N}$ we have

$$\ll \widehat{\nabla}_X^n Y - \nabla_X^n Y, Z \gg_n = 0.$$

Since Z is arbitrary, the Hausdorffness implies that

$$\widehat{\nabla}_X^n Y = \nabla_X^n Y.$$

(Existence). Fix X, Y , the function $K_n(X, Y, Z)$ is smooth since it is the sum of smooth functions. The mapping $K_n(X, Y, Z) \mapsto \mathcal{L}_Z K_n(X, Y, Z)$ is a continuous derivation so by Theorem 5 for each n there is a uniquely defined vector field which we call $\nabla_X^n Y$ such that

$$\ll \nabla_X^n Y, Z \gg_n = \frac{1}{2} K_n(X, Y, Z).$$

Showing that $\nabla_X^n Y$ satisfies the properties (1) – (3) in Definition 1 is standard. Therefore, it is omitted. \square

The preceding theorem and the ones we shall prove strongly depend on the nuclearity property of manifolds and the MC^k -differentiability. They are not true for Fréchet manifolds even for Banach manifolds with weak Riemannian metrics in general.

Henceforth, we assume that M is a connected nuclear Fréchet manifold of class MC^∞ with a Riemann-Finsler structure $(\|\cdot\|_n)_{n \in \mathbb{N}}$. Let $x \in M$ and let $B(0_x, r)$ be the open ball in $T_x M$ centered at 0_x with radius r with respect to the Finsler metric ρ (8). The injectivity radius of M at x , $i(x)$, is the least upper bound of numbers $r > 0$, such that \exp_x is a diffeomorphism on $B(0_x, r)$.

Theorem 7. *Let $x \in M$, and let $\varepsilon > 0$ be such that $\mathcal{U} = \exp_x(B(0_x, \varepsilon))$ is a normal neighborhood of x . Then for any $y \in \mathcal{U}$ there exists a unique geodesic $\ell : [0, 1] \rightarrow M$ joining x and y such that for all $n \in \mathbb{N}$*

$$L_n(\ell) \leq \varepsilon.$$

Proof. Let $x \in M$ and let $0_x \in T_x M$ be the zero vector. On an open neighborhood N of 0_x in $T_x M$ define the mapping $\varphi(v) = (x, \exp_x(v))$. By virtue of Proposition 1 in local charts, the Jacobin matrix of φ at 0_x is

$$\varphi_* = \begin{bmatrix} id & 0 \\ * & id \end{bmatrix},$$

which is invertible. Thus, by the inverse function theorem (3) φ is a diffeomorphism from some neighborhood W of 0_x onto its image. We can shrink W and assume that $W = \bigcup_{p \in V} B(0_p, \varepsilon)$ for some open neighborhood V of x . Then, for $y \in \mathcal{U}$ there exists a unique $v \in W$ such that $\varphi(v) = (x, y)$. That is, there exists a unique $v \in B(0_x, \varepsilon)$ such that $\exp_x v = y$. Now define $\ell(t) : [0, 1] \rightarrow M$ by $\ell(t) = \exp_x(tv)$, this is a geodesic connecting x to y and $\ell'(0) = v$ and entirely is contained in \mathcal{U} , since $B(0_x, \varepsilon)$ is star-shaped and so $tv \in B(0_x, \varepsilon)$ for $t \in [0, 1]$. Since ℓ is contained in \mathcal{U} then for all $n \in \mathbb{N}$ we have

$$\|\ell'(t)\|_{\ell(t)}^n \leq \varepsilon,$$

and so $L_n(\ell) \leq \varepsilon$.

To prove the uniqueness let α be another geodesic in \mathcal{U} connecting x, y . We may assume that $\alpha(0) = 1$ and $\alpha(1) = y$ after an appropriate reparameterization. Then by Proposition 2 we have $\alpha(t) = \exp(t\alpha'(0))$ for all $t \in [0, 1]$. Let

$$I = \exp_x^{-1}(\text{Img}(\alpha)).$$

It is a line segment contained in $B(0_x, \varepsilon)$ and its endpoints are 0_x and $a\alpha'(0)$ for some $a > 0$, because $\text{Img}(\alpha) \subset \mathcal{U}$ and the map \exp_x is a diffeomorphism so I is a connect closed subset in

$$\mathbb{A} = \{t\alpha'(0) \in T_x M \mid t \in (0, \infty)\}.$$

Now, we show that $a \geq 1$. If $a < 1$, then, the openness of \mathcal{U} yields there exists $b \in (0, 1]$ such that $b\alpha'(0) \in \mathcal{U}$. But

$$\exp_x(b\alpha'(0)) \notin \text{Img}(\alpha),$$

since \exp_x is bijective on $\mathbb{A} \cap \mathcal{U}$ and $\exp_x(I) = \text{Img}(\alpha)$. This is a contradiction because the image of the line segment connecting 0_x and $\alpha'(0)$ under \exp_x is $\text{Img}(\alpha)$. Thus, $a \geq 1$ and so $\alpha'(0) \in \mathcal{U}$. Therefore, $\exp_x(\alpha'(0)) = \alpha(1) = y$ and $\alpha'(0) = \exp_x^{-1}(y) = v$, whence $\alpha = \ell$. □

Let I_1, I_2 be open intervals in \mathbb{R} and let $\ell : I_1 \times I_2 \rightarrow M$, $(t, s) \mapsto \ell(t, s)$ be an MC^∞ -curve. Let $\partial_i \ell$, $i = 1, 2$, denote the ordinary partial derivative with respect to the i -th variable. Since the curves $t \mapsto \partial_i \ell$ and $s \mapsto \partial_i \ell$ are lifts in TM we can consider their covariant derivatives.

For each $n \in \mathbb{N}$, let $\nabla_1^n \partial_2 \ell$ be the covariant derivative of $\partial_2 \ell$ along the curve $\ell_s(t) = \ell(t, s)$ for a fixed s . Similarly, let $\nabla_2^n \partial_1 \ell$ be the covariant derivative of $\partial_1 \ell$ along the curve $\ell_t(s) = \ell(t, s)$ for each fixed t . By Formula (5) in a local chart U

$$\nabla_1^n \partial_2 \ell_U = \partial_1 \partial_2 \ell_U - S_U(\ell_U)(\partial_1 \ell_U, \partial_2 \ell_U),$$

and symmetry of S_U implies that

$$\nabla_1^n \partial_2 \ell = \nabla_2^n \partial_1 \ell. \tag{12}$$

therefore, for all $n \in \mathbb{N}$

$$\partial_2 \ll \partial_1 \ell, \partial_1 \ell \gg_n = 2 \ll \nabla_2^n \partial_1 \ell, \partial_1 \ell \gg_n, \quad (13)$$

so (12) follows that

$$\partial_2 \ll \partial_1 \ell, \partial_1 \ell \gg_n = 2 \ll \nabla_1^n \partial_2 \ell, \partial_1 \ell \gg_n. \quad (14)$$

Let $\varepsilon > 0$ and $x \in M$. Define a set $S_{x,\varepsilon} := \{v \in T_x M \mid \ll v, v \gg_n = \varepsilon^2 (\forall n \in \mathbb{N})\}$.

The following result generalizes the classical Gauss's lemma to the context of infinite dimensional MC^∞ -nuclear Fréchet manifolds equipped with Riemann-Finsler structures.

Theorem 8 (Gauss's lemma). *Let $x_0 \in M$ and let $(\mathcal{U}, \mathcal{W})$ be a normal neighborhood of x_0 . Then the geodesics through $x \in \mathcal{U}$ are F-orthogonal to the image of $S_{x,\varepsilon}$ under \exp_x , for small enough $\varepsilon > 0$.*

Proof. For $\varepsilon > 0$ small enough, the map \exp_x is defined on an open ball in $T_x M$ of radius slightly larger than ε . The proof is equivalent to prove that for any MC^∞ -curve $j : I \rightarrow S_{x,1}$, and $0 < s < \varepsilon$, if we define

$$\iota(s, t) = \exp(sj(t))$$

then for any arbitrary s_0, t_0 the following curves

$$t \rightarrow \exp_x(s_0 j(t)), \quad s \rightarrow \exp_x(sj(t_0))$$

are F-orthogonal. By proposition 2 for each t , the map $\iota_t : s \rightarrow \iota(s, t)$ is a geodesic so for all $n \in \mathbb{N}$

$$\nabla_1^n \partial_1 \iota = 0,$$

and

$$\partial_1 \ll \partial_1 \iota, \partial_1 \iota \gg_n = 2 \ll \nabla_1^n \partial_1 \iota, \partial_1 \iota \gg_n = 0, \quad \forall n \in \mathbb{N}.$$

Thus, the functions

$$s \mapsto \ll \partial_1 \iota(s, t), \partial_1 \iota(s, t) \gg_n \quad (15)$$

are constant for each t . Since $\partial_1 \iota(0, t) = j(t)$ and $\ll j(t), j(t) \gg_n = 1 (\forall n \in \mathbb{N})$ it follows that

$$\ll \partial_1 \iota, \partial_1 \iota \gg_n = 1 (\forall n \in \mathbb{N}).$$

Therefore, by (14)

$$\partial_1 \ll \partial_1 \iota, \partial_2 \iota \gg_n = \ll \nabla_1 \partial_1 \iota, \partial_2 \iota \gg_n + \frac{1}{2} \partial_2 \ll \partial_1 \iota, \partial_1 \iota \gg_n = 0, \quad \forall n \in \mathbb{N}.$$

Thereby, the functions $s \mapsto \ll \partial_1 \iota(s, t), \partial_2 \iota(s, t) \gg_n$ are constant for each fixed t . Let $s = 0$, then $\iota(0, t) = \exp_x(0) = x$ and therefore $\partial_2 \iota(0, t) = 0$ for all t . Thus,

$$\ll \partial_1 \iota, \partial_2 \iota \gg_n = 0 (\forall n \in \mathbb{N}),$$

that is $\partial_1 \iota$ and $\partial_2 \iota$ are F-orthogonal. This concludes the proof. \square

Theorem 9. *Let $x \in M$ and $\mathcal{U} = \exp_x(B(0_x, i(x)))$ be a normal neighborhood of x . Let $\ell : [0, 1] \rightarrow M$ be the unique geodesic in \mathcal{U} joining x to $y \in \mathcal{U}$. Then, for any other piecewise MC^1 -path $\iota : [0, 1] \rightarrow M$ joining x, y , we have*

$$L_n(\ell) \leq L_n(\iota), \quad \forall n \in \mathbb{N}.$$

If the equality holds, then ι must coincide with ℓ , up to reparametrization.

Proof. Consider an MC^1 -path $\iota : [0, 1] \rightarrow \mathcal{U}$ connecting x to y . Since \exp_x on $B(0_x, i(x))$ is a diffeomorphism we may find a unique curve

$$t \rightarrow v(t) : [0, 1] \rightarrow T_x M$$

with $\|v(t)\|_{v(t)}^n = 1 (\forall n \in \mathbb{N})$ and a curve $r(t) : (0, 1] \rightarrow (0, i(x))$ such that

$$\iota(t) = \exp_x(r(t)v(t)) := \mathbf{k}(r(t), t).$$

Locally, $r(t)$ and $v(t)$ are obtained by the inverse of the exponential map after a smooth projection so $r(t)$ and $v(t)$ are piecewise MC^1 . We may assume $r(t) \neq 0$, that is $\iota(t) \neq x$ for all $t \in (0, 1]$ since otherwise we may define t_0 to be the last value such that $\iota(t_0) = x$ and exchange ℓ with $\iota|_{[t_0, 1]}$. Now we have

$$\iota'(t) = \partial_1 \mathbf{k}(r(t), t)r'(t) + \partial_2 \mathbf{k}(r(t), t). \tag{16}$$

Also,

$$\partial_1 \mathbf{k} = (T_{rv(t)} \exp_x)(v(t)) \quad \text{and} \quad \partial_2 \mathbf{k} = (T_{rv(t)} \exp_x)(rv'(t)).$$

By Theorem 8, $\partial_1 \mathbf{k}$ and $\partial_2 \mathbf{k}$ are F-orthogonal. By the same arguments for proving (15) we have

$$\|\partial_1 \mathbf{k}\|_{v(t)}^n = 1 (\forall n \in \mathbb{N}),$$

and by (16) we obtain

$$\left(\|\iota'(t)\|_{\iota(t)}^n \right)^2 = |r'(t)|^2 + \left(\left\| \frac{d\mathbf{k}(t)}{dt} \right\|_{\mathbf{k}(t)}^n \right)^2 \geq r'(t)^2.$$

Therefore,

$$L_n(\iota) \geq \int_0^1 \|\iota(t)\|_{\iota(t)}^n dt \geq \int_0^1 |r'(t)| dt \geq r(1) - (\lim_{\epsilon \rightarrow 0} r(\epsilon) = \delta). \tag{17}$$

Let $y = \exp_x(rv)$ such that $0 < r < i(x)$ with $v \in T_x M$ and $\|v\|_x^n = 1 (\forall n \in \mathbb{N})$.

For $s, 0 < s < r$, the path $\iota(t)$ contains a segment joining $S_{x,s}$ and $S_{x,r}$ and remains between them. By (17) we have $L_n(\iota) \geq r - \delta$ and so if $\delta \rightarrow 0$ then $L_n(\iota) \geq r$. Theorem 7 implies that there exists $r_0 < i(x)$ such that $L_n(\alpha) \leq r_0$ (we may find $u \in T_x M$ such that $y = \exp(r_0 u)$) but $L_n(\iota) \geq r_0$, therefore for all n

$$L_n(\alpha) \leq L_n(\iota).$$

If $L_n(\alpha) = L_n(\iota)$ then in (17) we must have the equality as well and this happens if and only if $t \rightarrow v(t)$ is constant and $t \rightarrow r(t)$ is monotone. Thus, by a suitable reparametrization ι becomes a geodesic. Suppose this is the case, so $\iota : [0, r] \rightarrow M$ is the curve $t \rightarrow \exp_x(tv_0)$ and $\exp_x(rv_0) = y$ for some $v_0 \in T_x M$ with $\|v_0\|_x^n = 1 (\forall n \in \mathbb{N})$, but \exp_x is a diffeomorphism so $v = v_0$ and therefore $\alpha = \iota$. □

Let M be an MC^∞ -nuclear Fréchet manifold modeled on F with a Riemann-Finsler structure $(\|\cdot\|^n)_{n \in \mathbb{N}}$. Let a curve $\ell : [a, b] \rightarrow M$ be an MC^∞ -curve. We denote the local representatives of ℓ again by ℓ . In a local chart U , the coordinate of its canonical lift is $(\ell(t), \ell'(t))$. For each $n \in \mathbb{N}$ we define the energy functional E_n by

$$E_n(\ell) = \frac{1}{2} \int_a^b \ll \ell(t), \ell'(t) \gg_n dt.$$

Take an MC^∞ -proper variation $\mathbb{H} : (-\varepsilon, \varepsilon) \times [a, b] \rightarrow M$ of ℓ such that

$$\mathbb{H}(0, s) = \ell(s), \quad \mathbb{H}(t, a) = \ell(a), \quad \mathbb{H}(t, b) = \ell(b),$$

for all $t \in (-\varepsilon, \varepsilon)$.

Let $\mathbb{H}_t(s) = \mathbb{H}(t, s)$, a curve ℓ is called a critical point for E_n if

$$\frac{d}{dt}(E_n(\mathbb{H}_t)) \Big|_{t=0} = 0, \quad \forall n \in \mathbb{N}.$$

The partial derivative of local representative of $E_n : U \times F \rightarrow \mathbb{R}$ are

$$d_1 E_n(u, e)(f) = \lim_{h \rightarrow 0} \frac{1}{h} (E_n(u + hf, e) - E_n(u, e)),$$

$$d_2 E_n(u, e)(f) = \lim_{h \rightarrow 0} \frac{1}{h} (E_n(u, e + hf) - E_n(u, e)).$$

We will need the following result.

Theorem 10. [26, Theorem 6.3] *Let $\mathcal{L} \in C^\infty(TM, \mathbb{R})$ be a Lagrangian. Then a smooth curve $J(t)$ is critical for \mathcal{L} if and only if it satisfies the Euler-Lagrange equation*

$$(d_1 L)(j(t), j'(t)) - \frac{d}{dh} \Big|_{h=t} (d_2 L)(j(h), j'(h)) = 0, \quad (18)$$

in a local chart where L and $j(t)$ are, respectively, the local expressions of L and $J(t)$, and $d_i L (i \in 1, 2)$ are the partial derivatives of L .

We should mention that in the preceding theorem the used differentiability is equivalent to the Keller's differentiability, as we have seen functions $\ll \cdot, \cdot \gg_n$ are Keller's differentiable so we can apply it.

Theorem 11. *An MC^∞ -curve $\ell : [a, b] \rightarrow M$ is geodesic if and only if in a local chart it satisfies the Euler-Lagrange equations*

$$(d_1 E_n)(\ell(t), \ell'(t)) - \frac{d}{dh} \Big|_{h=t} (d_2 E_n)(\ell(h), \ell'(h)) = 0, \quad \forall n \in \mathbb{N}. \quad (19)$$

Proof. For an MC^∞ -variation $\mathbb{H} : (t, s) \mapsto \mathbb{H}(t, s)$, along \mathbb{H} define the vector fields

$$Y := d\mathbb{H}(\partial/\partial t), \quad X := d\mathbb{H}(\partial/\partial s).$$

For all $n \in \mathbb{N}$ we have

$$\begin{aligned}
 \frac{d}{dt}(E_n(\mathbb{H}_t)) &= \frac{1}{2} \left(\int_a^b \frac{d}{dt} \ll X, X \gg_n ds \right) \\
 &= \int_a^b \ll \nabla_Y^n X, X \gg_n ds \quad \text{since } \nabla^n \text{ is compatible} \\
 &= \int_a^b \ll \nabla_X^n Y, X \gg_n ds \quad \text{since } \nabla^n \text{ is torsion-free} \\
 &= \int_a^b \left(\frac{d}{ds} \ll Y, X \gg_n - \ll Y, \nabla_X^n X \gg_n \right) ds \\
 &= \ll Y, X \gg_n \Big|_a^b - \int_a^b \ll Y, \nabla_X^n X \gg_n ds
 \end{aligned}$$

Since the variation is proper we have

$$Y(a) = Y(b) = 0.$$

Moreover, $X(0, s) = \partial\mathbb{H}/\partial s(0, s) = \ell'(s)$, therefore,

$$\frac{d}{dt}(E(\mathbb{H}_t) |_{t=0}) = - \int_a^b \ll Y(0, s), (\nabla_{\ell'}^n \ell')(s) \gg_n ds.$$

The right side is zero if and only if ℓ is geodesic. That is, the critical points are geodesic and hence by Theorem 10 they need to satisfy the Euler-Lagrange equations (19). \square

Let N be a closed Einstein manifold of dimension n . The manifold of Riemannian metrics on N , \mathcal{M} , is a nuclear Fréchet manifold, it is also MC^∞ (see [5, 19]). The solution to the Ricci flow equation

$$\frac{dg(t)}{dt} = -2\text{Ric}(g(t))$$

is $g(t) = (1 - 2\lambda)tg_0$, where g_0 is a Riemannian metric and $\text{Ric}(g_0) = \lambda g_0$, see [1]. This is a curve on \mathcal{M} . In local charts, obviously $g(t)$ is C^1 and

$$g'(t) = -2\lambda g_0 \in \mathbb{L}_{\sigma, \varrho}([0, T], F),$$

where σ is the standard metric on \mathbb{R} , T is a time less than the finite singular time and

$$g' : [0, T] \rightarrow \mathbb{L}_{\sigma, \varrho}([0, T], F)$$

is constant and hence a continuous map into $\mathbb{L}_{\sigma, \varrho}([0, T], F)$. Thus, $g(t)$ is MC^1 and by induction it follows that $g(t)$ is MC^k with

$$g^{(k)} = 0, (k \geq 2).$$

Simple calculations show that for all n we have

$$(d_1 E_n)(g(t), g'(t)) \neq 0, \quad \frac{d}{dh} \Big|_{h=t} (d_2 E_n)(g(h), g'(h)) = 0.$$

So, the Euler-Lagrange equations do not hold, therefore, $g(t)$ is not geodesic. This result is proved in [9] by using the geodesic equation on the manifold of Riemannian metrics which is considered as the projective limit of Banach manifolds.

References

- [1] Chow, B., Lu, P., and Ni, L., *Hamilton's Ricci flow*, Graduate Studies in Mathematics 77, 2006.
- [2] Constantin, A., and Kolev, B., *Geodesic flow on the diffeomorphism group of the circle*, Comment. Math. Helv. **78** (2003), no. 4, 787-804.
- [3] Eftekharinasab, K., *On the generalization of the Darboux theorem*, Proceedings of the International Geometry Center, **12** (2019), no. 2, 1-10.
- [4] Eftekharinasab, K., *Geometry of bounded Fréchet manifolds*, Rocky Mount. J. Math., **46** (2016), no. 3, 895-913.
- [5] Eftekharinasab, K., *A simple proof of the short-time existence and uniqueness for Ricci flow*, Comptes Rendus de l'Académie Bulgare des Sciences, **72** (2019), no. 5, 569-572.
- [6] Eftekharinasab, K., *The Morse-Sard-Brown theorem for functionals on bounded-Fréchet-Finsler manifolds*, Communications in Mathematics, **23** (2015), no. 2, 101-112.
- [7] Esher, J., and Kolev, B., *Geometrical methods for equations of hydrodynamical type*, J. Nonlinear Math. Phys., **19** (2012), Suppl. 1, 1240013.
- [8] Freed, D.S., and Groisser, D., *The basic geometry of the manifold of Riemannian metrics and of its quotient by the diffeomorphism group*, Michigan Math. J. **36** (1989), 323-344.
- [9] Ghahremani-Gol, H., and Razavi, A., *Ricci flow and the manifold of Riemannian metrics*, Balk. J. Geom. Appl., **18** (2013), no. 2, 20-30.
- [10] Ghahremani-Gol, H., and Razavi, A., *The Ricci Flow as a Geodesic on the Manifold of Riemannian Metrics*, J. Contemp. Math. Anal., **51** (2016), no. 5, 38-48.
- [11] Gil-medrano, O., and Michor, P., *The Riemannian manifold of all Riemannian metrics*, Quarterly J. Math., **42** (1991), no. 2, 183-202.

- [12] Glöckner, H., *Implicit functions from topological vector spaces to Fréchet spaces in the presence of metric estimates*, ArXiv preprint, <http://arXiv.org/abs/math/0612673v5>.
- [13] Klingenberg, W., *Riemannian geometry*, De Gruyter Studies in Mathematics 1, Berlin, New York, 1982.
- [14] Kriegl, K., and Michor, P.W., *The convenient setting of global analysis*, Mathematical Surveys and Monographs 53, AMS, 1997.
- [15] Lang, S., *Fundamental of differential geometry*, Graduate Texts in Mathematics 191, Springer-Verlag, New York, 1999.
- [16] Lenells, L., *Riemannian geometry on the diffeomorphism group of the circle*, Ark. Mat., **45** (2007), no. 2, 297-325.
- [17] Michor, P., and Mumford, D., *Vanishing geodesic distance on spaces of submanifolds and diffeomorphisms*, Doc. Math., **10** (2005), 217-245.
- [18] Michor, P., and Mumford, D., *Riemannian geometries on spaces of plane curves*, J. Eur. Math. Soc. (JEMS), **8** (2006), no. 1, 1-48.
- [19] Müller, O., *A metric approach to Fréchet geometry*, J. Geom. Phys., **58** (2008), 1477-1500.
- [20] Neeb, K.-H., *A Cartan-Hadamard theorem for Banach-Finsler manifolds*, Geometriae Dedicata, **95** (2002) 115-156.
- [21] Neeb, K.-H., *Toward a Lie theory of locally convex groups*, Jpn. J. Math., **2** (2006), no. 2, 291-468
- [22] Omori, H., *Infinite dimensional Lie groups*, Translations of Mathematical Monographs **158**, 1997.
- [23] Palais, R., *Lusternik-Schnirelman theory on Banach manifolds*, Topology **5** (1966), no. 2, 115-132.
- [24] Subramaniam, T. N., *Slices for the actions of smooth tame Lie groups*, PhD thesis, Brandeis University, 1984.
- [25] Thomas, E. G. F., *Vector fields as derivations on nuclear manifolds*, Math. Nachr., **176** (1995), no. 1, 277-286.
- [26] Vallejo, J. A., *Euler-Lagrange equations for functionals defined on Fréchet manifolds*, J. Nonlinear Math. Phys., **16** (2009), no. 4, 443-454.

