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REFINEMENTS OF CAUCHY-BUNYAKOVSKY-SCHWARTZ INEQUALITY AND BERGSTRÖM INEQUALITY WITH APPLICATIONS

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Abstract

The present paper proposes some generalizations of Cauchy-Bunyakovsky-Schwarz inequality and some improvements of the Bergström inequality based on the newly obtained results. Also, improvements of the Bergström inequality based on Milne's inequality and Callebaut's inequality are presented. The newly acquired results are particularized and new applications are presented. In the last part of the paper, the current results are applied to attain improvements of the Nesbitt inequality.

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Key words: Cauchy-Bunyakovsky-Schwarz inequality, Milne's inequality, Callebaut's inequality, Bergström inequality, Nesbitt inequality

1 Introduction

The Cauchy-Bunyakovsky-Schwarz inequality, which is also known as the Cauchy-Schwarz inequality, states that for $a_1, a_2, ..., a_n \in \mathbb{R}$ and $b_1, b_2, ..., b_n \in \mathbb{R}$ we have

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2.$$

The equality holds for $\frac{a_1}{b_1} = \frac{a_2}{b_2} = ... = \frac{a_n}{b_n}$.

This inequality is used in numerous fields of mathematics and it is one of the most important and largely used inequalities. Thus, various mathematicians have made refinements and improvements to the Cauchy-Bunyakovsky-Schwarz (CBS) inequality. Among the numerous scientific articles which deal with this inequality we note the ones by Dragomir [3], [4], [5], De Buijin [1], Daykin, Eliezer and Carlitz [2], and Wigren [9].

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In [2], Daykin, Eliezer and Carlitz obtained a refinement of the (CBS) inequality as follows.

Theorem 1. *Let* $a_1, a_2, ..., a_n, b_1, b_2, ..., b_n \in \mathbb{R}$ *and let* $f, g : [0,\infty) \times [0,\infty) \to (0,\infty)$. *The inequality*

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \sum_{i=1}^{n} f(a_i, b_i) \sum_{i=1}^{n} g(a_i, b_i) \leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \tag{1}$$

holds if and only if

$$f(a,b)g(a,b) = a^2 b^2; \tag{2}$$

$$f(ka,kb) = k^2 f(a,b); \tag{3}$$

$$\frac{bf(a,1)}{af(b,1)} + \frac{af(b,1)}{bf(a,1)} \leq \frac{a}{b} + \frac{b}{a}, \tag{4}$$

for all $a, b, k > 0$.

Particularizing Theorem 1 for $f(a,b) = a^2 + b^2$ and $g(a,b) = \frac{a^2b^2}{a^2+b^2}$, the inequality known as *Milne's inequality* is obtained:

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \sum_{i=1}^{n} (a_i^2 + b_i^2) \sum_{i=1}^{n} \frac{a_i^2 b_i^2}{a_i^2 + b_i^2} \leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2. \tag{5}$$

Another generalization is obtained for choosing $f(a,b) = a^{1+\alpha}b^{1-\alpha}$ and $g(a,b) = a^{1-\alpha}b^{1+\alpha}$, $\alpha \in [0,1]$ which leads to the *inequality of Callebaut*

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \sum_{i=1}^{n} a_i^{1+\alpha} b_i^{1-\alpha} \sum_{i=1}^{n} a_i^{1-\alpha} b_i^{1+\alpha} \leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2. \tag{6}$$

On the other hand, the (CBS) inequality has numerous applications for a particular choice of a_i and b_i. Thus, taking $a_i = \dfrac{a_i}{\sqrt{b_i}}$, and $b_i = \sqrt{b_i}$, for $i = \overline{1,n}$ and $b_i \neq 0$, we obtain the following inequality

$$\frac{a_1^2}{b_1} + \frac{a_2^2}{b_2} + ... + \frac{a_n^2}{b_n} \geq \frac{(a_1 + a_2 + .. + a_n)^2}{b_1 + b_2 + ... + b_n} \tag{7}$$

known as *Bergström inequality.*

2 Improvement of Cauchy-Bunyakovsky-Schwarz inequality

Let us first prove a useful lemma.

Lemma 1. *Let a,b,c,d be positive real numbers. The inequality*

$$\frac{a}{b} + \frac{b}{a} \geq \frac{c}{d} + \frac{d}{c} \tag{8}$$

holds if and only if

$$(ad - bc)(ac - bd) \geq 0. \tag{9}$$

Proof. We can rewrite equation (8) as follows

$$\frac{a}{b} - \frac{c}{d} + \frac{b}{a} - \frac{d}{c} \geq 0$$

which leads to

$$(ad - bc)\left(\frac{1}{bd} - \frac{1}{ac}\right) \geq 0.$$

and hence we obtain relation (9). □

Next, we will prove a Theorem which reformulates Theorem 1.

Theorem 2. *Let* $h : [0, \infty) \times [0, \infty) \to (0, \infty)$ *which satisfies*

1. $h(ka, kb) = h(a, b)$;
2. $(ah(b, 1) - bh(a, 1))(ah(a, 1) - bh(b, 1)) \geq 0.$

The following inequality holds:

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \sum_{i=1}^{n} a_i b_i h(a_i, b_i) \sum_{i=1}^{n} \frac{a_i b_i}{h(a_i, b_i)} \leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2. \tag{10}$$

Proof. Let us consider $f(a_i, b_i) = a_i b_i h(a_i, b_i)$ and $g(a_i, b_i) = \frac{a_i b_i}{h(a_i, b_i)}$ in Theorem 1. We have to prove relations (2), (3) and (4).

It is obvious that $f(a_i, b_i) g(a_i, b_i) = a_i^2 b_i^2$.

Let us prove relation (3):

$$f(ka_i, kb_i) = ka_i kb_i h(ka_i, kb_i)$$

and since $h(ka_i, kb_i) = h(a_i.b_i)$ we get $f(ka_i, kb_i) = k^2 f(a_i, b_i)$.

Relation (4) becomes

$$\frac{abh(a, 1)}{abh(b, 1)} + \frac{abh(b, 1)}{abh(a, 1)} \leq \frac{a}{b} + \frac{b}{a}.$$

If we consider $c = h(a, 1)$ and $d = h(b, 1)$ in Lemma 1, we have that

$$(ah(b, 1) - bh(a, 1))(ah(a, 1) - bh(b, 1)) \geq 0,$$

which is true. □

Proposition 1. *Let* $a_i > 0$ *and* $b_i > 0$*, for all* $i = \overline{1, n}$*, then the following inequality holds:*

$$\left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \sum_{i=1}^{n} a_i b_i \frac{a_i^2 + b_i^2}{(a_i + b_i)^2} \sum_{i=1}^{n} a_i b_i \frac{(a_i + b_i)^2}{a_i^2 + b_i^2} \leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2. \tag{11}$$

Proof. We consider $h(x,y)=\frac{x^2+y^2}{(x+y)^2}$.

We verify conditions 1. and 2. from Theorem 2. It immediately follows that

$$h(kx,ky)=\frac{(kx)^2+(ky)^2}{(kx+ky)^2}=\frac{k^2(x^2+y^2)}{k^2(x+y)^2}$$

and simplifying with k^2, we get $h(kx,ky)=h(x,y)$.

In order to prove condition 2., let us consider $c=h(a,1)$ and $d=h(b,1)$ in Lemma 1. We need to prove that

$$(ah(b,1)-bh(a,1))(ah(a,1)-bh(b,1))\geq 0$$

which is equivalent to

$$\frac{1}{(a+1)^4(b+1)^4}(a(a+1)^2(b^2+1)-b(a^2+1)(b+1)^2)(a(a^2+1)(b+1)^2-b(a+1)^2(b^2+1))\geq 0.$$

Since $\frac{1}{(a+1)^4(b+1)^4}\geq 0$ we need to prove that

$$(a(a+1)^2(b^2+1)-b(a^2+1)(b+1)^2)(a(a^2+1)(b+1)^2-b(a+1)^2(b^2+1))\geq 0. \quad (12)$$

We compute

$$(a(a+1)^2(b^2+1)-b(a^2+1)(b+1)^2)=a^3b^2+2a^2b^2+ab^2+a^3+2a^2+a-a^2b^3-2a^2b^2-a^2b-b^3-2b^2-b=$$
$$=(a-b)(a^2b^2+a^2+b^2+2a+2b+1).$$

Similarly, we get

$$(a(1+a^2)(b+1)^2-b(a+1)^2(b^2+1))=(a-b)(a^2b^2+2a^2b+2ab^2+a^2+b^2+1).$$

Consequently, relation (12) is proved. □

Remark 1. *Considering* $h(x,y)=\dfrac{x^2+y^2}{xy}$ *in Theorem 2, we obtain* Milne's inequality.

Remark 2. *If we take* $h(x,y)=a^{\alpha}b^{-\alpha}$ *in Theorem 2, we obtain* Callebaut's inequality.

2.1 Particularizations of the improved C-B-S inequality

We can take relation $b_i=\dfrac{1}{a_i}, i=\overline{1,n}$ in (11) to obtain a new relation as follows

$$n^2\leq\sum_{i=1}^{n}\frac{a_i^2+\frac{1}{a_i^2}}{\left(a_i+\frac{1}{a_i}\right)^2}\sum_{i=1}^{n}\frac{\left(a_i+\frac{1}{a_i}\right)^2}{a_i^2+\frac{1}{a_i^2}}\leq\sum_{i=1}^{n}a_i^2\sum_{i=1}^{n}\frac{1}{a_i^2}, \quad (13)$$

and for $b_i=\dfrac{1}{n}, i=\overline{1,n}$, we obtain

$$\frac{1}{n}\left(\sum_{i=1}^{n}a_i\right)^2\leq\frac{1}{n^3}\sum_{i=1}^{n}\frac{a_i(n^2a_i^2+1)}{\left(a_i+\frac{1}{n}\right)^2}\sum_{i=1}^{n}\frac{na_i\left(a_i+\frac{1}{n}\right)^2}{n^2a_i^2+1}\leq\sum_{i=1}^{n}a_i^2. \quad (14)$$

3 Generalizations for three sequences

We shall give some results for three sequences of positive real numbers.

Lemma 2. *Let $a = (a_1, ..., a_n)$, $b = (b_1, ..., b_n)$ and $c = (c_1, ..., c_n)$ be three sequences of positive real numbers. The following inequality holds*

$$\left(\sum_{i=1}^{n} a_i b_i c_i\right)^2 \leq \sum_{i=1}^{n} a_i b_i \sum_{i=1}^{n} b_i c_i \sum_{i=1}^{n} c_i a_i. \tag{15}$$

Proof. We shall first prove by induction the following result

$$P(n) : \left(\sum_{i=1}^{n} a_i b_i c_i\right)^2 \leq \sum_{i=1}^{n} a_i b_i \sum_{i=1}^{n} b_i c_i \sum_{i=1}^{n} c_i a_i,$$

for all $n \geq 1$.

For $n = 1$ we obtain the equality. We show that $P(2)$ is true and we have

$$(a_1b_1c_1 + a_2b_2c_2)^2 \leq (a_1b_1 + a_2b_2)(b_1c_1 + b_2c_2)(c_1a_1 + c_2a_2)$$

$$a_1^2b_1^2c_1^2 + 2a_1b_1c_1a_2b_2c_2 + a_2^2b_2^2c_2^2 \leq$$

$$\leq a_1^2b_1^2c_1^2+a_1b_1^2c_1a_2c_2+a_1^2b_1c_1b_2c_2+a_1b_1c_2^2a_2b_2+a_1b_1c_1^2a_2b_2+a_2^2b_1b_2c_1c_2+a_1a_2b_2^2c_1c_2+a_2^2b_2^2c_2^2.$$

Notice that

$$(b_1\sqrt{a_1a_2c_1c_2} - b_2\sqrt{a_1a_2c_1c_2})^2+$$

$$+a_1^2b_1c_1b_2c_2 + a_1b_1c_2^2a_2b_2 + a_1b_1c_1^2a_2b_2 + a_2^2b_1b_2c_1c_2 \geq 0.$$

Since the sequences are positive, the inequality is proven.

We assume $P(n)$ is true:

$$\left(\sum_{i=1}^{n} a_i b_i c_i\right)^2 \leq \sum_{i=1}^{n} a_i b_i \sum_{i=1}^{n} b_i c_i \sum_{i=1}^{n} c_i a_i,$$

for all $n \geq 1$, and we prove that $P(n+1)$ is true

$$\left(\sum_{i=1}^{n} a_i b_i c_i + abc\right)^2 \leq \left(\sum_{i=1}^{n} a_i b_i + ab\right)\left(\sum_{i=1}^{n} b_i c_i + bc\right)\left(\sum_{i=1}^{n} c_i a_i + ca\right),$$

for all $n \geq 1$.

We make the following notations

$$\alpha = \sum_{i=1}^{n} a_i b_i; \quad \beta = \sum_{i=1}^{n} b_i c_i; \quad \gamma = \sum_{i=1}^{n} c_i a_i; \quad \delta = \sum_{i=1}^{n} a_i b_i c_i,$$

and thus we obtain

$$\delta^2 + 2abc\delta + a^2b^2c^2 \leq$$

$$\leq \alpha\beta\gamma + \alpha\beta ca + \alpha\gamma bc + \alpha abc^2 + \beta\gamma ab + \beta a^2bc + \gamma ab^2c + a^2b^2c^2.$$

We know that

$$\delta^2 \leq \alpha\beta\gamma,$$

and we need to prove that

$$2abc\delta \leq \alpha\beta ca + \alpha\gamma bc + \alpha abc^2 + \beta\gamma ab + \beta a^2bc + \gamma ab^2c.$$

If we consider the pairs $(\alpha\beta ca, \gamma ab^2c)$, $(\alpha\gamma bc, \beta a^2bc)$, $(\alpha abc^2, \beta\gamma ab)$, and we apply the geometric-arithmetic mean inequality we obtain

$$\frac{\alpha\beta ca + \gamma ab^2c}{2} \geq \sqrt{\alpha\beta ca\gamma ab^2c} = \sqrt{\alpha\beta\gamma}abc \geq \delta abc;$$

$$\frac{\alpha\gamma bc + \beta a^2bc}{2} \geq \sqrt{\alpha\gamma bc\beta a^2bc} = \sqrt{\alpha\beta\gamma}abc \geq \delta abc;$$

$$\frac{\alpha abc^2 + \beta\gamma ab}{2} \geq \sqrt{\alpha abc^2\beta\gamma ab} = \sqrt{\alpha\beta\gamma}abc \geq \delta abc.$$

Summing the inequities we get

$$\alpha\beta ca + \alpha\gamma bc + \alpha abc^2 + \beta\gamma ab + \beta a^2bc + \gamma ab^2c \geq 6\delta abc \geq 2\delta abc.$$

Thus, the lemma is proved

$$\left(\sum_{i=1}^{n} a_ib_ic_i\right)^2 \leq \sum_{i=1}^{n} a_ib_i \sum_{i=1}^{n} b_ic_i \sum_{i=1}^{n} c_ia_i.$$

□

Theorem 3. *Let $a = (a_1, ..., a_n)$, $b = (b_1, ..., b_n)$ and $c = (c_1, ..., c_n)$ be thethree sequences of positive real numbers and $f, g : [0, \infty) \times [0, \infty) \to (0, \infty)$. The inequality*

$$\left(\sum_{i=1}^{n} a_ib_ic_i\right)^2 \leq$$

$$\leq \sqrt{\sum_{i=1}^{n} f(a_i, b_i) \sum_{i=1}^{n} g(a_i, b_i) \sum_{i=1}^{n} f(b_i, c_i) \sum_{i=1}^{n} g(b_i, c_i) \sum_{i=1}^{n} f(c_i, a_i) \sum_{i=1}^{n} g(c_i, a_i)} \leq$$

$$\leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \sum_{i=1}^{n} c_i^2 \tag{16}$$

holds if and only if

$$f(a, b)g(a, b) = a^2b^2$$

$$f(ka, kb) = k^2 f(a, b)$$

$$\frac{bf(a, 1)}{af(b, 1)} + \frac{af(b, 1)}{bf(a, 1)} \leq \frac{a}{b} + \frac{b}{a}$$

for all $a, b, k > 0$.

Proof. Applying Theorem 1 for the pairs $(a,b),(b,c),(c,a)$ we get

$$\left(\sum_{i=1}^n a_i b_i\right)^2 \le \sum_{i=1}^n f(a_i,b_i)\sum_{i=1}^n g(a_i,b_i) \le \sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 \tag{17}$$

$$\left(\sum_{i=1}^n b_i c_i\right)^2 \le \sum_{i=1}^n f(b_i,c_i)\sum_{i=1}^n g(b_i,c_i) \le \sum_{i=1}^n b_i^2 \sum_{i=1}^n c_i^2 \tag{18}$$

$$\left(\sum_{i=1}^n c_i a_i\right)^2 \le \sum_{i=1}^n f(c_i,a_i)\sum_{i=1}^n g(c_i,a_i) \le \sum_{i=1}^n c_i^2 \sum_{i=1}^n a_i^2 \tag{19}$$

Multiplying relations (17), (18) and (19), we get

$$\left(\sum_{i=1}^n a_i b_i \sum_{i=1}^n b_i c_i \sum_{i=1}^n c_i a_i\right)^2 \le$$

$$\le \sum_{i=1}^n f(a_i,b_i)\sum_{i=1}^n g(a_i,b_i)\sum_{i=1}^n f(b_i,c_i)\sum_{i=1}^n g(b_i,c_i)\sum_{i=1}^n f(c_i,a_i)\sum_{i=1}^n g(c_i,a_i) \le$$

$$\le \left(\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 \sum_{i=1}^n c_i^2\right)^2,$$

which is equivalent to

$$\sum_{i=1}^n a_i b_i \sum_{i=1}^n b_i c_i \sum_{i=1}^n c_i a_i \le$$

$$\le \sqrt{\sum_{i=1}^n f(a_i,b_i)\sum_{i=1}^n g(a_i,b_i)\sum_{i=1}^n f(b_i,c_i)\sum_{i=1}^n g(b_i,c_i)\sum_{i=1}^n f(c_i,a_i)\sum_{i=1}^n g(c_i,a_i)} \le$$

$$\le \sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 \sum_{i=1}^n c_i^2.$$

Using the result in Lemma 1, we obtain

$$\left(\sum_{i=1}^n a_i b_i c_i\right)^2 \le \sum_{i=1}^n a_i b_i \sum_{i=1}^n b_i c_i \sum_{i=1}^n c_i a_i \le$$

$$\le \sqrt{\sum_{i=1}^n f(a_i,b_i)\sum_{i=1}^n g(a_i,b_i)\sum_{i=1}^n f(b_i,c_i)\sum_{i=1}^n g(b_i,c_i)\sum_{i=1}^n f(c_i,a_i)\sum_{i=1}^n g(c_i,a_i)} \le$$

$$\le \sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 \sum_{i=1}^n c_i^2,$$

which gets us the desired result

$$\left(\sum_{i=1}^n a_i b_i c_i\right)^2 \le$$

$$\leq \sqrt{\sum_{i=1}^{n} f(a_i, b_i) \sum_{i=1}^{n} g(a_i, b_i) \sum_{i=1}^{n} f(b_i, c_i) \sum_{i=1}^{n} g(b_i, c_i) \sum_{i=1}^{n} f(c_i, a_i) \sum_{i=1}^{n} g(c_i, a_i)} \leq$$

$$\leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \sum_{i=1}^{n} c_i^2.$$

□

3.1 Applications

If we combine Milne's inequality (5) and Theorem 2 we obtain the following inequality

$$\left(\sum_{i=1}^{n} a_i b_i c_i\right)^2 \leq$$

$$\leq \sqrt{\sum_{i=1}^{n}(a_i^2 + b_i^2) \sum_{i=1}^{n} \frac{a_i^2 b_i^2}{a_i^2 + b_i^2} \sum_{i=1}^{n}(b_i^2 + c_i^2) \sum_{i=1}^{n} \frac{b_i^2 c_i^2}{b_i^2 + c_i^2} \sum_{i=1}^{n}(c_i^2 + a_i^2) \sum_{i=1}^{n} \frac{c_i^2 a_i^2}{c_i^2 + a_i^2}}$$

$$\leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \sum_{i=1}^{n} c_i^2.$$

We can particularize the inequality for $b_i = \frac{1}{a_i}, i = \overline{1,n}$ and we obtain the new inequality

$$\left(\sum_{i=1}^{n} c_i\right)^2 \leq$$

$$\leq \sqrt{\sum_{i=1}^{n}(a_i^2 + \frac{1}{a_i^2}) \sum_{i=1}^{n} \frac{1}{a_i^2 + \frac{1}{a_i^2}} \sum_{i=1}^{n} \frac{a_i^2 c_i^2 + 1}{a_i^2} \sum_{i=1}^{n} \frac{c_i^2}{c_i^2 a_i^2 + 1} \sum_{i=1}^{n}(c_i^2 + a_i^2) \sum_{i=1}^{n} \frac{c_i^2 a_i^2}{c_i^2 + a_i^2}} \leq$$

$$\leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} \frac{1}{a_i^2} \sum_{i=1}^{n} c_i^2. \tag{20}$$

Another particularization, for $b_i = \frac{1}{n}, i = \overline{1,n}$, gives us the following inequality

$$\frac{1}{n^2}\left(\sum_{i=1}^{n} a_i c_i\right)^2 \leq$$

$$\leq \frac{1}{n^2}\sqrt{\sum_{i=1}^{n}(n^2 a_i^2 + 1) \sum_{i=1}^{n} \frac{a_i^2}{n^2 a_i^2 + 1} \sum_{i=1}^{n}(n^2 c_i^2 + 1) \sum_{i=1}^{n} \frac{c_i^2}{n^2 c_i^2 + 1} \sum_{i=1}^{n}(c_i^2 + a_i^2) \sum_{i=1}^{n} \frac{c_i^2 a_i^2}{c_i^2 + a_i^2}} \leq$$

$$\leq \frac{1}{n} \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} c_i^2. \tag{21}$$

Another result can be obtained by taking into consideration Callebaut's inequality (6) and Theorem 2:

$$\left(\sum_{i=1}^{n} a_i b_i c_i\right)^2 \le$$

$$\le \sqrt{\sum_{i=1}^{n} a_i^{1+\alpha} b_i^{1-\alpha} \sum_{i=1}^{n} a_i^{1-\alpha} b_i^{1+\alpha} \sum_{i=1}^{n} b_i^{1+\alpha} c_i^{1-\alpha} \sum_{i=1}^{n} b_i^{1-\alpha} c_i^{1+\alpha} \sum_{i=1}^{n} c_i^{1+\alpha} a_i^{1-\alpha} \sum_{i=1}^{n} c_i^{1-\alpha} a_i^{1+\alpha}}$$

$$\le \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \sum_{i=1}^{n} c_i^2.$$

A similar choice for $b_i = \dfrac{1}{a_i}, i = \overline{1,n}$, gives us

$$\left(\sum_{i=1}^{n} c_i\right)^2 \le$$

$$\le \sqrt{\sum_{i=1}^{n} a_i^{2\alpha} \sum_{i=1}^{n} \frac{1}{a_i^{2\alpha}} \sum_{i=1}^{n} \frac{c_i^{1-\alpha}}{a_i^{1+\alpha}} \sum_{i=1}^{n} \frac{c_i^{1+\alpha}}{a_i^{1-\alpha}} \sum_{i=1}^{n} c_i^{1+\alpha} a_i^{1-\alpha} \sum_{i=1}^{n} c_i^{1-\alpha} a_i^{1+\alpha}} \le \tag{22}$$

$$\le \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} \frac{1}{a_i^2} \sum_{i=1}^{n} c_i^2,$$

and for $b_i = \dfrac{1}{n}, i = \overline{1,n}$, we get

$$\frac{1}{n^2}\left(\sum_{i=1}^{n} a_i c_i\right)^2 \le$$

$$\le \frac{1}{n^2}\sqrt{\sum_{i=1}^{n} a_i^{1+\alpha} \sum_{i=1}^{n} a_i^{1-\alpha} \sum_{i=1}^{n} c_i^{1-\alpha} \sum_{i=1}^{n} c_i^{1+\alpha} \sum_{i=1}^{n} c_i^{1+\alpha} a_i^{1-\alpha} \sum_{i=1}^{n} c_i^{1-\alpha} a_i^{1+\alpha}} \le \tag{23}$$

$$\le \frac{1}{n} \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} c_i^2.$$

4 Improvements of the Bergström inequality

We shall give some improvements of the *Bergström inequality*.

4.1 Improvements based on Milne's inequality

Proposition 2. *Let $a_i > 0$ and $b_i > 0$, for all $i = \overline{1,n}$, then the following inequality holds:*

$$\sum_{i=1}^{n} \frac{a_i^2}{b_i} \geq \frac{\sum_{i=1}^{n}\left(\frac{a_i^2+b_i^2}{b_i}\right)\sum_{i=1}^{n}\left(\frac{a_i^2 b_i^2}{a_i^2+b_i^2}\right)}{\sum_{i=1}^{n} b_i} \geq \frac{\left(\sum_{i=1}^{n} a_i\right)^2}{\sum_{i=1}^{n} b_i}. \tag{24}$$

Proof. If we consider $a_i = \frac{x_i}{\sqrt{y_i}}$ and $b_i = \sqrt{y_i}$ in relation (5), we get

$$\sum_{i=1}^{n} \frac{x_i^2}{y_i} \sum_{i=1}^{n} y_i \geq \sum_{i=1}^{n}\left(\frac{x_i^2+y_i^2}{y_i}\right)\sum_{i=1}^{n}\left(\frac{x_i^2 y_i^2}{x_i^2+y_i^2}\right) \geq \left(\sum_{i=1}^{n} \frac{x_i^2}{\sqrt{y_i}}\sqrt{y_i}\right)^2.$$

and thus, we get equation (24). □

Proposition 3. *Let $a_i \geq 2b_i$, $b_i > 0$ and $c_i = \sqrt{a_i + 2b_i}$, $d_i = \sqrt{a_i - 2b_i}$, for all $i = \overline{1,n}$, then the following inequality holds:*

$$2\frac{\sum_{i=1}^{n} \frac{b_i}{(c_i-d_i)^3}\sum_{i=1}^{n}(c_i-d_i)}{\sum_{i=1}^{n} \frac{a_i}{c_i-d_i}} \geq \sum_{i=1}^{n} \frac{b_i^2}{a_i} \geq \frac{1}{8}\frac{\left(\sum_{i=1}^{n}(c_i+d_i)\right)^2}{\sum_{i=1}^{n} \frac{a_i}{c_i-d_i}}. \tag{25}$$

Proof. Let

$$\begin{cases} x_i^2 + y_i^2 = a_i \\ x_i y_i = b_i \end{cases}$$

in inequality (24). For x_i and y_i from Proposition 1 we get

$$\sum_{i=1}^{n} \frac{(c_i+d_i)^2}{4}\frac{2}{c_i-d_i} \geq \frac{\sum_{i=1}^{n} \frac{2a_i}{c_i-d_i}\sum_{i=1}^{n}\frac{b_i^2}{a_i}}{\sum_{i=1}^{n}\frac{c_i-d_i}{2}} \geq \frac{1}{4}\frac{\left(\sum_{i=1}^{n} c_i+d_i\right)^2}{\sum_{i=1}^{n}\frac{c_i-d_i}{2}},$$

from which we obtain

$$2\frac{\sum_{i=1}^{n} \frac{b_i}{(c_i-d_i)^3}\sum_{i=1}^{n}(c_i-d_i)}{\sum_{i=1}^{n} \frac{a_i}{c_i-d_i}} \geq \sum_{i=1}^{n} \frac{b_i^2}{a_i} \geq \frac{1}{8}\frac{\left(\sum_{i=1}^{n}(c_i+d_i)\right)^2}{\sum_{i=1}^{n} \frac{a_i}{c_i-d_i}}. \tag{26}$$

□

Observation 3.1. *Taking $b_i = x_i$ and $a_i = y_i$ in (25) we obtain a new upper bound*

$$\sum_{i=1}^{n} \frac{x_i^2}{y_i} \leq 2 \frac{\displaystyle\sum_{i=1}^{n} \frac{x_i^2}{(\sqrt{y_i + 2x_i} - \sqrt{y_i - 2x_i})^3} \sum_{i=1}^{n} (\sqrt{y_i + 2x_i} - \sqrt{y_i - 2x_i})}{\displaystyle\sum_{i=1}^{n} \frac{y_i}{x_i} (\sqrt{y_i + 2x_i} - \sqrt{y_i - 2x_i})}. \tag{27}$$

Observation 3.2. *Combining equation (27) with the result in Proposition 1, we get*

$$2 \frac{\displaystyle\sum_{i=1}^{n} \frac{x_i^2}{(\sqrt{y_i + 2x_i} - \sqrt{y_i - 2x_i})^3} \sum_{i=1}^{n} (\sqrt{y_i + 2x_i} - \sqrt{y_i - 2x_i})}{\displaystyle\sum_{i=1}^{n} \frac{y_i}{x_i} (\sqrt{y_i + 2x_i} - \sqrt{y_i - 2x_i})} \geq \sum_{i=1}^{n} \frac{x_i^2}{y_i} \geq$$

$$\frac{\displaystyle\sum_{i=1}^{n} \left(\frac{x_i^2 + y_i^2}{y_i} \right) \sum_{i=1}^{n} \left(\frac{x_i^2 y_i^2}{x_i^2 + y_i^2} \right)}{\displaystyle\sum_{i=1}^{n} y_i} \geq \frac{\left(\displaystyle\sum_{i=1}^{n} x_i \right)^2}{\displaystyle\sum_{i=1}^{n} y_i}.$$

4.1.1 Particularizations of Milne's inequality

We can particularize relation (5) for $b_i = \frac{1}{a_i}, i = \overline{1, n}$ to obtain a new relation as follows

$$n^2 \leq \sum_{i=1}^{n} \left(a_i^2 + \frac{1}{a_i^2} \right) \sum_{i=1}^{n} \frac{1}{a_i^2 + \frac{1}{a_i^2}} \leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} \frac{1}{a_i^2}, \tag{28}$$

and for $b_i = \frac{1}{n}, i = \overline{1, n}$, we obtain

$$\frac{1}{n} \left(\sum_{i=1}^{n} a_i \right)^2 \leq \frac{1}{n} \sum_{i=1}^{n} (n^2 a_i^2 + 1) \sum_{i=1}^{n} \frac{a_i^2}{n^2 a_i^2 + 1} \leq \sum_{i=1}^{n} a_i^2. \tag{29}$$

Moreover, if we take $b_i = \frac{1}{a_i}, i = \overline{1, n}$, in (24) we get

$$\sum_{i=1}^{n} a_i^3 \geq \frac{\displaystyle\sum_{i=1}^{n} \frac{a_i^4 + 1}{a_i} \sum_{i=1}^{n} \frac{a_i^3}{a_i^4 + 1}}{\displaystyle\sum_{i=1}^{n} \frac{1}{a_i}} \geq \frac{\left(\displaystyle\sum_{i=1}^{n} a_i \right)^2}{\displaystyle\sum_{i=1}^{n} \frac{1}{a_i}}.$$

and for $b_i = \frac{1}{n}, i = \overline{1, n}$, we have

$$n \sum_{i=1}^{n} a_i^2 \geq \sum_{i=1}^{n} (n^2 a_i^2 + 1) \sum_{i=1}^{n} \frac{a_i^2}{n^2 a_i^2 + 1} \geq \left(\sum_{i=1}^{n} a_i \right)^2.$$

4.2 Improvements based on Callebaut's inequality

Proposition 4. *Let $a_i > 0$ and $b_i > 0$, for all $i = \overline{1,n}$, and $\alpha \in (0,1)$ then the following inequality holds:*

$$\sum_{i=1}^{n} \frac{a_i^2}{b_i} \geq \frac{\sum_{i=1}^{n} a_i^{1+\alpha} b_i^{-\alpha} \sum_{i=1}^{n} a_i^{1-\alpha} b_i^{\alpha}}{\sum_{i=1}^{n} b_i} \geq \frac{\left(\sum_{i=1}^{n} a_i\right)^2}{\sum_{i=1}^{n} b_i}. \tag{30}$$

Proof. In Callebaut's inequality (6), we consider $a_i = \frac{x_i}{\sqrt{y_i}}$ and $b_i = \sqrt{y_i}$ and we get

$$\left(\sum_{i=1}^{n} x_i\right)^2 \leq \sum_{i=1}^{n} x_i^{1+\alpha} y_i^{1-\alpha-1-\alpha} \sum_{i=1}^{n} x_i^{1-\alpha} y_i^{1+\alpha-1+\alpha} \leq \sum_{i=1}^{n} \frac{x_i^2}{y_i} \sum_{i=1}^{n} y_i$$

which leads to the desired inequality. □

4.2.1 Particularizations of Callebaut's inequality

We can particularize relation (6) for $b_i = \frac{1}{a_i}, i = \overline{1,n}$ to obtain the inequality

$$n^2 \leq \sum_{i=1}^{n} a_i^{2\alpha} \sum_{i=1}^{n} \frac{1}{a_i^{2\alpha}} \leq \sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} \frac{1}{a_i^2}.$$

and taking $b_i = \frac{1}{n}, i = \overline{1,n}$, we have

$$\frac{1}{n}(\sum_{i=1}^{n} a_i)^2 \leq \frac{1}{n} \sum_{i=1}^{n} a_i^{1+\alpha} \sum_{i=1}^{n} a_i^{1-\alpha} \leq \sum_{i=1}^{n} a_i^2.$$

Particularizing relation (30) for $b_i = \frac{1}{a_i}$, we get

$$\sum_{i=1}^{n} a_i^3 \geq \frac{\sum_{i=1}^{n} a_i^{1+2\alpha} \sum_{i=1}^{n} a_i^{1-2\alpha}}{\sum_{i=1}^{n} \frac{1}{a_i}} \geq \frac{\left(\sum_{i=1}^{n} a_i\right)^2}{\sum_{i=1}^{n} \frac{1}{a_i}}.$$

and for $b_i = \frac{1}{n}, i = \overline{1,n}$, we have

$$n \sum_{i=1}^{n} a_i^2 \geq \sum_{i=1}^{n} a_i^{1+\alpha} \sum_{i=1}^{n} a_i^{1-\alpha} \geq \left(\sum_{i=1}^{n} a_i\right)^2.$$

5 Application - refinements of Nesbitt inequality

Consider the *Nesbitt* inequality

$$\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}\geq\frac{3}{2}. \tag{31}$$

Equation (31) can be rewritten as

$$\frac{a^2}{a(b+c)}+\frac{b^2}{b(a+c)}+\frac{c^2}{c(a+b)}\geq\frac{3}{2}.$$

Using inequality (24) with $a_1=a$, $a_2=b$, $a_3=c$ and $b_1=a(b+c)$, $b_2=b(a+c)$ and $b_3=c(a+b)$, we get a refinement of the Nesbitt inequality as follows

$$\begin{gathered}\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}=\frac{a^2}{a(b+c)}+\frac{b^2}{b(a+c)}+\frac{c^2}{c(a+b)}\geq\\ \frac{1}{2(a+b+c)}\left[\frac{a}{b+c}(1+(b+c)^2)+\frac{b}{a+c}(1+(a+c)^2)+\frac{c}{a+b}(1+(a+b)^2)\right]\cdot\\ \cdot\left[\frac{a^2(b+c)^2}{1+(b+c)^2}+\frac{b^2(a+c)^2}{1+(a+c)^2}+\frac{c^2(a+b)^2}{1+(a+b)^2}\right]\geq\\ \frac{(a^2+b^2+c^2)}{2(ab+bc+ac)}\geq\frac{3}{2}.\end{gathered} \tag{32}$$

Moreover, from inequality (30) we get the further refinement of Nesbitt inequality

$$\begin{gathered}\frac{a}{b+c}+\frac{b}{a+c}+\frac{c}{a+b}=\frac{a^2}{a(b+c)}+\frac{b^2}{b(a+c)}+\frac{c^2}{c(a+b)}\geq\\ \frac{1}{2(a+b+c)}\left[\frac{a}{(b+c)^\alpha}+\frac{b}{(a+c)^\alpha}+\frac{c}{(a+b)^\alpha}\right]\cdot\\ \cdot\left[\frac{a^2(b+c)^2}{1+(b+c)^2}+\frac{b^2(a+c)^2}{1+(a+c)^2}+\frac{c^2(a+b)^2}{1+(a+b)^2}\right]\geq\\ \frac{(a^2+b^2+c^2)}{2(ab+bc+ac)}\geq\frac{3}{2}.\end{gathered} \tag{33}$$

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