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# SOME RESULTS ON $LP$-SASAKIAN MANIFOLDS

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**Abstract**

The object of the present paper is to characterize $LP$-Sasakian manifolds satisfying Ricci pseudosymmetry and Ricci generalized pseudosymmetry. Beside this we prove that if $R(X,\xi).P = P(X,\xi).R$ holds, where $R$ and $P$ denote the curvature tensor and projective curvature tensor respectively, then the manifold becomes an Einstein manifold. Then we prove that $divR = 0$ and $divC = 0$ are equivalent if the scalar curvature is invariant under the characteristic vector field $\xi$, where '$div$' denotes divergence. Finally, we characterize 3-dimensional $LP$-Sasakian manifolds admitting Yamabe solitons and prove that the scalar curvature is constant and the potential vector field $V$ is Killing.

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## 1 Introduction

In 1989, the notion of Lorentzian Para-Sasakian manifold ( briefly $LP$-Sasakian manifold) was introduced by Matsumoto[13]. Then Mihai and Rosca[15] introduced the same notion separately and several results on this manifold have been obtained by them. $LP$-Sasakian manifolds have also been studied by Matsumoto and Mihai[14], Matsumoto, Mihai and Rosca[15], De and Saikh([6],[7]), De, Al-Aqeel and Shaikh[5], Ozgur[17], Ozgur and Murathan[18] and many others.

We define endomorphism $R(X,Y)$ and $X \wedge_A Y$ by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{1}$$

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and

$$(X \wedge_A Y)Z = A(Y,Z)X - A(X,Z)Y \tag{2}$$

respectively, where $X, Y, Z \in \chi(M), \chi(M)$ is the set of all differentiable vector fields on the manifold $M$, $A$ is the symmetric (0,2)-type tensor, $R$ is the Riemannian curvature tensor of type (1,3) and $\nabla$ is the Levi-Civita connection.

For the (0,k)-tensor field $T, k \geqslant 1$, on $(M^n, g)$ we define the tensors $R.T$ and $Q(g,T)$ by

$$\begin{aligned} & (R(X,Y).T)(X_1, X_2, ......, X_k) \\ = & -T(R(X,Y)X_1, X_2, ......, X_k) - T(X_1, R(X,Y)X_2, ....., X_k) \\ - & ............ - T(X_1, X_2, ........, R(X,Y)X_k) \end{aligned} \tag{3}$$

and

$$\begin{aligned} & Q(g,T)(X_1, X_2, ......, X_k) \\ = & -T((X \wedge Y)X_1, X_2, ......, X_k) - T(X_1, (X \wedge Y)X_2, ....., X_k) \\ - & ............ - T(X_1, X_2, ......., (X \wedge Y)X_k) \end{aligned} \tag{4}$$

respectively[22].

If the tensor $R.S$ and $Q(g,S)$ are linearly dependent then $M^n$ is called Ricci pseudo- symmetric[22]. This is equivalent to

$$R.S = fQ(g,S), \tag{5}$$

holding on the set $U_s = \{x \in M : S \neq 0 \text{ at } x\}$, where $f$ is some function on $U_s$. Analogously, if the tensors $R.R$ and $Q(S,R)$ are linearly independent, then $M^n$ is called Ricci generalized pseudo-symmetric[22]. This is equivalent to

$$R.R = fQ(S,R), \tag{6}$$

holding on the set $U_R = \{x \in M : R \neq 0 \text{ at } x\}$ , where $f$ is some function on $U_R$. A very important subclass of this class of manifolds realizing the condition is

$$R.R = Q(S,R). \tag{7}$$

Every three dimensional manifold satisfies the above equation identically. The condition $R.R = Q(S,R)$ also appears in the theory of plane gravitational waves. Further more we define the tensor $R.R$ and $R.S$ on $(M^n, g)$ by

$$\begin{aligned} (R(X,Y).R)(U,V)W &= R(X,Y)R(U,V)W - R(R(X,Y)U,V)W \\ &- R(U,R(X,Y)V)W - R(U,V)R(X,Y)W \end{aligned} \tag{8}$$

and

$$(R(X,Y).S)(U,V) = -S(R(X,Y)U,V) - S(U,R(X,Y)V) \tag{9}$$

respectively.

Recently, Kowalczyk[12] studied semi-Riemannian manifolds satisfying $Q(S,R) = 0$ and $Q(S,g) = 0$, where $S, R$ are the Ricci tensor and curvature tensor respectively.

In a Riemannian or semi-Riemannian manifold of dimension n, $divR$ is obtained from the Bianchi identity and given by

$$(divR)(X,Y)Z = (\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z), \tag{10}$$

where $R$ denotes the curvature tensor, $S$ is the Ricci tensor, $\nabla$ is the Riemannian connection and '*div*' denotes the divergence.

Also it is known that

$$\begin{aligned}(divC)(X,Y)Z &= \frac{n-2}{n-3}[\{(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)\} \\ &\quad -\frac{1}{2(n-1)}\{dr(X)g(Y,Z) - dr(Y)g(X,Z)\}],\end{aligned} \tag{11}$$

where $C$ is the Weyl curvature tensor of type (1,3), $r$ is the scalar curvature.

From the above definitions, it follows that $divR = 0$ implies $divC = 0$. However, the converse, is not necessarily true.

In an $n$-dimensional Riemannian manifold the projective curvature tensor $P$ is defined by [16]

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}\{S(Y,Z)X - S(X,Z)Y\}, \tag{12}$$

for $X, Y, Z \in T(M)$, where $R$ is the curvature tensor and $S$ is the Ricci tensor. Infact, $M$ is projectively flat ( that is, $P = 0$) if and only if the manifold is of constant curvature[24]. Thus, the projective curvature tensor is a measure of the failure of a Riemannian manifold to be of constant curvature.

In 1982 Szabo([20],[21]) studied Riemannian spaces satisfying $R(X,Y).R = 0$. In [3] De and Samui studied $P.R = 0, R.P = 0$ and $P.S = 0$ in an $LP$-Sasakian manifold.

In [11], Hamilton introduced the notion of Yamabe solition. According to Hamilton, a Riemannian metric $g$ of an $n$-dimensional Riemannian manifold $(M, g)$ is said to be a Yamabe soliton if it satisfies

$$\mathcal{L}_V g = (\lambda - r)g, \tag{13}$$

for a smooth vector field $V$ and a real number $\lambda$, where $r$ is the scalar curvature of $g$ and $\mathcal{L}$ denotes the Lie-derivative operator. The vector field is called soliton field of the Yamabe soliton. A Yamabe soliton is said to be shrinking, steady or expanding according to $\lambda > 0, \lambda = 0$ or $\lambda < 0$, respectively. Yamabe solitons have been studied by many authors in different contexts (see, [10], [19], [23], [24]).

On the other hand, Yamabe solitons have been defined in pseudo-Riemannian manifold, in particular in Lorentzian manifold in the same way as in the Riemannian manifolds[2].

Let us now briefly review conformal vector fields. A vector field $V$ on an $m$-dimensional Riemannian manifold $(M, g)$ is said to be conformal if,

$$\mathcal{L}_V g = 2\rho g \tag{14}$$

for a smooth function $\rho$ on $M$. A conformal vector field satisfies,

$$(\mathcal{L}_V S)(X, Y) = -(m-2)g(\nabla_X D\rho, Y) + (\Delta\rho)g(X, Y), \tag{15}$$

$$\mathcal{L}_V r = -2\rho r + 2(m-1)\Delta\rho, \tag{16}$$

where $D$ is the gradient operator and $\Delta = -divD$ is the Laplacian operator of $g$ [25].

The present paper is organized as follows :
After preliminaries we characterize Ricci pseudosymmetric and Ricci generalized pseudosymmetric $LP$-Sasakian manifolds in section 3 and 4 respectively. Next, in section 5, we study the curvature condition $R(X, \xi).P = P(X, \xi).R$ in an $LP$-Sasakian manifold. Then we prove that in an $LP$-Sasakian manifold $divR = 0$ and $divC = 0$ are equivalent under certain restriction on the scalar curvature. Finally, we study Yamabe solitons in a 3-dimensional $LP$-Sasakian manifold.

## 2 Preliminaries

An $n$-dimensional differentiable manifold $M$ is called an $LP$-Sasakian manifold ([8],[14]) if it admits a (1,1) tensor field $\phi$, a covariant vector field $\xi$, a 1-from $\eta$ and a Lorentzian metric $g$ satisfying :

$$\phi^2(X) = X + \eta(X)\xi, \eta(\xi) = -1, \tag{17}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{18}$$

$$g(X, \xi) = \eta(X), \nabla_X \xi = \phi X, \tag{19}$$

$$(\nabla_X \phi)Y = g(X, Y)\xi + 2\eta(X)\eta(Y)\xi, \tag{20}$$

where $\nabla$ denotes the operator of covariant differentiation with respect to the Lorentzian metric $g$.

It can be easily seen that in an $LP$-Sasakian manifold, the following relations hold:

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \tag{21}$$

$$rank(\phi) = n - 1. \tag{22}$$

Again if we put

$$\Omega(X, Y) = g(X, \phi Y), \tag{23}$$

for any vector fields X and Y , then the tensor field $\Omega(X, Y)$ is symmetric (0,2) tensor field[13]. Also, since the vector field $\eta$ is closed in an $LP$-Sasakian manifold, we have ([8],[13]),

$$\Omega(X, Y) = (\nabla_X \eta)Y, \quad \Omega(X, \xi) = 0, \tag{24}$$

for any vector fields X and Y.

An $LP$-Sasakian manifold $M$ is said to be $\eta$-Einstein if its Ricci tensor S is of the from

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y), \tag{25}$$

for any vector fields $X, Y$ where a,b are smooth functions on $M$. Let $M$ be an n-dimensional $LP$-Sasakian manifold with the structure $(\phi, \xi, \eta, g)$. Then we have([8],[14]):

$$g(R(X, Y)Z, \xi) = \eta(R(X, Y)Z) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y), \tag{26}$$

$$R(\xi, X)Y = g(X, Y)\xi - \eta(Y)X, \tag{27}$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{28}$$

$$R(\xi, X)\xi = X + \eta(X)\xi, \tag{29}$$

$$S(X, \xi) = (n - 1)\eta(X), \tag{30}$$

$$S(\phi X, \phi Y) = S(X, Y) + (n - 1)\eta(X)\eta(Y), \tag{31}$$

for any vector fields $X, Y, Z$, where $R$ is the curvature tensor and $S$ is the Ricci tensor.

## 3 Ricci pesudo-symmetric $LP$-Sasakian manifolds

In this section we study Ricci pseudosymmetric $LP$-Sasakian manifolds, that is, the manifold satisfies the condition

$$R.S = fQ(g, S).$$

Assume that $M$ is a Ricci pseudo-symmetric $LP$-Sasakian manifold and $X, Y, U, V \in \chi(M)$. We have from (5)

$$(R(X, Y).S)(U, V) = fQ(g, S)(X, Y; U, V), \tag{32}$$

which is equivalent to

$$(R(X,Y).S)(U,V) = f((X \wedge_g Y)(U,V). \tag{33}$$

Using (9) and (4) in the above equation, we get

$$\begin{aligned} -S(R(X,Y)U,V) - S(U,R(X,Y)V) &= f[-g(Y,U)S(X,V) + g(X,U)S(Y,V) \\ &- g(Y,V)S(U,X) + g(X,V)S(U,Y)]. \end{aligned} \tag{34}$$

Putting $X = U = \xi$ in (34) and using (26)-(29) yields

$$(1-f)[S(Y,V) - (n-1)g(Y,V)] = 0. \tag{35}$$

Then either $f = 1$ or, the manifold is an Einstein manifold of the from

$$S(Y,V) = (n-1)g(Y,V). \tag{36}$$

By the above discussion we have the following :

**Proposition 1.** *Every $n$-dimensional Ricci pseudo-symmetric $LP$-Sasakian manifold is of the from $R.S = Q(g,S)$, provided the manifold is non-Einstein.*

Conversely, if the manifold is an Einstein manifold of the from (36), then from (34) it follows that $R.S = fQ(g,S)$. This leads to the following :

**Theorem 1.** *An $n$-dimensional $LP$-Sasakian manifold is Ricci pseudo-symmetric if and only if the manifold is an Einstein manifold, provided $f \neq 1$.*

## 4 Ricci generalized pseudo-symmetric $LP$-Sasakian manifolds

This section deals with Ricci generalized pseudosymmetric $LP$-Sasakian manifolds. Let us suppose that $M$ is an $n$-dimensional Ricci generalized pseudosymmetric $LP$-Sasakian manifolds. Then from (6) we have

$$R.R = fQ(S,R), \tag{37}$$

that is,

$$(R(X,Y).R)(U,V)W = f((X \wedge_S Y).R)(U,V)W. \tag{38}$$

Using (8) and (4) we get from (38)

$$\begin{aligned} & R(X,Y)R(U,V)W - R(R(X,Y)U,V)W \\ - & R(U,R(X,Y)V)W - R(U,V)R(X,Y)W \\ = & f[(X \wedge_S Y)R(U,V)W - R((X \wedge_S Y)U,V)W \\ - & R(U,(X \wedge_S Y)V)W - R(U,V)(X \wedge_S Y)W]. \end{aligned} \tag{39}$$

Applying (2) in (39), we get

$$\begin{aligned} & R(X,Y)R(U,V)W - R(R(X,Y)U,V)W \\ - \; & R(U,R(X,Y)V)W - R(U,V)R(X,Y)W \\ = \; & f[S(Y,R(U,V)W)X - S(X,R(U,V)W)Y \\ - \; & S(Y,U)R(X,V)W + S(X,U)R(X,Y)W \\ - \; & S(Y,V)R(U,X)W + S(X,V)R(U,Y)W \\ - \; & S(Y,W)R(U,V)X + S(X,W)R(U,V)Y]. \end{aligned} \tag{40}$$

Substituting $X = U = \xi$ and using (27)-(30) in the above equation, we have

$$\begin{aligned} & g(V,W)Y + g(V,W)\eta(Y)\xi - g(Y,V)\eta(W)\xi + \eta(V)\eta(W)Y \\ - \; & R(Y,V)W - g(V,W)\eta(Y)\xi + \eta(W)\eta(Y)V + g(Y,W)\eta(V)\xi \\ - \; & \eta(W)\eta(V)Y - g(Y,W)V - g(Y,W)\eta(V)\xi + g(V,Y)\eta(W)\xi - \eta(Y)\eta(W)V \\ = \; & f[(n-1)g(V,W)\eta(Y)\xi - S(Y,V)\eta(W)\xi + (n-1)g(V,W)Y \\ + \; & (n-1)\eta(W)\eta(V)Y - (n-1)g(V,W)\eta(Y)\xi + (n-1)\eta(W)\eta(Y)V \\ - \; & (n-1)R(Y,V)W + (n-1)g(Y,W)\eta(V)\xi - (n-1)\eta(W)\eta(V)Y - S(Y,W)V \\ - \; & \eta(V)S(Y,W)\xi + (n-1)g(V,Y)\eta(W)\xi - (n-1)\eta(Y)\eta(W)V]. \end{aligned} \tag{41}$$

Taking the inner product of (41) with $Z$ we obtain

$$\begin{aligned} & g(V,W)g(Y,Z) + g(Y,Z)\eta(V)\eta(W) - g(R(Y,V)W,Z) \\ - \; & g(Y,Z)\eta(W)\eta(V) - g(Y,W)g(V,Z) \\ = \; & f[(n-1)g(V,W)g(Y,Z) - S(Y,V)\eta(W)\eta(Z) \\ - \; & (n-1)g(R(Y,V)W,Z) + (n-1)g(Y,W)\eta(V)\eta(Z) \\ - \; & S(Y,W)g(V,Z) - S(Y,W)\eta(V)\eta(Z) + (n-1)g(V,Y)\eta(W)\eta(Z)]. \end{aligned} \tag{42}$$

Putting $Y = Z = e_i$ in (42), where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and taking summation over '$i$'$(1 \leq i \leq n)$, we get

$$(1 - nf)[S(V,W) - (n-1)g(V,W)] = 0.$$

Then either $f = \frac{1}{n}$ or, the manifold is an Einstein manifold of the from

$$S(V,W) = (n-1)g(V,W).$$

This leads the following :

**Theorem 2.** *An $n$-dimensional Ricci generalized pseudo-symmetric LP-Sasakian manifold is an Einstein manifold, provided $nf \neq 1$.*

By the above discussion we have the following :

**Proposition 2.** *Every $n$-dimensional Ricci generalized pseudo-symmetric LP-Sasakian manifold is of the from $R.R = \frac{1}{n}Q(S,R)$, provided the manifold is non-Einstein.*

## 5 $LP$-Sasakian manifolds satisfying $R(X,\xi).P = P(X,\xi).R$

This section is devoted to characterizing $LP$-Sasakian manifolds satisfying the curvature condition $R(X,\xi).P = P(X,\xi).R$.
Suppose

$$(R(X,\xi).P)(U,V)W = (P(X,\xi).R)(U,V)W.$$

Then we get

$$\begin{aligned} & R(X,\xi)P(U,V)W - P(R(X,\xi)U,V)W \\ - & P(U,R(X,\xi)V)W - P(U,V)R(X,\xi)W \\ = & P(X,\xi)R(U,V)W - R(P(X,\xi)U,V)W \\ - & R(U,P(X,\xi)V)W - R(U,V)P(X,\xi)W. \end{aligned} \tag{43}$$

Substituting $U = W = \xi$ and using (27)-(30) in the above equation, we get

$$S(X,V) = (n-1)g(X,V).$$

Hence, we have

**Theorem 3.** *An LP-Sasakian manifold satisfying the curvature condition $R(X,\xi).P = P(X,\xi).R$, is an Einstein manifold.*

## 6 $LP$-Sasakian manifolds with $divC = 0$

From the definition of $divR$ and $divC$, it follows that $divR = 0$ implies $divC = 0$. But the converse, is not generally, true. In this section we prove that $divC = 0$ implies $divR = 0$.
Let us assume that $divC = 0$. Then from (11) we have

$$(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) = \frac{1}{2(n-1)}[dr(X)g(Y,Z) - dr(Y)g(X,Z)]. \tag{44}$$

Using (30) we have

$$(\nabla_X S)(Y,\xi) - (\nabla_Y S)(X,\xi) = 2(n-1)d\eta(X,Y). \tag{45}$$

But in an $LP$-Sasakian manifold $d\eta = 0$[11]. Then (45) yields

$$(\nabla_X S)(Y,\xi) - (\nabla_Y S)(X,\xi) = 0. \tag{46}$$

Substituting $Z = \xi$ in (44) and using (46), we have

$$dr(X)\eta(Y) - dr(Y)\eta(X) = 0.$$

Replacing $X$ by $\xi$ in the above equation, it follows

$$dr(Y) = -dr(\xi)\eta(Y).$$

Suppose the scalar is invariant under the characteristic vector $\xi$, then from the above expression we get $r$ is constant.
Hence from (44) we get

$$(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) = 0,$$

which implies

$$divR = 0.$$

Conversely, let us suppose $divR = 0$. This implies $r$ = constant.
Hence $divR = 0$ implies $divC = 0$.
Thus, we state the following :

**Theorem 4.** *In an LP-Sasakian manifold $divR = 0$ and $divC = 0$ are equivalent, provided the scalar curvature $r$ is invariant under the characteristic vector filed $\xi$.*

## 7 A 3-dimensional $LP$-Sasakian metric as a Yamabe soliton

Before proving the main theorem, we state and prove the following lemma.

**Lemma 1.** *For an LP-Sasakian manifold, equation (13) implies*

$$\text{(i) } (\mathcal{L}_V \eta)(\xi) = \tfrac{r-\lambda}{2}$$

*and*

$$\text{(ii) } \eta(\mathcal{L}_V \xi) = \tfrac{\lambda-r}{2}.$$

*Proof.* In an $LP$-Sasakian manifold

$$g(\xi,\xi) = -1. \tag{47}$$

Lie-differentiating (13) along V and using (47) we obtain (ii).
Next, Lie-differentiating $\eta(\xi) = -1$ along V gives (i). This completes the proof. □

Let us consider a Yamabe soliton that is of type $(g,\xi)$ on an $LP$-Sasakian manifold, that is, $V = \xi$.
From (13) we have

$$\mathcal{L}_V g = (\lambda - r)g. \tag{48}$$

Substituting $V = \xi$ in (48), we obtain

$$(\mathcal{L}_\xi g)(X,Y) = (\lambda - r)g(X,Y). \tag{49}$$

Now putting $X = Y = \xi$ in (49), we get $\lambda = r$, since $\nabla_\xi \xi = 0$.
Now using $\lambda = r$ in (49), we get $\xi$ is a Killing vector field.
In view of the above, we can state the following theorem :

**Theorem 5.** *If an LP-Sasakian manifold admits Yamabe soliton $(g, \xi)$, then $\xi$ is a Killing vector field.*

Recalling that the Ricci tensor $S$ of a 3-dimensional $LP$-Sasakian manifold is given by[4]

$$S(X,Y) = \frac{1}{2}\{(r-2)g(X,Y) + (6-r)\eta(X)\eta(Y)\}. \tag{50}$$

As $V$ is a conformal vector field with $\rho = \frac{r-\lambda}{2}$, equations (15) and (16) can be rewritten as,

$$(\mathcal{L}_V S)(X,Y) = \frac{1}{2}[g(\nabla_X Dr, Y) - (\Delta r)g(X,Y)], \tag{51}$$

$$\mathcal{L}_V r = -2\Delta r - r(\lambda - r). \tag{52}$$

Applying Lie-derivative of (50) and using (13), (51) and (52) yields

$$\begin{aligned} g(\nabla_X Dr, Y) &= -[\Delta r + 2(\lambda - r)]g(X,Y) \\ &+ [2\Delta r + r(\lambda - r)]\eta(X)\eta(Y) \\ &+ (6-r)[(\mathcal{L}_V\eta)X\eta(Y) + (\mathcal{L}_V\eta)Y\eta(X)]. \end{aligned} \tag{53}$$

In a 3-dimensional $LP$-Sasakian manifold, holds[9]

$$\xi r = -(r-6)trace(\phi). \tag{54}$$

Let us suppose that

$$trace(\phi) = 0, \tag{55}$$

that is, the characteristic vector field $\xi$ is harmonic[1].
Differentiating (54) along an arbitrary vector field $X$ and using (55),(17)-(19), we infer

$$g(\nabla_X Dr, \xi) = -(\phi X)r. \tag{56}$$

Substituting $\xi$ for Y in (53), using (56) and Lemma 7.1 provided the equation :

$$-(\phi X)r = [-3\Delta r - \frac{(\lambda - r)(10 + r)}{2}]\eta(X) - (6-r)(\mathcal{L}_V\eta)X. \tag{57}$$

Putting $X = \xi$ in the above equation, using (17),(19) and Lemma 7.1, we get

$$\Delta r = -\frac{8}{3}(\lambda - r). \tag{58}$$

Using (58) in (57) gives

$$(6-r)(\mathcal{L}_V\eta)X = (\phi X)r + \frac{(\lambda - r)(6-r)}{2}\eta(X). \tag{59}$$

Equations (58) and (59) transform equation (53) as

$$\nabla_X Dr = \frac{2}{3}(\lambda - r)[X + \eta(X)\xi] + g(X, \phi Dr)\xi + \eta(X)\phi Dr. \tag{60}$$

At this point, we assume $\{e_i\}$(i= 1,2,3) to be a local orthonormal frame on $M$. Using (60) we compute $S(X, Dr) = g(R(e_i, X)Dr, e_i)$, and then using (17),(19), skew-symmetry of $\phi$ and equation (20) we obtain,

$$S(X, Dr) = -\eta(X)g(\phi\nabla_{ei}Dr, e_i),$$

where '$i$' is summed over 1,2,3. Then use of (60) in the right hand side of the foregoing equation shows that $S(X, Dr) = 0$. Using this in (50) immediately yields $(Xr)(r-2) = 0$, which implies $r =$ constant. Hence from (58) we obtain $r = \lambda$. Thus equation (13) implies the potential vector field $V$ is Killing.
This leads to the following :

**Theorem 6.** *If a 3-dimensional LP-Sasakian manifold admits Yamabe soliton, then the scalar curvature is constant and the potential vector field $V$ is Killing, provided the characteristic vector field $\xi$ is harmonic.*

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